Ternary linear codes and quadrics

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Abstract

For an $[n, k, d]_3$ code $\mathcal{C}$ with $gcd(d, 3) = 1$, we define a map w_G from $\Sigma = \mathrm{PG}(k-1, 3)$ to the set of weights of codewords of $\mathcal{C}$ through a generator matrix G. A t-flat Π in Σ is called an $(i, j)_t$ flat if $(i, j) = (|\Pi \cap F_0|, |\Pi \cap F_1|)$, where $F_0 = \{P \in \Sigma \mid w_G(P) \equiv 0 \pmod 3\}$, $F_1 = \{P \in \Sigma \mid w_G(P) \not\equiv 0, d \pmod 3\}$. We give geometric characterizations of $(i, j)_t$ flats, which involve quadrics. As an application to the optimal linear codes problem, we prove the non-existence of a $[305, 6, 202]_3$ code, which is a new result.

1 Introduction

Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements. A linear code $\mathcal{C}$ of length n, dimension k and minimum (Hamming) distance d over $\mathbb{F}_q$ is referred to as an $[n, k, d]_q$ code. Linear codes over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$ are called binary, ternary and quaternary linear codes, respectively. The *weight* of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. The weight distribution of $\mathcal{C}$ is the list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i. The weight distribution with $(A_0, A_d, ...) = (1, \alpha, ...)$ is also expressed as $0^1 d^\alpha \cdots$. We only consider *non-degenerate* codes having no coordinate which is identically zero. An $[n, k, d]_q$ code $\mathcal{C}$ with a generator matrix G is called (l, s)*-extendable* (to $\mathcal{C}'$) if there exist l vectors $h_1, \ldots, h_l \in \mathbb{F}_q^k$ so that the extended matrix $[G, h_1^{\mathrm{T}}, \cdots, h_l^{\mathrm{T}}]$ generates an $[n+l, k, d+s]_q$ code $\mathcal{C}'$ ([10]). Then $\mathcal{C}'$ is called an (l, s)*-extension* of $\mathcal{C}$. $\mathcal{C}$ is simply called *extendable* if $\mathcal{C}$ is $(1, 1)$-extendable.

We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A j*-flat* is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes*,

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respectively. We refer to [7], [8] and [9] for geometric terminologies. We investigate linear codes over $\mathbb{F}_q$ through the projective geometry.

We assume that $k \geq 3$. Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix $G = [g_0, g_1, \cdots, g_{k-1}]^{\mathrm{T}}$. Put $\Sigma =$PG$(k-1, q)$, the projective space of dimension $k-1$ over $\mathbb{F}_q$. We consider the mapping w_G from Σ to $\{i \mid A_i > 0\}$, the set of weights of codewords of $\mathcal{C}$. For $P = \mathbf{P}(p_0, p_1, \ldots, p_{k-1}) \in \Sigma$ we define *the weight of P with respect to G*, denoted by $w_G(P)$, as

$$w_G(P) = wt(\sum_{i=0}^{k-1} p_i g_i).$$

Our geometric method is just the dual version of that introduced first in [11] to investigate the extendability of $\mathcal{C}$. See also [14], [15], [16], [18] for the extendability of ternary linear codes. Let

$$\begin{aligned} F &= \{P \in \Sigma \mid w_G(P) \not\equiv d \pmod{q}\}, \\ F_d &= \{P \in \Sigma \mid w_G(P) = d\}. \end{aligned}$$

Recall that a hyperplane H of Σ is defined by a non-zero vector $h = (h_0, \ldots, h_{k-1}) \in \mathbb{F}_q^k$ as $H = \{P = \mathbf{P}(p_0, \ldots, p_{k-1}) \in \Sigma \mid h_0 p_0 + \cdots + h_{k-1} p_{k-1} = 0\}$. h is called a *defining vector of H*, which is uniquely determined up to non-zero multiple. It would be possible to investigate the $(l, 1)$-extendability of linear codes from the geometrical structure of F or F_d as follows.

Theorem 1.1 ([12]). *$\mathcal{C}$ is $(l, 1)$-extendable if and only if there exist l hyperplanes $H_1, \ldots, H_l$ of Σ such that $F_d \cap H_1 \cap \cdots \cap H_l = \emptyset$. Moreover, the extended matrix of G by adding the defining vectors of $H_1, \ldots, H_l$ as columns generates an $(l, 1)$-extension of $\mathcal{C}$. Hence, $\mathcal{C}$ is $(l, 1)$-extendable if there exists a $(k-1-l)$-flat contained in F.*

The mapping w_G is *trivial* if $F = \emptyset$. For example, w_G is trivial if $\mathcal{C}$ attains the Griesmer bound and if q divides d when q is prime [17]. When w_G is trivial, there seems no clue to investigate the extendability of $\mathcal{C}$ except for computer search, see [10]. To avoid such cases we assume $\gcd(d, q) = 1$; d and q are relatively prime. Then, F forms a blocking set with respect to lines [12], that is, every line meets F in at least one point. The aim of this paper is to give a geometric characterization of F for $q = 3$. An application to the optimal linear codes problem is also given in Section 4.

2 Main theorems

Let $\mathcal{C}$ be an $[n, k, d]_3$ code with $k \geq 3$, $\gcd(3, d) = 1$. The *diversity* (Φ_0, Φ_1) of $\mathcal{C}$ was defined in [11] as the pair of integers:

$$\Phi_0 = \frac{1}{2} \sum_{3|i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \not\equiv 0, d \pmod{3}} A_i,$$

where the notation $x|y$ means that x is a divisor of y. Let

$$\begin{aligned} F_0 &= \{P \in \Sigma \mid w_G(P) \equiv 0 \pmod 3\}, \\ F_2 &= \{P \in \Sigma \mid w_G(P) \equiv d \pmod 3\}, \\ F_1 &= F \setminus F_0,\ F_e = F_2 \setminus F_d. \end{aligned}$$

Then we have $\Phi_s = |F_s|$ for $s = 0, 1$.

A t-flat Π of Σ with $|\Pi \cap F_0| = i$, $|\Pi \cap F_1| = j$ is called an $(i,j)_t$ *flat*. An $(i,j)_1$ flat is called an (i,j)-*line*. An (i,j)-*plane*, an (i,j)-*solid* and so on are defined similarly. We denote by $\mathcal{F}_j$ the set of j-flats of Σ. Let Λ_t be the set of all possible (i,j) for which an $(i,j)_t$ flat exists in Σ. Then we have

$$\begin{aligned} \Lambda_1 &= \{(1,0),(0,2),(2,1),(1,3),(4,0)\}, \\ \Lambda_2 &= \{(4,0),(1,6),(4,3),(4,6),(7,3),(4,9),(13,0)\}, \\ \Lambda_3 &= \{(13,0),(4,18),(13,9),(10,15),(16,12),(13,18),(22,9),(13,27),(40,0)\}, \\ \Lambda_4 &= \{(40,0),(13,54),(40,27),(31,45),(40,36),(40,45),(49,36),(40,54),(67,27), \\ &\quad (40,81),(121,0)\}, \\ \Lambda_5 &= \{(121,0),(40,162),(121,81),(94,135),(121,108),(112,126),(130,117), \\ &\quad (121,135),(148,108),(121,162),(202,81),(121,243),(364,0)\}, \end{aligned}$$

see [11]. Let $\Pi_t \in \mathcal{F}_t$. Denote by $c_{i,j}^{(t)}$ the number of $(i,j)_{t-1}$ flats in Π_t and let ${\varphi_s}^{(t)} = |\Pi_t \cap F_s|$, $s = 0,1$. $({\varphi_0}^{(t)}, {\varphi_1}^{(t)})$ is called the *diversity of* Π_t and the list of $c_{i,j}^{(t)}$'s is called its *spectrum*. Thus Λ_t is the set of all possible diversities of Π_t. It holds that $(\varphi_0, \varphi_1) \in \Lambda_t$ implies $(3\varphi_0 + 1, 3\varphi_1) \in \Lambda_{t+1}$ ([15]). We call $(\varphi_0, \varphi_1) \in \Lambda_t$ is *new* if $((\varphi_0 - 1)/3, \varphi_1/3) \notin \Lambda_{t-1}$. For example, $(4,3),(4,6) \in \Lambda_2$ and $(10,15),(16,12) \in \Lambda_3$ are new. We define that $(0,2),(2,1) \in \Lambda_1$ are new for convenience. Let $\theta_j = |\mathrm{PG}(j,3)| = (3^{j+1}-1)/2$. We set $\theta_j = 0$ for $j < 0$. New diversities of Λ_t and the corresponding spectra for $t \geq 2$ are given as follows.

Lemma 2.1 ([15]). *New diversities and the corresponding spectra for $t \geq 2$ are*

(1) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)$ *with spectrum*

$$(c^{(t)}_{\theta_{t-2}-3^{T+1},\theta_{t-2}+\theta_T+1}, c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_T}, c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1})$$
$$= (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1, \theta_{t-1} + \theta_T + 1)$$

and

$(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)$ *with spectrum*

$$(c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_T}, c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1}, c^{(t)}_{\theta_{t-2}+3^{T+1},\theta_{t-2}-\theta_T})$$
$$= (\theta_{t-1} - \theta_T, \theta_{t-1} - \theta_T, \theta_{t-1} + 3^{T+1})$$

when t is odd, where $T = (t-3)/2$.

(2) $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1})$ *with spectrum*

$$(c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+1}}, c^{(t)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1}, c^{(t)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U})$$
$$= (\theta_{t-1}, \theta_{t-1} - \theta_{U+1}, \theta_{t-1} + \theta_{U+1} + 1),$$

and

$(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)$ *with spectrum*

$(c^{(t)}_{\theta_{t-2}-3^{U+1}, \theta_{t-2}+\theta_U+1}, c^{(t)}_{\theta_{t-2}+3^{U+1}, \theta_{t-2}-\theta_U}, c^{(t)}_{\theta_{t-2}, \theta_{t-2}+\theta_{U+1}+1})$
$= (\theta_{t-1} - \theta_{U+1}, \theta_{t-1} + \theta_{U+1} + 1, \theta_{t-1})$

when t is even, where $U = (t-4)/2$.

Let us recall some known results on quadrics in $\mathrm{PG}(r,3)$, $r \geq 2$, from [9]. Let $f \in \mathbb{F}_3[x_0, \ldots, x_r]$ be a quadratic form which is non-degenerate, that is, f is not reducible to a form in fewer than $r+1$ variables by a linear transformation. We define

$$V_i(f) = \{P = \mathbf{P}(p_0, \ldots, p_{r-1}) \in \mathrm{PG}(r,3) \mid f(p_0, \ldots, p_{r-1}) = i\}$$

for $i = 0, 1, 2$. Then, $V_0(f)$ is a *non-singular quadric*. Let

$\mathcal{P}_r^i = V_i(x_0^2 + x_1x_2 + \cdots + x_{r-1}x_r)$ for r even;

$\mathcal{E}_r^i = V_i(x_0^2 + x_1^2 + x_2x_3 + \cdots + x_{r-1}x_r)$, $\mathcal{H}_r^i = V_i(x_0x_1 + x_2x_3 + \cdots + x_{r-1}x_r)$ for r odd.

The quadrics $\mathcal{P}_r^0$, $\mathcal{H}_r^0$ and $\mathcal{E}_r^0$ are called *parabolic, hyperbolic* and *elliptic*, respectively. It is well known for any non-singular quadric $\mathcal{Q}$ in $\mathrm{PG}(r,3)$ that $\mathcal{Q} \sim \mathcal{P}_r^0$ for r even and that $\mathcal{Q} \sim \mathcal{H}_r^0$ or $\mathcal{Q} \sim \mathcal{E}_r^0$ for r odd (see Section 5.2 in [8]), where $\mathcal{Q}_1 \sim \mathcal{Q}_2$ means that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are projectively equivalent.

Theorem 2.2. *Let Π_t be a t-flat in Σ with new diversity, $t \geq 2$.*
(1) $F_0 \cap \Pi_t \sim \mathcal{P}_t^0$ when t is even.
(2) $F_0 \cap \Pi_t \sim \mathcal{E}_t^0$ if $\varphi_0^{(t)} = \theta_{t-1} - 3^{T+1}$ and $F_0 \cap \Pi_t \sim \mathcal{H}_t^0$ if $\varphi_0^{(t)} = \theta_{t-1} + 3^{T+1}$ when t is odd, where $T = (t-3)/2$.

We define $2V_i(f) = V_i(2f)$ for $i = 1, 2$. We prove the following theorem in the next section.

Theorem 2.3. *Let Π_t be a t-flat in Σ with new diversity, $t \geq 2$.*
(1) $F_i \cap \Pi_t \sim \mathcal{P}_t^i$ or $2\mathcal{P}_t^i$ for $i = 1, 2$ when t is even.
(2) $F_i \cap \Pi_t \sim \mathcal{E}_t^i$ if $\varphi_0^{(t)} = \theta_{t-1} - 3^{T+1}$ and $F_i \cap \Pi_t \sim \mathcal{H}_t^i$ if $\varphi_0^{(t)} = \theta_{t-1} + 3^{T+1}$ for $i = 1, 2$ when t is odd, where $T = (t-3)/2$.

The geometric characterizations of t-flats whose diversities are not new are already known. We summarize them here. For $t \geq 2$ we set Λ_t^- and Λ_t^+ as

$$\begin{aligned} \Lambda_t^- &= \{(\theta_{t-1}, 0), (\theta_{t-2}, 2 \cdot 3^{t-1}), (\theta_{t-1}, 2 \cdot 3^{t-1}), (\theta_{t-1} + 3^{t-1}, 3^{t-1}), (\theta_{t-1}, 3^t), (\theta_t, 0)\} \\ \Lambda_t^+ &= \Lambda_t \setminus \Lambda_t^-. \end{aligned}$$

Then Λ_t^- is included in Λ_t for all $t \geq 2$, $\Lambda_2^+ = \{(4,3)\}$, and $\mathcal{C}$ is extendable if $(\Phi_0, \Phi_1) \in \Lambda_{k-1}^-$ ([11]). It is also known that Π_t contains a (4,3)-plane if and only if its diversity is in Λ_t^+. Obviously, A $(\theta_t, 0)_t$ flat is contained in F_0.

Theorem 2.4 ([11]). *Let Π_t be a $(\varphi_0, \varphi_1)_t$ flat in Σ with $(\varphi_0, \varphi_1) \in \Lambda_t^-$, $t \geq 2$.*
(1) $\Pi_t \cap F_0$ forms a hyperplane of Π_t if $(\varphi_0, \varphi_1) = (\theta_{t-1}, 0)$ or $(\theta_{t-1}, 3^t)$.
(2) There are two $(\theta_{t-2}, 3^{t-1})_{t-1}$ flats in Π_t meeting in a $(\theta_{t-2}, 0)_{t-2}$ flat if $(\varphi_0, \varphi_1) = (\theta_{t-2}, 2 \cdot 3^{t-1})$.
(3) There are two $(\theta_{t-1}, 0)_{t-1}$ flats and a $(\theta_{t-2}, 3^{t-1})_{t-1}$ flat through a fixed $(\theta_{t-2}, 0)_{t-2}$ flat in Π_t if $(\varphi_0, \varphi_1) = (\theta_{t-1} + 3^{t-1}, 3^{t-1})$.

Recall that $(i, j) \in \Lambda_t$ implies $(3i + 1, 3j) \in \Lambda_{t+1}$, so $(3^\nu i + \theta_{\nu-1}, 3^\nu j) \in \Lambda_{t+\nu}$ for $\nu = 1, 2, \cdots$. $(\varphi_0, \varphi_1) \in \Lambda_t$ is *ν-descendant* if $(\varphi_0, \varphi_1) = (3^\nu i + \theta_{\nu-1}, 3^\nu j)$ for some new $(i, j) \in \Lambda_{t-\nu}$. For example, $(13, 9) \in \Lambda_3$ is 1-descendant since (4,3) is new in Λ_2.

Let Π_t be a $(\varphi_0, \varphi_1)_t$ flat with $(\varphi_0, \varphi_1) = (\theta_{t-1}, 2 \cdot 3^{t-1})$ or $(\varphi_0, \varphi_1) \in \Lambda_t^+$. Assume that (φ_0, φ_1) is not new in Λ_t. Then (φ_0, φ_1) is ν-descendant for some positive integer ν. A t-flat whose diversity is ν-descendant can be characterized with axis.

An s-flat S in Π_t is called the *axis of Π_t of type (a, b)* if every hyperplane of Π_t not containing S has the same diversity (a, b) and if there is no hyperplane of Π_t through S whose diversity is (a, b). Then the spectrum of Π_t satisfies $c_{a,b}^{(t)} = \theta_t - \theta_{t-1-s}$ and the axis is unique if it exists ([14]).

Theorem 2.5 ([16]). *Let Π_t be a $(\varphi_0, \varphi_1)_t$ flat in Σ with $(\varphi_0, \varphi_1) = (\theta_{t-1}, 2 \cdot 3^{t-1})$ or $(\varphi_0, \varphi_1) \in \Lambda_t^+$, $t \geq 3$, and let ν be a positive integer. Then, (φ_0, φ_1) is ν-descendant in Λ_t if and only if Π_t contains a $(\theta_{\nu-1}, 0)_{\nu-1}$ flat which is the axis of Π_t.*

If Π_t has a $(\theta_{\nu-1}, 0)_{\nu-1}$ flat L which is the axis of type (a, b), then for any point P in L and a point Q of an $(a, b)_{t-1}$ flat H in Π_t, $\langle P, Q \rangle$ is a (4,0)-line, a (1,3)-line or a (1,0)-line if $Q \in F_0$, $Q \in F_1$, $Q \in F_2$, respectively, where $\langle P, Q \rangle$ is the line through P and Q. In this paper, $\langle \chi_1, \chi_2, \cdots \rangle$ stands for the smallest flat containing subsets $\chi_1, \chi_2, \cdots$ of Σ.

Proof of Theorem 2.2. When $t = 2$, Π_2 is a (4,3)-plane or a (4,6)-plane, and $F_0 \cap \Pi_2$ forms a 4-arc (a set of 4 points no three of which are collinear, see [11]), which is projectively equivalent to a conic $\mathcal{P}_2^0$ by Theorem 8.14 in [8].

When $t = 3$, Π_3 is a (10,15)-solid or a (16,12)-solid. If Π_3 is a (10,15)-solid, then it follows from the spectrum that $F_0 \cap \Pi_3$ forms a 10-cap (a set of 10 points no three of which are collinear), whence we have $F_0 \cap \Pi_3 \sim \mathcal{E}_3^0$ by Theorem 16.1.7 in [7]. Similarly, if Π_3 is a (16,12)-solid, we obtain $F_0 \cap \Pi_3 \sim \mathcal{H}_3^0$ from the spectrum of Π_3 by Theorem 16.2.1 in [7].

Assume $t \geq 4$. Since every line in Σ meets F_0 in $0, 1, 2$ or $\theta_1 = 4$ points, and since every point P of $F_0 \cap \Pi_t$ is on a (2,1)-line when Π_t has new diversity (see Section 3 for the exact number of (2,1)-lines through P in Σ), $F_0 \cap \Pi_t$ forms a non-singular $\varphi_0^{(t)}$-set of type $(0, 1, 2, \theta_1)$, see Section 22.10 in [9]. It can be easily shown by induction on t that a maximal flat contained in $F_0 \cap \Pi_t$ is a T-flat when Π_t has diversity $(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)$ with t odd, $T = (t-3)/2$, for Π_t contains a hyperplane whose diversity is 1-descendant to new $(\theta_{t-3} - 3^T, \theta_{t-3} + \theta_{T-1} + 1) \in \Lambda_{t-2}$. Hence our assertion follows from Theorem 22.11.6 in [9] and Lemma 2.1.

3 Focal points and focal hyperplanes

For $i = 1, 2$, a point $P \in F_i$ is called a *focal point* of a hyperplane H (or P is *focal to* H) if the following three conditions hold:

(a) $\langle P, Q\rangle$ is a $(0, 2)$-line for $Q \in F_i \cap H$,

(b) $\langle P, Q\rangle$ is a $(2, 1)$-line for $Q \in F_{3-i} \cap H$,

(c) $\langle P, Q\rangle$ is a $(1, 6-3i)$-line for $Q \in F_0 \cap H$.

Such a hyperplane H is called a *focal hyperplane* of P (or H is *focal to* P). Note that for any point Q of H, the two points on the line $\langle P, Q\rangle$ other than P, Q are contained in the same set F_j for some $0 \le j \le 2$ with $Q \notin F_j$. Hence, a focal hyperplane of a given point is uniquely determined if it exists. Conversely, a focal point of a given hyperplane H' is uniquely determined if it exists and if every point of $F_0 \cap H'$ is contained in a $(2, 1)$-line in H'. Note that every point of $F_0 \cap \Pi_t$ is contained in a $(2, 1)$-line in Π_t if $({\varphi_0}^{(t)}, {\varphi_1}^{(t)})$ is new. From the one-to-one correspondence between focal points and focal hyperplanes, we get the following.

Lemma 3.1. *Let $t \ge 2$, $i = 1$ or 2 and let Π_t be a t-flat with ${\varphi_s}^{(t)} = |\Pi_t \cap F_s|$ for $s = 0, 1, 2$, satisfying ${\varphi_i}^{(t)} = c^{(t)}_{a,b}$ and that (a, b) is new in Λ_{t-1}. Then, every point of $\Pi_t \cap F_i$ has a focal (a, b)-hyperplane in Π_t if and only if every (a, b)-hyperplane of Π_t has a focal point in $\Pi_t \cap F_i$.*

We note from Lemma 2.1 that the condition ${\varphi_i}^{(t)} = c^{(t)}_{a,b}$ in Lemma 3.1 holds for $i = 1, 2$ for some new $(a, b) \in \Lambda_{t-1}$ if $({\varphi_0}^{(t)}, {\varphi_1}^{(t)})$ is new in Λ_t.

Lemma 3.2. *Let δ be a $(4, 3)$-plane. Then, every point of $\delta \cap F_1$ and of $\delta \cap F_2$ has a focal $(0, 2)$-line and a focal $(2, 1)$-line, respectively, and vice versa.*

Proof. Recall from [11] that $K = \delta \cap F_0$ forms a 4-arc in δ and that δ has spectrum $(c^{(2)}_{1,0}, c^{(2)}_{0,2}, c^{(2)}_{2,1}) = (4, 3, 6)$. The set of internal points of K (on no unisecant of K [8]) is $\delta \cap F_1$ and the set of external points of K (on two unisecants of K [8]) is $\delta \cap F_2$. For $Q \in \delta \cap F_1$, there exists a unique $(0, 2)$-line ℓ in δ not containing Q. Then ℓ is the focal line of Q. For $R \in \delta \cap F_2$, there is a unique (2,1)-line ℓ_1 through R. Let Q' be the point of F_1 in ℓ_1 and let ℓ_2 be the (2,1)-line through Q' other than ℓ_1. Then ℓ_2 is the focal line of R. The converses follow by Lemma 3.1. □

See Fig. 1 for the configuration of a $(4, 3)$-plane (Q and R are focal to ℓ_1 and ℓ_2, respectively). Replacing $\delta \cap F_1$ and $\delta \cap F_2$ for a $(4, 3)$-plane yields a (4,6)-plane with spectrum $(c^{(2)}_{1,3}, c^{(2)}_{0,2}, c^{(2)}_{2,1}) = (4, 3, 6)$, see Fig. 2. Hence we get the following.

Lemma 3.3. *Let δ be a $(4, 6)$-plane. Then, every point of $\delta \cap F_2$ and of $\delta \cap F_1$ has a focal $(0, 2)$-line and a focal $(2, 1)$-line, respectively, and vice versa.*

For a flat S in a $(\varphi_0, \varphi_1)_t$ flat Π_t, let $r^{(s)}_{i,j}$ be the number of $(i, j)_s$ flats through S in Π_t. We summarize the lists of $r^{(s)}_{i,j}$'s to Table 3.1 for $(\varphi_0, \varphi_1)_t = (10, 15)_3, (16, 12)_3$.

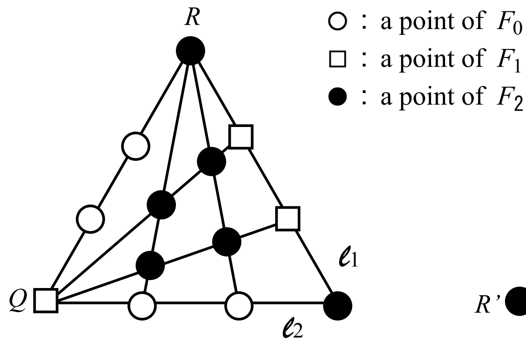

Fig. 1. $(4,3)$-plane

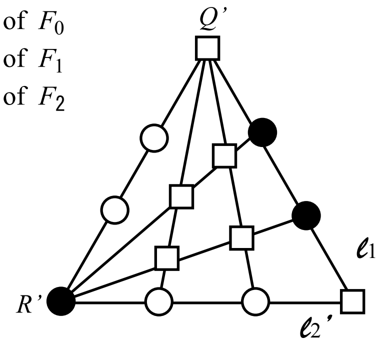

Fig. 2. $(4,6)$-plane

Table 3.1.

Π_t	S	$r^{(s)}_{i,j}$ = # of $(i,j)_s$ flats through S in Π_t
$(10,15)_3$	$P \in F_0$	$r^{(1)}_{1,0} = r^{(1)}_{1,3} = 2$, $r^{(1)}_{2,1} = 9$
$(10,15)_3$	$Q \in F_1$	$r^{(1)}_{0,2} = 6$, $r^{(1)}_{2,1} = 3$, $r^{(1)}_{1,3} = 4$
$(10,15)_3$	$R \in F_2$	$r^{(1)}_{1,0} = 4$, $r^{(1)}_{0,2} = 6$, $r^{(1)}_{2,1} = 3$
$(10,15)_3$	$(1,0)_1$	$r^{(2)}_{1,6} = 1$, $r^{(2)}_{4,3} = 3$
$(10,15)_3$	$(0,2)_1$	$r^{(2)}_{1,6} = 2$, $r^{(2)}_{4,3} = r^{(2)}_{4,6} = 1$
$(10,15)_3$	$(2,1)_1$	$r^{(2)}_{4,3} = r^{(2)}_{4,6} = 2$
$(10,15)_3$	$(1,3)_1$	$r^{(2)}_{1,6} = 1$, $r^{(2)}_{4,6} = 3$
$(16,12)_3$	$P \in F_0$	$r^{(1)}_{1,0} = r^{(1)}_{1,3} = 1$, $r^{(1)}_{2,1} = 9$, $r^{(1)}_{4,0} = 2$
$(16,12)_3$	$Q \in F_1$	$r^{(1)}_{0,2} = 3$, $r^{(1)}_{2,1} = 6$, $r^{(1)}_{1,3} = 4$
$(16,12)_3$	$R \in F_2$	$r^{(1)}_{1,0} = 4$, $r^{(1)}_{0,2} = 3$, $r^{(1)}_{2,1} = 6$
$(16,12)_3$	$(1,0)_1$	$r^{(2)}_{4,3} = 3$, $r^{(2)}_{7,3} = 1$
$(16,12)_3$	$(0,2)_1$	$r^{(2)}_{4,3} = r^{(2)}_{4,6} = 2$
$(16,12)_3$	$(2,1)_1$	$r^{(2)}_{4,3} = r^{(2)}_{4,6} = 1$, $r^{(2)}_{7,3} = 2$
$(16,12)_3$	$(1,3)_1$	$r^{(2)}_{4,6} = 3$, $r^{(2)}_{7,3} = 1$
$(16,12)_3$	$(4,0)_1$	$r^{(2)}_{7,3} = 4$

Lemma 3.4. *Let Δ be a $(10,15)$-solid. Then, every point of $\Delta \cap F_1$ and of $\Delta \cap F_2$ has a focal $(4,6)$-plane and a focal $(4,3)$-plane, respectively, and vice versa.*

Proof. We prove that every point $R \in \Delta \cap F_2$ has a focal $(4,3)$-plane. It follows from Table 3.1 that there are exactly four (1,0)-lines through R in Δ, say $\ell_1, \dots, \ell_4$. Let P_i be the point $\ell_i \cap F_0$ for $i = 1, \dots, 4$ and let δ be a plane containing P_1, P_2, P_3. Since Δ has spectrum $(c^{(3)}_{1,6}, c^{(3)}_{4,3}, c^{(3)}_{4,6}) = (10, 15, 15)$, δ is a (4,3)-plane or a (4,6)-plane. Let P be the point of $\delta \cap F_0$ other than P_1, P_2, P_3, and put $\ell = \langle P, R \rangle$. Then $\delta_i = \langle \ell, P_i \rangle$ is a (4,3)-plane for $i = 1, 2, 3$, since it contains a (1,0)-line ℓ_i. Thus, ℓ is contained in three (4,3)-planes. Hence ℓ is a (1,0)-line by Table 3.1, and we have $P = P_4$ and $\ell = \ell_4$. Since the line $\langle P, P_i\rangle$ is a (2,1)-line and since $\ell_1, \ldots, \ell_4$ are (1,0)-lines, R is focal to $\langle P, P_i\rangle$ in δ_i for $i = 1, 2, 3$. Now, let ℓ_P be the line through P in δ other than $\langle P, P_i\rangle$, $i = 1, 2, 3$. Then $\langle \ell, \ell_P\rangle$ is a (1,6)-plane by Table 3.1, and ℓ_P is a (1,0)-line or a (1,3)-line, for a (1,6)-plane has spectrum $(c^{(2)}_{1,0}, c^{(2)}_{0,2}, c^{(2)}_{1,3}) = (2, 9, 2)$ [11]. Suppose ℓ_P is a (1,3)-line. Let Q be the point $\ell_P \cap \langle P_1, P_2\rangle$ and put $m = \langle Q, R\rangle$. Then m is a (0,2)-line since $\langle \ell, \ell_P\rangle$ is a (1,6)-plane. On the other hand, since $\delta_{12} = \langle R, P_1, P_2\rangle$ is a (4,3)-plane satisfying that R is focal to $\langle P_1, P_2\rangle$ in δ_{12}, m must be a (2,1)-line, a contradiction. Hence ℓ_P is a (1,0)-line and is focal to R in the plane $\langle R, \ell_P\rangle$, and our assertion follows. □

The following lemma can be also proved similarly using Table 3.1.

Lemma 3.5. *Let Δ be a $(16, 12)$-solid. Then, every point of $\Delta \cap F_1$ and of $\Delta \cap F_2$ has a focal $(4, 3)$-plane and a focal $(4, 6)$-plane, respectively, and vice versa.*

Easy counting arguments yield the following.

Lemma 3.6. *For even $t \geq 4$, let Π^1_t, Π^2_t be flats with parameters $(\theta_{t-1}, \theta_{t-1} - \theta_{U+1})_t$, $(\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)_t$, $U = (t-4)/2$. For odd $t \geq 5$, let Π^3_t, Π^4_t be flats with parameters $(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)_t$, $(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)_t$, $T = (t-3)/2$. Then Table 3.2 holds.*

Table 3.2.

Π_t	S	$r^{(s)}_{i,j} = \#$ of $(i, j)_s$ flats through S in Π_t
Π^1_t	Π^3_{t-3}	$r^{(t-2)}_{\theta_{t-3}-3^{U+1},\theta_{t-3}+\theta_U+1} = 4,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}-\theta_U} = 6,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}+\theta_U+1} = 3$
Π^1_t	Π^4_{t-3}	$r^{(t-2)}_{\theta_{t-3}-3^{U+1},\theta_{t-3}+\theta_U+1} = 4,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}-\theta_U} = 3,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}+\theta_U+1} = 6$
Π^1_t	Π^1_{t-2}	$r^{(t-1)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+1}} = 2,\ r^{(t-1)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = r^{(t-1)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = 1$
Π^1_t	Π^2_{t-2}	$r^{(t-1)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = r^{(t-1)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = 2$
Π^2_t	Π^3_{t-3}	$r^{(t-2)}_{\theta_{t-3},\theta_{t-3}-\theta_U} = 6,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}+\theta_U+1} = 3,\ r^{(t-2)}_{\theta_{t-3}+3^{U+1},\theta_{t-3}-\theta_U} = 4$
Π^2_t	Π^4_{t-3}	$r^{(t-2)}_{\theta_{t-3},\theta_{t-3}-\theta_U} = 3,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}+\theta_U+1} = 6,\ r^{(t-2)}_{\theta_{t-3}+3^{U+1},\theta_{t-3}-\theta_U} = 4$
Π^2_t	Π^1_{t-2}	$r^{(t-1)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = r^{(t-1)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = 2$
Π^2_t	Π^2_{t-2}	$r^{(t-1)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = r^{(t-1)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = 1,\ r^{(t-1)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+1}+1} = 2$
Π^3_t	Π^1_{t-3}	$r^{(t-2)}_{\theta_{t-3},\theta_{t-3}-\theta_T} = 4,\ r^{(t-2)}_{\theta_{t-3}-3^T,\theta_{t-3}+\theta_{T-1}+1} = 6,\ r^{(t-2)}_{\theta_{t-3}+3^T,\theta_{t-3}-\theta_{T-1}} = 3$
Π^3_t	Π^2_{t-3}	$r^{(t-2)}_{\theta_{t-3}-3^T,\theta_{t-3}+\theta_{T-1}+1} = 6,\ r^{(t-2)}_{\theta_{t-3}+3^T,\theta_{t-3}-\theta_{T-1}} = 3,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}+\theta_T+1} = 4$
Π^3_t	Π^3_{t-2}	$r^{(t-1)}_{\theta_{t-2}-3^{T+1},\theta_{t-2}+\theta_T+1} = 2,\ r^{(t-1)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = r^{(t-1)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = 1$
Π^3_t	Π^4_{t-2}	$r^{(t-1)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = r^{(t-1)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = 2$
Π^4_t	Π^1_{t-3}	$r^{(t-2)}_{\theta_{t-3},\theta_{t-3}-\theta_T} = 4,\ r^{(t-2)}_{\theta_{t-3}-3^T,\theta_{t-3}+\theta_{T-1}+1} = 3,\ r^{(t-2)}_{\theta_{t-3}+3^T,\theta_{t-3}-\theta_{T-1}} = 6$
Π^4_t	Π^2_{t-3}	$r^{(t-2)}_{\theta_{t-3}-3^T,\theta_{t-3}+\theta_{T-1}+1} = 3,\ r^{(t-2)}_{\theta_{t-3}+3^T,\theta_{t-3}-\theta_{T-1}} = 6,\ r^{(t-2)}_{\theta_{t-3},\theta_{t-3}+\theta_T+1} = 4$
Π^4_t	Π^3_{t-2}	$r^{(t-1)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = r^{(t-1)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = 2$
Π^4_t	Π^4_{t-2}	$r^{(t-1)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = r^{(t-1)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = 1,\ r^{(t-1)}_{\theta_{t-2}+3^{T+1},\theta_{t-2}-\theta_T} = 2$

We prove the following four lemmas by induction on t. More precisely, we show Lemma 3.7 and Lemma 3.8 for even t using Lemmas 3.7 - 3.10 as the induction hypothesis for $t-2$ or $t-1$, and we show Lemma 3.9 and Lemma 3.10 for odd t using Lemmas 3.7 - 3.10 as well, where Lemmas 3.2 - 3.5 give the induction basis.

Lemma 3.7. *Let Π_t be a $(\theta_{t-1}, \theta_{t-1} - \theta_{U+1})_t$ flat for even $t \geq 4$, where $U = (t-4)/2$. Then, every point of $\Pi_t \cap F_1$ and of $\Pi_t \cap F_2$ has a focal $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flat and a focal $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat, respectively, and vice versa.*

Proof. We prove that arbitrary $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat π in Π_t has a focal point in $F_2 \cap \Pi_t$. Let δ be a $(\theta_{t-4} - 3^U, \theta_{t-4} + \theta_{U-1} + 1)_{t-3}$ flat in π. Then, from Table 3.2, there are exactly three $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flats through δ in Π_t, precisely two of which are contained in π. Let Δ be the $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flat through δ not contained in π. From Table 3.2, in Π_t, there are two $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flats through Δ, say π_1, π_2, and two $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flats through Δ, say π_3, π_4. Let $\Delta_i = \pi \cap \pi_i$ for $i = 1, \ldots, 4$. Then, $\Delta_1, \cdots, \Delta_4$ are the $(t-2)$-flats through δ in π, consisting two $(\theta_{t-3}, \theta_{t-3} - \theta_U)_{t-2}$ flats and two $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flats from Table 3.2. It also follows from Table 3.2 that a $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flat cannot contain two $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flats meeting in a $(\theta_{t-4} - 3^U, \theta_{t-4} + \theta_{U-1} + 1)_{t-3}$ flat. Hence, Δ_3, Δ_4 are $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flats and Δ_1, Δ_2 are $(\theta_{t-3}, \theta_{t-3} - \theta_U)_{t-2}$ flats. From the induction hypothesis for $t-2$, δ has a focal point $R \in F_2$ in Δ. To show that R is focal to π, It suffices to prove that R is focal to Δ_i in π_i for $i = 1, \ldots, 4$. Since the diversity of π_i is new in Λ_{t-1} and since R is focal to δ, it follows from the induction hypothesis for $t-1$ that R has the focal $(t-2)$-flat Δ_i' through δ in π_i for $i = 1, \ldots, 4$. For $i = 1, 2$, Δ_i' is a $(\theta_{t-3}, \theta_{t-3} - \theta_U)_{t-2}$ flat, and Δ_i is the only $(\theta_{t-3}, \theta_{t-3} - \theta_U)_{t-2}$ flat through δ in π_i from Table 3.2. Hence $\Delta_i' = \Delta_i$. For $i = 3, 4$, Δ_i' is a $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flat, and Δ_i is the only $(\theta_{t-3}, \theta_{t-3} + \theta_U + 1)_{t-2}$ flat through δ other than Δ in π_i from Table 3.2. Hence we have $\Delta_i' = \Delta_i$ as well. Thus R is focal to Δ_i in π_i for $i = 1, \ldots, 4$.
Similarly, it can be proved using Table 3.2 that every $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flat in Π_t has a focal point in $F_1 \cap \Pi_t$. The converses follow from Lemma 3.1. □

Replacing $\Pi_t \cap F_1$ and $\Pi_t \cap F_2$ for a $(\theta_{t-1}, \theta_{t-1} - \theta_{U+1})_t$ flat Π_t yields a $(\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)_t$ flat in which every $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat and every $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flat have a focal point in $F_1 \cap \Pi_t$ and in $F_2 \cap \Pi_t$, respectively. Hence we get the following.

Lemma 3.8. *Let Π be a $(\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)_t$ flat for even $t \geq 4$, where $U = (t-4)/2$. Then, every point of $\Pi \cap F_1$ and of $\Pi \cap F_2$ has a focal $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat and a focal $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flat, respectively, and vice versa.*

Lemma 3.9. *Let Π be a $(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)_t$ flat for odd $t \geq 5$, where $T = (t-3)/2$. Then, every point of $\Pi \cap F_1$ and of $\Pi \cap F_2$ has a focal $(\theta_{t-2}, \theta_{t-2} - \theta_T)_{t-1}$ flat and a focal $(\theta_{t-2}, \theta_{t-2} + \theta_T + 1)_{t-1}$ flat, respectively, and vice versa.*

Proof. We prove that arbitrary $(\theta_{t-2}, \theta_{t-2} - \theta_T)_{t-1}$ flat π in Π_t has a focal point in $F_2 \cap \Pi_t$. Let δ be a $(\theta_{t-4}, \theta_{t-4} + \theta_{T-1} + 1)_{t-3}$ flat in π. Then, from Table 3.2, there are exactly

three $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flats through δ in Π_t, precisely two of which are contained in π. Let Δ be the $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flat through δ not contained in π. From Table 3.2, in Π_t, there are two $(\theta_{t-2}, \theta_{t-2}-\theta_T)_{t-1}$ flats through Δ, say π_1, π_2, and two $(\theta_{t-2}, \theta_{t-2}+\theta_T+1)_{t-1}$ flats through Δ, say π_3, π_4. Let $\Delta_i = \pi \cap \pi_i$ for $i = 1, \ldots, 4$. Then, $\Delta_1, \cdots, \Delta_4$ are the $(t-2)$-flats through δ in π, consisting two $(\theta_{t-3}-3^T, \theta_{t-3}+\theta_{T-1}+1)_{t-2}$ flats and two $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flats from Table 3.2. It also follows from Table 3.2 that a $(\theta_{t-2}, \theta_{t-2}+\theta_T+1)_{t-1}$ flat cannot contain two $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flats meeting in a $(\theta_{t-4}, \theta_{t-4}+\theta_{T-1}+1)_{t-3}$ flat. Hence, Δ_3, Δ_4 are $(\theta_{t-3}-3^T, \theta_{t-3}+\theta_{T-1}+1)_{t-2}$ flats and Δ_1, Δ_2 are $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flats. From the induction hypothesis for $t-2$, δ has a focal point $R \in F_2$ in Δ. To show that R is focal to π, It suffices to prove that R is focal to Δ_i in π_i for $i = 1, \ldots, 4$. Since the diversity of π_i is new in Λ_{t-1} and since R is focal to δ, it follows from the induction hypothesis for $t-1$ that R has the focal $(t-2)$-flat Δ'_i through δ in π_i for $i = 1, \ldots, 4$. For $i = 1, 2$, Δ'_i is a $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flat, and Δ_i is the only $(\theta_{t-3}+3^T, \theta_{t-3}-\theta_{T-1})_{t-2}$ flat through δ other than Δ in π_i from Table 3.2. Hence we have $\Delta'_i = \Delta_i$. For $i = 3, 4$, Δ'_i is a $(\theta_{t-3}-3^T, \theta_{t-3}+\theta_{T-1}+1)_{t-2}$ flat, and Δ_i is the only $(\theta_{t-3}-3^T, \theta_{t-3}+\theta_{T-1}+1)_{t-2}$ flat through δ in π_i from Table 3.2. Hence $\Delta'_i = \Delta_i$ as well. Thus R is focal to Δ_i in π_i for $i = 1, \ldots, 4$.
Similarly, it can be proved using Table 3.2 that every $(\theta_{t-2}, \theta_{t-2}+\theta_T+1)_{t-1}$ flat in Π_t has a focal point in $F_1 \cap \Pi_t$. The converses follow from Lemma 3.1. □

The following lemma can be also proved similarly using Table 3.2.

Lemma 3.10. *Let Π be a $(\theta_{t-1}+3^{T+1}, \theta_{t-1}-\theta_T)_t$ flat for odd $t \geq 5$, where $T = (t-3)/2$. Then, every point of $\Pi \cap F_1$ and of $\Pi \cap F_2$ has a focal $(\theta_{t-2}, \theta_{t-2}+\theta_T+1)_{t-1}$ flat and a focal $(\theta_{t-2}, \theta_{t-2}-\theta_T)_{t-1}$ flat, respectively, and vice versa.*

Recall that $(2,1)$ and $(0,2)$ are new in Λ_1. We have shown the following theorem by Lemmas 3.2 - 3.10.

Theorem 3.11. *Let Π be a t-flat with new diversity in Λ_t, $t \geq 2$. Then, every point of $\Pi \cap F_1$ or $\Pi \cap F_2$ has a unique focal hyperplane whose diversity is new in Λ_{t-1}. Conversely, every hyperplane with new diversity in Λ_{t-1} has a unique focal point in $\Pi \cap F_1$ or in $\Pi \cap F_2$.*

Table 3.3. The focal line of $R \in F_2 \cap \delta$

plane δ	(4,0)	(1,6)	(4,3)	(4,6)	(7,3)
focal line	(4,0)	(1,0)	(2,1)	(0,2)	(1,3)

Table 3.4. The focal line of $Q \in F_1 \cap \delta$

plane δ	(1,6)	(4,3)	(4,6)	(7,3)	(4,9)
focal line	(1,3)	(0,2)	(2,1)	(1,0)	(4,0)

Let δ be an (i,j)-plane with $i+j<\theta_2$ and take $R\in\delta\cap F_2$. Then, it follows from the geometric configurations of $F_0\cap\delta$, $F_1\cap\delta$, $F_2\cap\delta$ that R has the unique focal line in δ as in Table 3.3. This can be proved for t-flats as follows for $t\ge 3$.

Let Π_t be a $(\varphi_0,\varphi_1)_t$ flat with $t\ge 3$. By Theorem 3.11, every point of $F_2\cap\Pi_t$ or $F_1\cap\Pi_t$ has the unique focal hyperplane of Π_t provided (φ_0,φ_1) is new in Λ_{t-1}.
Assume that (φ_0,φ_1) is not new in Λ_{t-1}. Then, there is a $((\varphi_0-1)/3,\varphi_1/3)_{t-1}$ flat π in Π_t. Let L be the axis of Π_t and let P be a point of L out of π. Then, for a point $Q\in\pi$, the line $\langle P,Q\rangle$ is a $(4,0)$-line, a $(1,3)$-line or a $(1,0)$-line if $Q\in F_0$, $Q\in F_1$ or $Q\in F_2$, respectively. Assume that $F_2\cap\Pi_t\neq\emptyset$ and that $R\in F_2\cap\pi$ is focal to a $(t-2)$-flat Δ in π. Then, it is easy to see that R is focal to $\langle P,\Delta\rangle$. Thus, every point of $F_2\cap\Pi_t$ has the unique focal hyperplane of Π_t.

Theorem 3.12. *Let Π_t be a $(\varphi_0,\varphi_1)_t$ flat with $\varphi_0+\varphi_1<\theta_t$, $t\ge 2$. Then, for any point R of $F_2\cap\Pi_t$,*
(1) *R has the unique focal $(a,b)_{t-1}$ flat in Π_t with*

$$a=(4\theta_{t-1}-\varphi_0-2\varphi_1)/3,\ b=(2\varphi_0+\varphi_1-2\theta_{t-1})/3.$$

(2) *The numbers of (i,j)-lines through R in Π_t are*

$$r^{(1)}_{1,0}=a,\ r^{(1)}_{2,1}=b,\ r^{(1)}_{0,2}=\theta_{t-1}-a-b.$$

We also get the following similarly (see Table 3.4 for $t=2$).

Theorem 3.13. *Let Π_t be a $(\varphi_0,\varphi_1)_t$ flat with $\varphi_1>0$, $t\ge 2$. Then, for any point Q of $F_1\cap\Pi_t$,*
(1) *Q has the unique focal $(a,b)_{t-1}$ flat in Π_t with*

$$a=(\varphi_0+2\varphi_1-2\theta_{t-1}-2)/3,\ b=(4\theta_{t-1}-2\varphi_0-\varphi_1+1)/3.$$

(2) *The numbers of (i,j)-lines through Q in Π_t are*

$$r^{(1)}_{1,3}=a,\ r^{(1)}_{0,2}=b,\ r^{(1)}_{2,1}=\theta_{t-1}-a-b.$$

Now, assume $P\in F_0$. To count $r^{(1)}_{i,j}$ for P when (φ_0,φ_1) is new, we employ the following lemmas.

Lemma 3.14 ([16]). *Let Π be a t-flat in Σ with even $t\ge 4$, $U=(t-4)/2$.*
(1) *If Π is a $(\theta_{t-1},\theta_{t-1}-\theta_{U+1})_t$ flat, then Π contains four $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $\ell=\{P_1,P_2,P_3,P_4\}$ which is the axis of Δ of type $(\theta_{t-4},\theta_{t-4}-\theta_U)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$ for $1\le i\le 4$.*
(2) *If Π is a $(\theta_{t-1},\theta_{t-1}+\theta_{U+1}+1)_t$ flat, then Π contains four $(\theta_{t-2},\theta_{t-2}+\theta_{U+1}+1)_{t-1}$ flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}+\theta_{U+1}+1)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $\ell=\{P_1,P_2,P_3,P_4\}$ which is the axis of Δ of type $(\theta_{t-4},\theta_{t-4}+\theta_U+1)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}+\theta_U+1)$ for $1\le i\le 4$.*

Lemma 3.15 ([16]). *Let Π be a t-flat in Σ with odd $t \geq 5$, $T = (t-3)/2$.*
(1) If Π is a $(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)_t$ flat, then Π contains four $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flats $\pi_1 \cdots \pi_4$ through a fixed $(\theta_{t-3} + 3^{T+1}, \theta_{t-3} - \theta_T)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $\ell = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{t-4} + 3^T, \theta_{t-4} - \theta_{T-1})$ and that P_i is the axis of π_i of type $(\theta_{t-3} + 3^T, \theta_{t-3} - \theta_{T-1})$ for $1 \leq i \leq 4$.
(2) If Π is a $(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)_t$ flat, then Π contains four $(\theta_{t-3} - 3^T, \theta_{t-3} + \theta_{T-1} + 1)_{t-1}$ flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{t-3} - 3^{T+1}, \theta_{t-3} + \theta_T + 1)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $\ell = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{t-4} - 3^T, \theta_{t-4} + \theta_{T-1} + 1)$ and that P_i is the axis of π_i of type $(\theta_{t-3} - 3^T, \theta_{t-3} + \theta_{T-1} + 1)$ for $1 \leq i \leq 4$.

Since F_0 is projectively equivalent to a non-singular quadric $\mathcal{Q}$ by Theorem 2.2 and since $G(\mathcal{Q})$, the group of projectivities fixing $\mathcal{Q}$, acts transitively on $\mathcal{Q}$ (see Theorem 22.6.4 of [9]), we may assume that $P = P_1$ in Lemmas 3.14 or 3.15. Since P is the axis of π_1 but not of π_2, π_3, π_4, we get the following.

Theorem 3.16. *Let Π_t be a t-flat with new diversity, $t \geq 4$, and let P_1 and π_1 be as in Lemma 3.14 or Lemma 3.15. Assume that P_1 is the axis of π_1 of type (a,b). Then, for any point P of $F_0 \cap \Pi_t$, the numbers of (i,j)-lines through P in Π_t are*

$$r_{4,0}^{(1)} = a,\ r_{1,3}^{(1)} = b,\ r_{1,0}^{(1)} = \theta_{t-2} - a - b,\ r_{2,1}^{(1)} = 3^{t-1}.$$

Proof of Theorem 2.3. We first prove for $t = 2$ as the induction basis. Let Π_2 be a $(4,3)$-plane. Recall that $F_0 \cap \Pi_2$ forms a 4-arc, say K, and the set of internal points of K in Π_2 is $F_1 \cap \Pi_2$. On the other hand, $\mathcal{P}_2^2 = \{\mathbf{P}(0,1,2), \mathbf{P}(1,1,1), \mathbf{P}(1,2,2)\}$ is the set of internal points of the conic $\mathcal{P}_2^0 = V_0(x_0^2 + x_1x_2)$ in $\mathrm{PG}(2,3)$. Hence, taking a projectivity τ from Π_2 to $\mathrm{PG}(2,3)$ with $\tau(F_1 \cap \Pi_2) = \mathcal{P}_2^2 = 2\mathcal{P}_2^1$, we get $F_i \cap \Pi_2 \sim 2\mathcal{P}_2^i$ for $i = 0,1,2$. When Π_2 is a $(4,6)$-plane, we have $F_i \cap \Pi_2 \sim \mathcal{P}_2^i$ for $i = 0,1,2$ since $F_2 \cap \Pi_2$ is the set of internal points of a 4-arc $F_0 \cap \Pi_2$ in this case.

Now, let t be odd ≥ 3 and $T = (t-3)/2$. Let Π_t be a $(\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)_t$ flat and π be a $(\theta_{t-2}, \theta_{t-2} + \theta_T + 1)_{t-1}$ flat in Π_t which is focal to $Q \in F_1 \cap \Pi_t$. We prove $F_i \cap \Pi_t \sim \mathcal{E}_t^i$ for $i = 0,1,2$. We have $F_i \cap \pi \sim \mathcal{P}_{t-1}^i$ for $i = 0,1,2$ by the induction hypothesis for $t-1$. Let π' be the hyperplane $V_0(x_0)$ in $\mathrm{PG}(t,3)$ and take $f = x_1^2 + x_2x_3 + \cdots + x_{t-1}x_t$. We consider $V_i(f) \cap \pi'$ ($\sim \mathcal{P}_{t-1}^i$) and $\mathcal{E}_t^i = V_i(x_0^2 + x_1^2 + x_2x_3 + \cdots + x_{t-1}x_t)$ for $i = 1,2$. Note that $Q' = \mathbf{P}(1,0,\cdots,0) \in \mathcal{E}_t^1 \setminus \pi'$ and $\mathcal{E}_t^i \cap \pi' = V_i(f) \cap \pi'$. Since $F_i \cap \pi \sim \mathcal{P}_{t-1}^i$ for $i = 1,2$, we can take a projectivity τ from Π_t to $\mathrm{PG}(t,3)$ satisfying $\tau(F_i \cap \pi) = V_i(f) \cap \pi'$ for $i = 1,2$ and $\tau(Q) = Q'$. For $P' = \mathbf{P}(0, p_1, \cdots, p_t) \in \mathcal{E}_t^i \cap \pi'$, the two points $\mathbf{P}(1, p_1, \cdots, p_t)$ and $\mathbf{P}(2, p_1, \cdots, p_t)$ on the line $\langle P', Q' \rangle$ other than P', Q' belong to $\mathcal{E}_t^{i+1}$, where $i+1$ is calculated modulo 3. Thus, we have $\tau(F_i \cap \Pi_t) = \mathcal{E}_t^i$ for $i = 0,1,2$.

Next, let Π_t be a $(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)_t$ flat for odd $t \geq 3$, $T = (t-3)/2$. Let R be a point of F_2 and π be a $(\theta_{t-2}, \theta_{t-2} + \theta_T + 1)_{t-1}$ flat which is focal to R. We prove $F_i \cap \Pi_t \sim \mathcal{H}_t^i$ for $i = 1,2$. We have $F_i \cap \pi \sim \mathcal{P}_{t-1}^i$ for $i = 0,1,2$ by the induction hypothesis for $t-1$. Let π' be the hyperplane $V_0(x_0 - x_1)$ in $\mathrm{PG}(t,3)$ and take $f = x_1^2 + x_2x_3 + \cdots + x_{t-1}x_t$ as above. We consider $V_i(f) \cap \pi'$ ($\sim \mathcal{P}_{t-1}^i$) and $\mathcal{H}_t^i = V_i(x_0x_1 + x_2x_3 + \cdots + x_{t-1}x_t)$ for

$i = 1, 2$. Note that $R' = \mathbf{P}(1, 2, 0, \cdots, 0) \in \mathcal{H}_t^2 \setminus \pi'$ and $\mathcal{H}_t^i \cap \pi' = V_i(f) \cap \pi'$. Since $F_i \cap \pi \sim \mathcal{P}_{t-1}^i$ for $i = 1, 2$, we can take a projectivity τ from Π_t to $\mathrm{PG}(t, 3)$ satisfying $\tau(F_i \cap \pi) = V_i(f) \cap \pi'$ for $i = 1, 2$ and $\tau(R) = R'$. For $P' = \mathbf{P}(p_1, p_1, p_2, \cdots, p_t) \in \mathcal{H}_t^i \cap \pi'$, the two points $\mathbf{P}(p_1 + 1, p_1 - 1, p_2, \cdots, p_t)$ and $\mathbf{P}(p_1 - 1, p_1 + 1, p_2, \cdots, p_t)$ on the line $\langle P', R' \rangle$ other than P', R' belong to $\mathcal{H}_t^{i+2}$, where $i + 2$ is calculated modulo 3. Hence, we have $\tau(F_i \cap \Pi_t) = \mathcal{H}_t^i$ for $i = 0, 1, 2$.

For even $t \geq 4$, we first assume Π_t is a $(\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)_t$ flat, where $U = (t-4)/2$. Let Q be a point of F_1 and π be a $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat which is focal to Q. We prove $F_i \cap \Pi_t \sim \mathcal{P}_t^i$ for $i = 1, 2$. We have $F_i \cap \pi \sim \mathcal{P}_{t-1}^i$ for $i = 0, 1, 2$ by the induction hypothesis for $t - 1$. Let π' be the hyperplane $V_0(x_0)$ in $\mathrm{PG}(t, 3)$ and take $f = x_1x_2 + x_3x_4 + \cdots + x_{t-1}x_t$. We consider $V_i(f) \cap \pi' (\sim \mathcal{H}_{t-1}^i)$ and $\mathcal{P}_t^i = V_i(x_0^2 + x_1x_2 + \cdots + x_{t-1}x_t)$ for $i = 1, 2$. Note that $Q' = \mathbf{P}(1, 0, \cdots, 0) \in \mathcal{P}_t^1 \setminus \pi'$ and $\mathcal{P}_t^i \cap \pi' = V_i(f) \cap \pi'$. Since $F_i \cap \pi \sim \mathcal{H}_{t-1}^i$ for $i = 1, 2$, we can take a projectivity τ from Π_t to $\mathrm{PG}(t, 3)$ satisfying $\tau(F_i \cap \pi) = V_i(f) \cap \pi'$ for $i = 1, 2$ and $\tau(Q) = Q'$. For $P' = \mathbf{P}(0, p_1, p_2, \cdots, p_t) \in \mathcal{P}_t^i \cap \pi'$, the two points $\mathbf{P}(1, p_1, p_2, \cdots, p_t)$ and $\mathbf{P}(2, p_1, p_2, \cdots, p_t)$ on the line $\langle P', Q' \rangle$ other than P', Q' belong to $\mathcal{P}_t^{i+1}$, where $i+1$ is calculated modulo 3. Hence, we have $\tau(F_i \cap \Pi_t) = \mathcal{P}_t^i$ for $i = 0, 1, 2$.

Next, let Π_t be a $(\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)_t$ flat for even $t \geq 4$, $U = (t-4)/2$. Let R be a point of F_2 and π be a $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_U + 1)_{t-1}$ flat which is focal to R. We prove $F_i \cap \Pi_t \sim \mathcal{P}_t^i$ for $i = 1, 2$. We have $F_i \cap \pi \sim \mathcal{P}_{t-1}^i$ for $i = 0, 1, 2$ by the induction hypothesis for $t - 1$. Let π' be the hyperplane $V_0(x_0 - x_1 - x_2)$ in $\mathrm{PG}(t, 3)$ and take $f = x_1^2 + x_2^2 + x_3x_4 + \cdots + x_{t-1}x_t$. We consider $V_i(f) \cap \pi' (\sim \mathcal{E}_{t-1}^i)$ and $\mathcal{P}_t^i = V_i(x_0^2 + x_1x_2 + \cdots + x_{t-1}x_t)$ for $i = 1, 2$. Note that $R' = \mathbf{P}(1, 1, 1, 0, \cdots, 0) \in \mathcal{P}_t^1 \setminus \pi'$ and $\mathcal{P}_t^i \cap \pi' = V_i(f) \cap \pi'$. Since $F_i \cap \pi \sim \mathcal{E}_{t-1}^i$ for $i = 1, 2$, we can take a projectivity τ from Π_t to $\mathrm{PG}(t, 3)$ satisfying $\tau(F_i \cap \pi) = V_i(f) \cap \pi'$ for $i = 1, 2$ and $\tau(R) = R'$. For $P' = \mathbf{P}(p_1 + p_2, p_1, p_2, \cdots, p_t) \in \mathcal{P}_t^i \cap \pi'$, the two points $\mathbf{P}(p_1 + p_2 + 1, p_1 + 1, p_2 + 1, p_3 \cdots, p_t)$ and $\mathbf{P}(p_1 + p_2 + 2, p_1 + 2, p_2 + 2, p_3 \cdots, p_t)$ on the line $\langle P', R' \rangle$ other than P', R' belong to $\mathcal{P}_t^{i+2}$, where $i + 2$ is calculated modulo 3. Hence, we have $\tau(F_i \cap \Pi_t) = \mathcal{P}_t^i$ for $i = 0, 1, 2$.

4 An application to optimal linear codes problem

One of the fundamental problems in coding theory is the *optimal linear codes problem*, which is the problem to optimize one of the parameters n, k, d for given the other two over a given field $\mathbb{F}_q$, see [4], [5]. Here, we consider one version of the problem to determine $n_q(k, d)$, the minimum value of n for which an $[n, k, d]_q$ code exists. $[n_q(k, d), k, d]_q$ codes are called *optimal*. $n_3(k, d)$ has been determined for all d for $k \leq 5$, but not for many values of d for the case $k \geq 6$. For example, $n_3(6, 202)$ is not determined yet so far since Hamada [3] proved the following in 1993.

Lemma 4.1 ([3]). (1) $n_3(6, 203) = 307$. (2) $n_3(6, 202) = 305$ *or* 306.

In this section, we show how our investigations in the previous section can be applied to consider such problems by proving the non-existence of a $[305, 6, 202]_3$ code, which is a new result.

Theorem 4.2. *A* $[305, 6, 202]_3$ *code does not exist.*

Corollary 4.3. $n_3(6, 202) = 306$.

We first introduce the usual geometric method. Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix G attaining the Griesmer bound:

$$n \geq g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and assume that $\mathcal{C}$ satisfies $d \leq q^{k-1}$. We mainly deal with such codes in this section. Then, any two columns of G are linearly independent, see, e.g., Theorem 5.1 of [4]. Hence the set of n columns of G can be considered as an n-set C_1 in $\Sigma = \mathrm{PG}(k-1, q)$ such that every hyperplane meets C_1 in at most $n-d$ points and that some hyperplane meets C_1 in exactly $n-d$ points, see Theorem 2.3 of [5]. On the other hand, each column of G was considered as a defining vector of a hyperplane of Σ in Section 1. So, the geometric structures found in the previous sections can be applied to the dual space Σ^* of Σ.

A line l with $t = |l \cap C_1|$ is called a *t-line*. A *t-plane*, a *t-solid* and so on are defined similarly. Let $\mathcal{F}_j$ be the set of j-flats in Σ. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = max\{|\Delta \cap C_1| \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \leq j \leq m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n - d,\ \gamma_{k-1} = n$.

Denote by a_i the number of i-hyperplanes Π in Σ. Note that $a_i = A_{n-i}/2$ for $0 \leq i \leq n-d$ and that $a_{n-d} > 0$. The list of a_i's is called the *spectrum* of $\mathcal{C}$ (or C_1). We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following.

Lemma 4.4. *Let* $(a_0, a_1, \ \ldots\ , a_{n-d})$ *be the spectrum of* $\mathcal{C}$. *Then*

$$(1) \sum_{i=0}^{n-d} a_i = \theta_{k-1}. \quad (2) \sum_{i=1}^{n-d} i a_i = n\theta_{k-2}. \quad (3) \sum_{i=2}^{n-d} \binom{i}{2} a_i = \binom{n}{2} \theta_{k-3}.$$

One can get the following from the three equalities of Lemma 4.4:

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2} \theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2} \theta_{k-3}. \tag{4.1}$$

Lemma 4.5. *Let Π be an i-hyperplane through a t-secundum Δ with $t = \gamma_{k-3}(\Pi)$. Then*
(1) $t \le \gamma_{k-2} - \dfrac{n-i}{q} = \dfrac{i + q\gamma_{k-2} - n}{q}$.
(2) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \ge i - \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.
(3) $t = \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ *if an* $[i, k-1, d_1]_q$ *code with* $d_1 \ge i - \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor + 1$ *does not exist.*

(4) *Let* c_j *be the number of* j*-hyperplanes through* Δ *other than* Π*. Then the following equality holds:*

$$\sum_j (\gamma_{k-2} - j) c_j = i + q\gamma_{k-2} - n - qt. \tag{4.2}$$

(5) *For a* γ_{k-2}*-hyperplane* Π_0 *with spectrum* $(\tau_0, \cdots, \tau_{\gamma_{k-3}})$*,* $\tau_t > 0$ *holds if* $i + q\gamma_{k-2} - n - qt < q$.

Proof. (1) Counting the points of C_1 on the hyperplanes through Δ, we get $n \le q(\gamma_{k-2} - t) + i$.
(2) Π gives an $[i, k-1, d_0]_q$ code with $d_0 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ by (1).
(3) If $t \le \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor - 1$, then Π gives an $[i, k-1, d_1]_q$ code with $d_1 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor + 1$. Hence our assertion follows from (1).
(4) (4.2) follows from $\sum_j c_j = q$ and $\sum_j (j-t) c_j = n - i$.
(5) It holds that $c_{\gamma_{k-2}} > 0$ when the right hand side of (4.2) is at most $q-1$. □

An f-set F in $\mathrm{PG}(k-1, q)$ satisfying

$$m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{k-2}\}$$

is called an $\{f, m; k-1, q\}$-*minihyper*. Put $C_0 = \Sigma \setminus C_1$. Note that C_0 forms a $\{\theta_{k-1} - n, \theta_{k-2} - (n-d); k-1, q\}$-minihyper.

Lemma 4.6. *Let* F *be a* $\{18 = \theta_2 + \theta_1 + \theta_0, 5 = \theta_1 + \theta_0; 4, 3\}$*-minihyper corresponding to a* $[103, 5, 68]_3$ *code* $\mathcal{C}_{103}$*. Then*
(1) *there exist a plane* δ*, a line* ℓ *and a point* P *which are mutually disjoint such that*

$$F = \delta \cup \ell \cup \{P\}.$$

(2) *The spectrum of* $\mathcal{C}_{103}$ *is* $(a_{25}, a_{26}, a_{31}, a_{32}, a_{34}, a_{35}) = (1, 3, 4, 9, 35, 69)$.

Proof. (1) follows from Theorem 3.1 of [2]. (2) can be easily calculated from the fact that δ, ℓ and P are mutually disjoint. □

The following lemma can also be obtained from Theorem 3.1 of [2].

Lemma 4.7. (1) *The spectrum of a* $[81, 5, 54]_3$ *code is* $(a_0, a_{27}) = (1, 120)$.
(2) *The spectrum of a* $[80, 5, 53]_3$ *code is* $(a_0, a_{26}, a_{27}) = (1, 40, 80)$.

Lemma 4.8. *Let* F *be a* $\{21 = \theta_2 + 2\theta_1, 6 = \theta_1 + 2\theta_0; 4, 3\}$*-minihyper corresponding to a* $[100, 5, 66]_3$ *code* $\mathcal{C}_{100}$*. Then, either*
(*a*) *there exist a plane* δ *and two lines* ℓ_1, ℓ_2 *all of which are skew such that*

$$F = \delta \cup \ell_1 \cup \ell_2,$$

and $\mathcal{C}_{100}$ *has spectrum* $(a_{25}, a_{28}, a_{31}, a_{34}) = (4, 1, 24, 92)$, *or*

(*b*) *there exist two skew lines* $\ell_1 = \{Q_0, Q_1, Q_2, Q_3\}$ *and* $\ell_2 = \{R_0, R_1, R_2, R_3\}$ *and a plane* δ *containing* ℓ_1 *with* $\ell_2 \cap \delta = R_0$ *such that*

$$F = (\delta \setminus Q_0) \cup \langle Q_1, R_1\rangle \cup \langle Q_2, R_2\rangle \cup \langle Q_3, R_3\rangle,$$

and $\mathcal{C}_{100}$ *has spectrum* $(a_{19}, a_{28}, a_{31}, a_{34}) = (1, 3, 27, 90)$.

Proof. See Theorem 5.10(2) of [2]. Each spectrum can be calculated by hand from the geometrical structure. □

Lemma 4.9. *Let* F *be a* $\{30 = 2\theta_2 + \theta_1, 9 = 2\theta_1 + \theta_0; 4, 3\}$*-minihyper corresponding to a* $[91, 5, 60]_3$ *code* $\mathcal{C}_{91}$*. Then*
(1) *There exist two skew lines* $\ell_1 = \{P_1, P_2, P_3, P_4\}$ *and* $\ell_2 = \{Q_1, Q_2, R, S\}$ *such that* $F = (\delta_1 \setminus Q_1) \cup (\delta_2 \setminus Q_2) \cup \langle P_1, R\rangle \cup \langle P_2, R\rangle \cup \langle P_3, S\rangle \cup \langle P_4, S\rangle$, *where* $\delta_1 = \langle \ell_1, Q_1\rangle$, $\delta_2 = \langle \ell_1, Q_2\rangle$.
(2) *The spectrum of* $\mathcal{C}_{91}$ *is* $(a_{10}, a_{28}, a_{31}) = (1, 30, 90)$.

Proof. (1) follows from Theorem 5.13(1) of [2].
(2) F is contained in a solid, say Δ, and there are ten 1-planes and thirty 4-planes in Δ. Hence (2) follows. □

Lemma 4.10 ([1]). (1) *The spectrum of a* $[26, 4, 17]_3$ *code is* $(a_0, a_8, a_9) = (1, 13, 26)$.
(2) *The spectrum of a* $[31, 4, 20]_3$ *code is*
(a) $(a_4, a_9, a_{10}, a_{11}) = (1, 9, 12, 18)$ *or* (b) $(a_7, a_8, a_{10}, a_{11}) = (2, 6, 11, 21)$.

As an application of Theorem 3.13, we prove the following.

Lemma 4.11. *A* $[90, 5, 59]_3$ *code is extendable.*

Proof. Let $\mathcal{C}$ is a $[90, 5, 59]_3$ code and let Δ be a γ_3-solid, which gives a $[31, 4, 20]_3$ code by Lemma 4.5. Then Δ has no j-planes for $j \notin \{4, 7, 8, 9, 10, 11\}$ by Lemma 4.10(2), so we have

$$a_i = 0 \text{ for all } i \notin \{9, 10, 18, 19, 24, 25, 26, 27, 28, 30, 31\}$$

by Lemma 4.5 and the $n_3(4,d)$ table (see [6]). Now, it holds that $F_0 = \{i\text{-solids} \mid i \equiv 0 \pmod 3\}$, $F_1 = \{26\text{-solids}\}$. Suppose that $\mathcal{C}$ is not extendable. Then the diversity (Φ_0, Φ_1) of $\mathcal{C}$ satisfies

$$(\Phi_0, \Phi_1) \in \{(40,27),(31,45),(40,36),(40,45),(49,36)\}$$

by Theorem 2.7 of [11]. Let Δ_0 be a 26-solid in $\Sigma = \mathrm{PG}(4,3)$ and let Q be the corresponding point of F_1 in Σ^*. Then there are at most 18 $(2,1)$-lines through Q in Σ^* by Theorem 3.13(2). On the other hand, setting $(i,t) = (26,9)$ in Lemma 4.5, the equation (4.2) has the unique solution $(c_{30}, c_{31}) = (2,1)$ corresponding to a $(2,1)$-line through Q. Hence, by Lemma 4.10(1), there are at least 26 $(2,1)$-lines through Q, a contradiction. $\square$

Now, we are ready to prove Theorem 4.4. Let $\mathcal{C}$ be a putative $[305,6,202]_3$ code and let π_0 be a γ_4-hyperlane which gives a $[103,5,68]_3$ code by Lemma 4.5. Then π_0 has no j-solid for $j \notin \{25, 26, 31, 32, 34, 35\}$ by Lemma 4.6, so we have

$$a_i = 0 \text{ for all } i \notin \{74,80,81,89,90,91,92,98,99,100,101,102,103\}$$

by Lemma 4.5 and the $n_3(5,d)$ table (see [13]). For $s = 0,1,2$, it holds that

$$F_s = \{i\text{-hyperlanes} \mid i+1 \equiv s \pmod 3\}. \tag{4.3}$$

Let π be an i-hyperlane of $\Sigma = \mathrm{PG}(5,3)$. If $i = 81$, $C_1 \cap \pi$ gives a $[81,5,54]_3$ code by Lemma 4.5 and π has no solid contained in π_0 by Lemma 4.7(1), a contradiction. Hence $a_{81} = 0$. We obtain $a_{80} = 0$ by Lemma 4.7(2) similarly.

If $i = 91$, $C_1 \cap \pi$ gives a $[91,5,60]_3$ code by Lemma 4.5 and π has a 10-solid by Lemma 4.9. Setting $(i,t) = (91,10)$ in Lemma 4.5, the equation (4.2) has no solution, a contradiction. Hence $a_{91} = 0$. If $i = 90$, π corresponds to a $[90,5,59]_3$ code by Lemma 4.5 and π has a 9-solid or a 10-solid by Lemmas 4.9 and 4.11. Setting $i = 90$ and $t = 9$ or 10 in Lemma 4.5, the equation (4.2) has no solution. Thus $a_{90} = 0$.

Hence, from (4.1), we have

$$406a_{74} + 91a_{89} + 55a_{92} + 10a_{98} + 6a_{99} + 3a_{100} + a_{101} = 2182. \tag{4.4}$$

It follows from Lemma 4.1(1) that $\mathcal{C}$ is not extendable. Hence the diversity of $\mathcal{C}$ (Φ_0, Φ_1) is one of the following:

$$(121,81),(94,135),(121,108),(112,126),(130,117),(121,135),(148,108).$$

Hence, if $r^{(1)}_{1,0} + r^{(1)}_{0,2} \geq 90$, then it holds that

$$r^{(1)}_{1,0} + r^{(1)}_{0,2} = 94 \tag{4.5}$$

for a fixed point of $R \in F_2$ by Theorem 3.12, where $r^{(1)}_{i,j}$ denotes the number of (i,j)-lines through R in Σ^*.

If $i = 100$, $C_1 \cap \pi$ gives a $[100, 5, 66]_3$ code by Lemma 4.5 and $C_0 \cap \pi$ forms a minihyper of type (a) or (b) in Lemma 4.8. Let R_π be the point of F_2 in Σ^* corresponding to π. Setting $i = 100$ in Lemma 4.5, the equation (4.2) has the solutions as in Table 4.1, where 'line' stands for the corresponding line through R_π in Σ^*. For example, (4.2) has the unique solution $(c_{74}, c_{89}, c_{99}) = (1, 1, 1)$ when $t = 19$. Equivalently, by (4.3), a 19-solid in π corresponds to a $(2, 1)$-line through R_π in Σ^*. Now, (4.5) holds from Table 4.1 since the spectrum of a 100-hyperplane satisfies $\tau_{34} \geq 90$ by Lemma 4.8. If $C_0 \cap \pi$ forms a minihyper of type (a) in Lemma 4.8, we have $\tau_{34} = 92$. Hence there are at most two $(1, 0)$-lines through R_π in Σ^* which correspond to the solutions of (4.2) with $t \neq 34$. Let δ be the plane contained in $C_0 \cap \pi$. Since all of the solids in π through δ are 25-solids and since there are at most two $(1, 0)$-lines through R_π in Σ^* corresponding to the solution $(c_{74}, c_{103}) = (1, 2)$ in Table 4.1 for $t = 25$, δ corresponds to a $(7, 3)$-plane δ^* through R_π in Σ^* by Theorem 3.12. In δ^*, there are one $(1, 0)$-line and three $(2, 1)$-lines through R_π. Hence, estimating the left hand side of (4.4), we get

$$2182 \leq 406 + 182 \cdot 3 + 101 + 55 + 20 \cdot 23 + 92 + 3 = 1663,$$

from the spectrum of $C_1 \cap \pi$ of type (a), a contradiction. If $C_0 \cap \pi$ forms a minihyper of type (b) in Lemma 4.8, we have $\tau_{34} = 90$. Hence there are at most four $(1, 0)$-lines through R_π in Σ^* which correspond to the solutions of (4.2) with $t \neq 34$. Let δ be the plane given in (b) of Lemma 4.8. Since the solids in π through δ consist of one 19-solid and three 28-solids and since the solution in Table 4.1 for $t = 19$ corresponds to a $(2, 1)$-line, δ corresponds to a $(7, 3)$-plane δ^* through R_π in Σ^* by Theorem 3.12. Hence, estimating the left hand side of (4.4), we get

$$2182 \leq 503 + 101 \cdot 2 + 97 + 55 \cdot 3 + 20 \cdot 24 + 90 + 3 = 1540,$$

from the spectrum of $C_1 \cap \pi$ of type (b), a contradiction. Hence $a_{100} = 0$.

Table 4.1. Solutions of (4.2) for $i = 100$

t	c_{74}	c_{89}	c_{92}	c_{98}	c_{99}	c_{100}	c_{101}	c_{102}	c_{103}	line
19	1	1			1					$(2,1)$
25	1								2	$(1,0)$
		2						1		$(2,1)$
		1	1		1					$(2,1)$
28		1		1				1		$(2,1)$
		1			1		1			$(2,1)$
		1				2				$(1,0)$
			1	1	1					$(2,1)$
31			1						2	$(1,0)$
				2				1		$(2,1)$
				1	1		1			$(2,1)$
				1		2				$(1,0)$
					2	1				$(0,2)$
34							1		2	$(1,0)$
								2	1	$(0,2)$

Table 4.2. Solutions of (4.2) for $i = 103$

t	c_{74}	c_{89}	c_{92}	c_{98}	c_{99}	c_{101}	c_{102}	c_{103}	line
25	1					1	1		$(2,1)$
		2			1				$(2,1)$
26	1							2	$(1,0)$
		2					1		$(2,1)$
		1	1		1				$(2,1)$
31		1						2	$(1,0)$
			1			1	1		$(2,1)$
				2	1				$(2,1)$
32			1					2	$(1,0)$
				2			1		$(2,1)$
				1	1	1			$(2,1)$
34				1				2	$(1,0)$
					1		1	1	$(0,2)$
						2	1		$(2,1)$
35						1		2	$(1,0)$
							2	1	$(0,2)$

Next, we prove the non-existence of a $(13,0)$-plane in Σ^* which consists of collinear four points corresponding to 89-hyperplanes and nine points corresponding to 92-hyperplanes. Let δ^* be such a plane containing a $(4,0)$-line l_0 consisting the points corresponding to 89-hyperplanes of Σ. Take a point P of l_0 which corresponds to a 89-hyperplane π_P and let l_1, l_2, l_3 be the other lines on δ^* through P. Setting $i = 89$ in Lemma 4.5, l_0 corresponds to the solution $c_{89} = 3$ for $t = 17$ in (4.2) and l_1, l_2, l_3 correspond to the solution $c_{92} = 3$ for $t = 20$ in (4.2). It follows that there exists a u-plane δ_0 in π_P such that there are one 17-solid and three 20-solids in π_P through δ_0, so $(20-u)3+17 = 89$, giving a contradiction.

Finally, assume $i = 103$. Then, $C_1 \cap \pi$ gives a $[103, 5, 68]_3$ code by Lemma 4.5 and $C_0 \cap \pi$ forms a minihyper consisting of a plane δ, a line ℓ and a point P which are mutually disjoint by Lemma 4.6. Let R_π be the point of F_2 in Σ^* corresponding to π. Setting $i = 103$ in Lemma 4.5, the equation (4.2) has the solutions as in Table 4.2, where 'line' stands for the corresponding line through R_π in Σ^*. Since there are one 25-solid (corresponding to a $(2,1)$-line) and three 26-solids (corresponding to a $(2,1)$-line or a $(1,0)$-line) through δ in π, δ corresponds to a $(7,3)$-plane, say δ^*, through R_π by Theorem 3.12. Hence, there are one $(1,0)$-line and three $(2,1)$-lines through R_π in δ^*. Furthermore, the solids in π through ℓ are four 31-solids containing $\langle \ell, P \rangle$ and nine 32-solids, all of which correspond to $(1,0)$-lines or $(2,1)$-lines through R_π. If all of the lines are $(1,0)$-lines, then ℓ corresponds to a $(13,0)$-solid in Σ^* containing the $(13,0)$-plane which consists of collinear four points corresponding to 89-hyperplanes and nine points corresponding to 92-hyperplanes, a contradiction. Hence, by Theorem 3.12, ℓ corresponds to a $(22,9)$-solid containing four $(1,0)$-lines and nine $(2,1)$-lines through R_π. Recall that the spectrum of π is $(\tau_{25}, \tau_{26}, \tau_{31}, \tau_{32}, \tau_{34}, \tau_{35}) = (1, 3, 4, 9, 35, 69)$. Estimating the left hand side of (4.4) we get

$$2182 \leq 407 + 406 + 182 \cdot 2 + 91 \cdot 4 + 20 \cdot 9 + 10 \cdot 35 + 1 \cdot 69 = 2140,$$

a contradiction. This completes the proof of Theorem 4.2. □

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