Some Results on Chromatic Polynomials of Hypergraphs

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Abstract

In this paper, chromatic polynomials of (non-uniform) hypercycles, unicyclic hypergraphs, hypercacti and sunflower hypergraphs are presented. The formulae generalize known results for r-uniform hypergraphs due to Allagan, Borowiecki/Łazuka, Dohmen and Tomescu.

Furthermore, it is shown that the class of (non-uniform) hypertrees with m edges, where m_r edges have size r, $r \geq 2$, is chromatically closed if and only if $m \leq 4$, $m_2 \geq m-1$.

1 Notation and preliminaries

Most of the notation concerning graphs and hypergraphs is based on Berge [4].

A *hypergraph* $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ consists of a finite non-empty set $\mathcal{V}$ of vertices and a family $\mathcal{E}$ of edges which are non-empty subsets of $\mathcal{V}$ of *cardinality* at least 2. An edge e of cardinality $r(e)$ is called an *r-edge*. $\mathcal{H}$ is *r-uniform* if each edge $e \in \mathcal{E}$ is an r-edge. The *degree* $d_{\mathcal{H}}(v)$ is the number of edges containing the vertex v. A vertex v is called *pendant* if $d_{\mathcal{H}}(v) = 1$.

$\mathcal{H}$ is said to be *simple* if all edges are distinct. $\mathcal{H}$ is is said to be *Sperner* if no edge is a subset of another edge. Uniform simple hypergraphs are Sperner. Simple 2-uniform hypergraphs are graphs.

A hypergraph $\mathcal{H}' = (\mathcal{W}, \mathcal{F})$ with $\mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{F} \subseteq \mathcal{E}$ is called a *subhypergraph* of $\mathcal{H}$. If $\mathcal{W} = \bigcup_{e \in \mathcal{F}} e$, then the subhypergraph is said to be *induced by* $\mathcal{F}$, abbreviated by $\mathcal{H}_{\mathcal{F}}$.

The *2-section* of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is the graph $[\mathcal{H}]_2 = (\mathcal{V}, [\mathcal{E}]_2)$ such that $\{u, v\} \in [\mathcal{E}]_2$, $u \neq v$, $u, v \in \mathcal{V}$ if and only if u, v are contained in a hyperedge of $\mathcal{H}$.

In a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ an alternating sequence $v_1, e_1, v_2, e_2, \ldots, e_m, v_{m+1}$, where $v_i \neq v_j$, $1 \leq i < j < m$, $v_i, v_{i+1} \in e_i$ is called a *chain*. Note that repeated edges are allowed in a chain. If also $e_i \neq e_j$, $1 \leq i < j \leq m$, we call it a *path* of *length* m. If $v_1 = v_{m+1}$, a chain is called *cyclic chain*, and a path is called *cycle*. The subhypergraph $\mathcal{C}$ induced by the edge set of a cycle of length m is called a *hypercycle*, short *m-hypercycle*. Observe that in case of graphs the notion chain and path, cyclic chain and cycle coincide whereas this is not the case for hypergraphs in general.

A hypergraph $\mathcal{H}$ is said to be *connected* if for every $v, w \in \mathcal{V}$ there exists a sequence of edges $e_1, \ldots, e_k$, $k \geq 1$ such that $v \in e_1$, $w \in e_k$ and $e_i \cap e_{i+1} \neq \emptyset$, for $1 \leq i < k$. The maximal subhypergraphs which are connected are called *components*. If a single vertex v or single edge e is a component then v or e is called *isolated*. We use the abbreviation $\mathbin{\dot\cup}$ for the disjoint union operation, especially of connected components.

According Acharya [1], the relation $\sim$ in $\mathcal{E}$ is an equivalence relation, where $e_1 \sim e_2$ if and only if $e_1 = e_2$ or there exists a cyclic chain containing e_1, e_2. A *block* of $\mathcal{H}$ is either an isolated vertex/edge or a subhypergraph induced by the edge set of an equivalence class. A block consisting of only one non-isolated edge is called a *bridge-block*.

Lemma 1.1 ([1, Theorem 1.1])**.** *Two distinct blocks of a hypergraph have at most one vertex in common.*

The *block-graph* $bc(\mathcal{H})$ of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is the bipartite graph created as follows. Take as vertices the blocks of $\mathcal{H}$ and the vertices in $\mathcal{V}$ which are common vertices of two blocks. Two vertices of $bc(\mathcal{H})$ are adjacent if and only if one vertex corresponds to a block B of $\mathcal{H}$ and the other vertex is a common vertex $c \in B$. Observe that in case of graphs we get the block-cutpoint-tree introduced by Harary and Prins [10].

Lemma 1.2 ([10, Theorem 1])**.** *If G is a connected graph, then $bc(G)$ is a tree*

A hypercycle $\mathcal{C}$ is said to be *elementary* if $d_{\mathcal{C}}(v_i) = 2$ for each $i \in \{1, 2, \ldots, m\}$ and each other vertex $u \in \bigcup_{i=1}^m e_i$ is pendant. This is equivalent to the fact that $\mathcal{C}$ contains only a unique cycle (sequence) up to permutation. A 2-uniform m-hypercycle (which is elementary per se) is called *m-gon*. A hypergraph is *linear* if any two of its edges do not intersect in more than one vertex. Elementary 2-hypercycles are not linear, whereas elementary m-hypercycles, $m \geq 3$, are linear.

A *hypertree* is a connected hypergraph without cycles. Obviously, a hypertree is linear. A *hyperstar* is a hypertree where all edges intersect in one vertex. A *hyperforest* consists of components each of which is a hypertree. A *unicyclic* hypergraph is a connected hypergraph containing exactly one cycle, i.e. one hypercycle which is elementary.

A *hypercactus* is a connected hypergraph, where each block is an elementary hypercycle or a bridge-block. Note that this is another approach to generalize the notion of cactus from graphs to hypergraphs as chosen by Sonntag [14, 15].

A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ of order n is called a *sunflower hypergraph* if there exist $\mathcal{X} \subset \mathcal{V}, |\mathcal{X}| = q$, $1 \leq q < n$ and a partition $\mathcal{V} \setminus \mathcal{X} = \dot{\bigcup}_{i=1}^m \mathcal{Y}_i$ such that $\mathcal{E} = \bigcup_{i=1}^m (\mathcal{X} \mathbin{\dot\cup} \mathcal{Y}_i)$. Each set $\mathcal{Y}_i$ is called a *petal*, the vertices in X are called *seeds*. Observe, if $|\mathcal{X}| = 1$ then $\mathcal{H}$ is a hyperstar and if $|\mathcal{X}| = 2$ then $\mathcal{H}$ is a 2-hypercycle.

A λ-coloring of $\mathcal{H}$ is a function $f\colon \mathcal{V} \to \{1,\dots,\lambda\}$, $\lambda \in \mathbb{N}$, such that for each edge $e \in \mathcal{E}$ there exist $u, v \in e$, $u \neq v$, $f(u) \neq f(v)$. The number of λ-colorings of $\mathcal{H}$ is given by a polynomial $P(\mathcal{H},\lambda)$ of degree n in λ, called the *chromatic polynomial* of $\mathcal{H}$.

Two hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ are said to be *chromatically equivalent*, written $\mathcal{H} \approx \mathcal{H}'$, if and only if $\mathrm{P}(\mathcal{H},\lambda)$=$\mathrm{P}(\mathcal{H}',\lambda)$. The equivalence class of $\mathcal{H}$ is abbreviated by $\langle\mathcal{H}\rangle$.

Extending a definition based on Dong, Koh and Teo [8, Chapter 3] from graphs to hypergraphs, a class $\mathfrak{H}$ of hypergraphs is called *chromatically closed* if for any $\mathcal{H} \in \mathfrak{H}$ the condition $\langle\mathcal{H}\rangle \subseteq \mathfrak{H}$ is satisfied. Let $\mathfrak{H}, \mathfrak{K}$ be two classes of hypergraphs, then $\mathfrak{H}$ is said to be *chromatically closed within the class* $\mathfrak{K}$, if for every $\mathcal{H} \in \mathfrak{H} \cap \mathfrak{K}$ we have $\langle\mathcal{H}\rangle \cap \mathfrak{K} \subseteq \mathfrak{H} \cap \mathfrak{K}$.

We use the following abbreviations throughout this paper. If $\mathcal{H}$ is isomorphic to $\mathcal{H}'$, we write $\mathcal{H} \cong \mathcal{H}'$. If $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, $\mathcal{H}_1 \cap \mathcal{H}_2 \cong K_n$, we write $\mathcal{H} = \mathcal{H}_1 \cup_n \mathcal{H}_2$. K_n denotes the complete graph of order n, especially K_1 is an isolated vertex. $\overline{K}_n$ denotes the hypergraph consisting of $n \geq 2$ isolated vertices. $\mathcal{S}_{(k_1)r_1,\dots,(k_m)r_m}$ denotes a hyperstar with k_i r_i-edges, $i = 1,\dots,m$. $\mathcal{C}_{r_1,\dots,r_m}$ denotes the elementary m-hypercycle, where e_i has size r_i, $i = 1,\dots,m$. If k_i consecutive edges of the hypercycle have the same size r_i, we write $\mathcal{C}_{(k_1)r_1,\dots,(k_m)r_m}$.

Explicit expressions of chromatic polynomials of hypergraphs were obtained by several authors. In most cases the hypergraphs are assumed to be uniform and linear.

The chromatic polynomials of r-uniform hyperforests and r-uniform elementary hypercycles were presented by Dohmen [7] and rediscovered by Allagan [3] who used a slightly different notation.

Theorem 1.1 ([7, Theorem 1.3.2, Theorem 1.3.4], [3, Theorem 1, Theorem 2])**.** *If $\mathcal{H} = (\mathcal{V},\mathcal{E})$ is an r-uniform hyperforest with m edges and c components, where $r \geq 2$, then*

$$P(\mathcal{H},\lambda) = \lambda^c(\lambda^{r-1} - 1)^m \tag{1.1}$$

If $\mathcal{H} = (\mathcal{V},\mathcal{E})$ is an r-uniform elementary m-hypercycle, where $r \geq 2$, $m \geq 3$, then

$$P(\mathcal{H},\lambda) = (\lambda^{r-1} - 1)^m + (-1)^m(\lambda - 1) \tag{1.2}$$

With the restriction that the hypergraphs are linear, Borowiecki/Łazuka [6] were able to show the converse of (1.1). Combined with the classical result of Read [13] concerning trees, we get

Theorem 1.2 ([6, Theorem 5], [13, Theorem 13])**.** *If $\mathcal{H}$ is a linear hypergraph and*

$$P(\mathcal{H},\lambda) = \lambda(\lambda^{r-1} - 1)^m, \text{ where } r \geq 2, m \geq 1 \tag{1.3}$$

then $\mathcal{H}$ is an r-uniform hypertree with m edges.

Similarly, results of Eisenberg [9], Łazuka [12] for graphs and Borowiecki/Łazuka [6] concerning r-uniform unicyclic hypergraphs, $r \geq 3$, can be summarized as follows:

Theorem 1.3 ([9], [12, Theorem 2], [6, Theorem 8]). *Let $\mathcal{H}$ be a linear hypergraph. $\mathcal{H}$ is an r-uniform unicyclic hypergraph with $m+p$ edges and a cycle of length p if and only if*

$$P(\mathcal{H},\lambda) = (\lambda^{r-1}-1)^{m+p} + (-1)^p(\lambda-1)(\lambda^{r-1}-1)^m, \tag{1.4}$$

where $r \geq 2$, $m \geq 0$ and $p \geq 3$.

In parallel Allagan [3, Corollary 3] discovered a slightly different formula for r-uniform unicyclic hypergraphs which can be easily transformed into (1.4).

Borowiecki/Łazuka [5, Theorem 5] were the first who studied a class of non-linear uniform hypergraphs which are named sunflower hypergraphs by Tomescu in [17]. In [18] Tomescu gave the following formula of the corresponding chromatic polynomial which we restate in a slightly different notation.

Theorem 1.4 ([18, Lemma 2.1]). *Let $\mathcal{S}(m,q,r)$ be an r-uniform sunflower hypergraph having m petals and q seeds, where $m \geq 1$, $1 \leq q \leq r-1$, then*

$$P(\mathcal{S}(m,q,r),\lambda) = \lambda(\lambda^{r-q}-1)^m + \lambda^{(r-q)m}(\lambda^q-\lambda) \tag{1.5}$$

The first formulae of chromatic polynomials of non-uniform hypergraphs were mentioned by Allagan [2]. He considered the special case of non-uniform elementary cycles $\mathcal{H}_m$ which are constructed from an m-gon, $m \geq 3$, by replacing a 2-edge by a $k+2$-edge, where $k \geq 1$.

Theorem 1.5 ([2, Theorem 1]). *The chromatic polynomial of the hypergraph $\mathcal{H}_m$, $m \geq 3$, has the form:*

$$P(\mathcal{H}_m,\lambda) = (\lambda-1)^m \sum_{i=0}^{k} \lambda^i + (-1)^m(\lambda-1). \tag{1.6}$$

Remark 1.1. (1.6) *can be restated as follows*

$$P(\mathcal{H}_m,\lambda) = (\lambda-1)^{m-1}(\lambda^{k+1}-1) + (-1)^m(\lambda-1) \tag{1.7}$$

Borowiecki/Łazuka [5] extended (1.1) by dropping the uniformity assumption.

Theorem 1.6 ([5, Theorem 8]). *If $\mathcal{H} = (\mathcal{V},\mathcal{E})$ is a hyperforest with m_r r-edges, where $2 \leq r \leq R$, and c components, then*

$$P(\mathcal{H},\lambda) = \lambda^c \prod_{r=2}^{R} (\lambda^{r-1}-1)^{m_r} \tag{1.8}$$

These results suggest to generalize (1.2), (1.4) and (1.5) to non-uniform hypergraphs.

Before we state our results, we remember three useful reduction methods concerning the calculation of chromatic polynomials of hypergraphs.

Given a hypergraph $\mathcal{H}$. If dropping an edge $e \in \mathcal{E}$ yields a hypergraph $\mathcal{H}'$ being chromatically equivalent to $\mathcal{H}$, then e is called *chromatically inactive*. Otherwise, e is said to be *chromatically active*. Dohmen [7] gave the following lemma:

Lemma 1.3 ([7, Theorem 1.2.1]). *A hypergraph $\mathcal{H}$ and the subhypergraph $\mathcal{H}'$ which results by dropping all chromatically inactive edges are chromatically equivalent.*

The next lemma generalizes Whitney's fundamental reduction theorem. It was already mentioned by Jones [11] in case where the added edge is a 2-edge.

Lemma 1.4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph, $X \subseteq \mathcal{V}$ an r-set, $r \geq 2$, such that $e \nsubseteq X$ for every $e \in \mathcal{E}$. Let $\mathcal{H}+X$ denote the hypergraph obtained by adding X as a new edge to $\mathcal{E}$ and dropping all chromatically inactive edges. Let $\mathcal{H}.X$ be the hypergraph obtained by contracting all vertices in X to a common vertex x and dropping all chromatically inactive edges. Then*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H}+X, \lambda) + P(\mathcal{H}.X, \lambda) \tag{1.9}$$

Proof. We extend the standard proof well-known in the case of graphs.

Let f be a λ-coloring of $\mathcal{H}$ and $X \subseteq \mathcal{V}$ an r-set, $r \geq 2$, such that $e \nsubseteq X$ for every $e \in \mathcal{E}$. Either (i) there exist $u, v \in X$ with $f(u) \neq f(v)$ or (ii) $f(u) = f(v)$ for all $u, v \in X$.

The λ-colorings of $\mathcal{H}$ for which (i) holds are also λ-colorings of $\mathcal{H}+X = (\mathcal{V}, \mathcal{E}+X)$ where $\mathcal{E}+X = \mathcal{E} \cup X \setminus \mathcal{E}_X$ where $\mathcal{E}_X = \{e \in \mathcal{E} \mid X \subset e\}$, and vice versa.

The λ-colorings of $\mathcal{H}$ for which (ii) holds are also λ-colorings of $\mathcal{H}.X = (\mathcal{V}.X, \mathcal{E}.X)$ where $\mathcal{V}.X = \mathcal{V} \setminus X \cup \{x\}, \mathcal{E}.X = \{e \setminus X \cup \{x\} \mid e \in \mathcal{E}\}$, and vice versa. Observe that $\mathcal{H}.X$ may contain parallel edges, of which all but one can be dropped as chromatically inactive edges. □

Corollary 1.1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. Let $\mathcal{H}-e$ denote the hypergraph obtained by deleting some $e \in \mathcal{E}$ and let $\mathcal{H}.e$ be the hypergraph by contracting all vertices in e to a common vertex x and dropping all chromatically inactive edges. Then*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H}-e, \lambda) - P(\mathcal{H}.e, \lambda) \tag{1.10}$$

Borowiecki/Łazuka [5] generalized an old result of Read [13].

Lemma 1.5 ([5, Theorem 6]). *If $\mathcal{H}$ is a hypergraph such that $\mathcal{H} = \bigcup_{i=1}^{k} \mathcal{H}_i$ for $k \geq 2$, where $\mathcal{H}_i \cap \mathcal{H}_j = K_p$ for $i \neq j$ and $\bigcap_{i=1}^{k} \mathcal{H}_i = K_p$, then*

$$P(\mathcal{H}, \lambda) = P(K_p, \lambda)^{1-k} \prod_{i=1}^{k} P(\mathcal{H}_i, \lambda). \tag{1.11}$$

2 The chromatic polynomials of non-uniform hypergraphs

Our first generalization concerns non-uniform elementary hypercycles. Note, that elementary 2-hypercycles are not linear whereas elementary m-hypercycles, $m \geq 3$, are linear.

Theorem 2.1. *If $\mathcal{C} = (\mathcal{V}, \mathcal{E})$ is an elementary m-hypercycle having m_r r-edges, where $2 \leq r \leq R$, then*

$$P(\mathcal{C}, \lambda) = \prod_{r=2}^{R} (\lambda^{r-1} - 1)^{m_r} + (-1)^m (\lambda - 1) \tag{2.1}$$

Our second generalization concerns non-uniform hypercacti.

Theorem 2.2. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypercactus with*

(1) k elementary p_i-hypercycles $\mathcal{C}_i = (\mathcal{W}_i, \mathcal{F}_i), i = 1, \ldots, k$, having p_{ir} r-edges, where $2 \leq r \leq R$

(2) m_r bridge-blocks of size r, $2 \leq r \leq R$.

Then

$$P(\mathcal{H}, \lambda) = \frac{1}{\lambda^{k-1}} \prod_{r=2}^{R} (\lambda^{r-1} - 1)^{m_r} \prod_{i=1}^{k} \left[\prod_{r=2}^{R} (\lambda^{r-1} - 1)^{p_{ir}} + (-1)^{p_i} (\lambda - 1) \right] \tag{2.2}$$

By converting (2.2), we get the following generalization of Theorem 1.3 concerning non-uniform unicyclic hypergraphs.

Corollary 2.1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a connected unicyclic hypergraph containing a p-hypercycle $\mathcal{C} = (\mathcal{W}, \mathcal{F})$ with p_r r-edges and containing m_r bridge-blocks of size r, where $2 \leq r \leq R$, then*

$$P(\mathcal{H}, \lambda) = \prod_{r=2}^{R} (\lambda^{r-1} - 1)^{m_r + p_r} + (-1)^p (\lambda - 1) \prod_{r=2}^{R} (\lambda^{r-1} - 1)^{m_r} \tag{2.3}$$

Our third generalization concerns non-uniform sunflower hypergraphs.

Theorem 2.3. *Let $\mathcal{S}$ be a sunflower hypergraph of order n containing m_r r-edges and q seeds, where $q + 1 \leq r \leq R$, then*

$$P(\mathcal{S}, \lambda) = \lambda \left[\lambda^{n-1} - \lambda^{n-q} + \prod_{r=q+1}^{R} (\lambda^{r-q} - 1)^{m_r} \right] \tag{2.4}$$

Especially in case of uniform hypergraphs we get an alternative expression of Theorem 1.4:

Corollary 2.2. *If $\mathcal{H}$ is an r-uniform sunflower hypergraph of order n and q seeds, then*

$$P(\mathcal{H}, \lambda) = \lambda \left[\lambda^{n-1} - \lambda^{n-q} + (\lambda^{r-q} - 1)^m \right] \tag{2.5}$$

Remark 2.1. *The proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3 are based on the fact that the chromatic polynomials can be restated as follows*

$$(1.8)\quad P(\mathcal{H},\lambda) = \lambda^c \prod_{x\in\mathcal{E}} (\lambda^{r(x)-1} - 1) \tag{2.6}$$

$$(2.1)\quad P(\mathcal{C},\lambda) = \prod_{x\in\mathcal{E}} (\lambda^{r(x)-1} - 1) + (-1)^m(\lambda - 1) \tag{2.7}$$

$$(2.2)\quad P(\mathcal{H},\lambda) = \frac{1}{\lambda^{|I|-1}} \prod_{x\in\mathcal{E}\setminus\mathcal{F}} (\lambda^{r(x)-1} - 1) \prod_{i\in I} \left[\prod_{x\in\mathcal{F}_i} (\lambda^{r(x)-1} - 1) + (-1)^{p_i}(\lambda - 1) \right],$$

$$\text{where } \mathcal{F} = \bigcup_{i\in I} \mathcal{F}_i, I = \{1, \ldots, k\} \tag{2.8}$$

$$(2.3)\quad P(\mathcal{H},\lambda) = \prod_{x\in\mathcal{E}} (\lambda^{r(x)-1} - 1) + (-1)^p(\lambda - 1) \prod_{x\in\mathcal{E}\setminus\mathcal{F}} (\lambda^{r(x)-1} - 1) \tag{2.9}$$

$$(2.4)\quad P(\mathcal{H},\lambda) = \lambda \left[\lambda^{n-1} - \lambda^{n-q} + \prod_{x\in\mathcal{E}} (\lambda^{r(x)-q} - 1) \right] \tag{2.10}$$

Proof of Theorem 2.1. We use induction on the sum $s(\mathcal{C})$ of the edge cardinalities of the elementary m-hypercycle $\mathcal{C}$.

The induction starts for each m separately.

For $m = 2$, the elementary m-hypercycle $\mathcal{C}$ with minimum $s(\mathcal{C})$ consists of two 3-edges e, f, which intersect in exactly two vertices u_1, u_2. Let $v \in e \setminus f$. Replacing the edge e by a 2-edge $k = \{u_1, v\}$ yields the hypergraph $\mathcal{C}+k$ which is obviously a hypertree with a 3-edge and a 2-edge. Contracting the vertices u, v yields the hypergraph $\mathcal{C}.k$, where e shrinks to the 2-edge $\{u_1, u_2\} \subset f$. Therefore f is chromatically inactive in $\mathcal{C}.k$ and can be dropped. The resulting chromatically equivalent Sperner hypergraph is isomorphic to $K_1 \uplus K_2$.

By Lemma 1.4 and (2.6), we have

$$P(\mathcal{C},\lambda) = \lambda(\lambda-1)(\lambda^2-1) + \lambda^2(\lambda-1) = (\lambda^2-1)^2 + (-1)^2(\lambda-1)$$

This proves the assertion.

For $m \geq 3$ the elementary m-hypercycle with minimal $s(\mathcal{C})$ is the m-gon.
Hence, (2.1) is the well-known formula

$$P(\mathcal{C},\lambda) = (\lambda-1)^m + (-1)^m(\lambda-1).$$

The induction step can be made for all $m \geq 2$ simultaneously.

Choose an edge e of the elementary cycle $\mathcal{C}$ with maximal cardinality. If $m = 2$, then $r(e) \geq 4$, if $m \geq 3$, then $r(e) \geq 3$. Let f be the predecessor edge in the cycle sequence. Let $u \in e \cap f$ and $v \in e \setminus f$. We create the two hypergraphs $\mathcal{C}+k$ and $\mathcal{C}.k$ as follows. We add the 2-edge $k = \{u, v\}$ and shrink the edge e to the edge e' by identifying u, v. e' remains chromatically active in $\mathcal{C}.k$.

Obviously, $\mathcal{C}+k$ is a hyperforest and has $r(e)-2$ components where $r(e)-3$ of these are isolated vertices. $\mathcal{C}.k$ is an elementary m-hypercycle where e is replaced by e' with size $r(e')=r(e)-1$. Observe that $\mathcal{C}$, $\mathcal{C}+k$ and $\mathcal{C}.k$ have the same number of edges m.
Since s($\mathcal{C}.k$)=s($\mathcal{C}$)-1, we can apply the induction hypothesis. By (1.9), (2.6) and (2.7), we have

$$\begin{aligned} P(\mathcal{C},\lambda) &= \lambda^{r(e)-2}(\lambda-1)\prod_{g\in\mathcal{E},g\neq e}(\lambda^{r(g)-1}-1)\\ &\qquad + (\lambda^{r(e')-1}-1)\prod_{x\in\mathcal{E},x\neq e'}(\lambda^{r(x)-1}-1)+(-1)^m(\lambda-1)\\ &= \lambda^{r(e)-2}(\lambda-1)\prod_{x\in\mathcal{E},x\neq e}(\lambda^{r(x)-1}-1)\\ &\qquad + (\lambda^{r(e)-2}-1)\prod_{x\in\mathcal{E},x\neq e}(\lambda^{r(x)-1}-1)+(-1)^m(\lambda-1)\\ &= \left[\lambda^{r(e)-2}(\lambda-1)+\lambda^{r(e)-2}-1\right]\prod_{x\in\mathcal{E},x\neq e}(\lambda^{r(x)-1}-1)+(-1)^m(\lambda-1)\\ &= (\lambda^{r(e)-1}-1)\prod_{x\in\mathcal{E},x\neq e}(\lambda^{r(x)-1}-1)+(-1)^m(\lambda-1)\\ &= \prod_{x\in\mathcal{E}}(\lambda^{r(x)-1}-1)+(-1)^m(\lambda-1) \end{aligned}$$

□

To simplify the proof of Theorem 2.2 we extend Lemma 1.2 to hypergraphs.

Lemma 2.1. *The block-graph $bc(\mathcal{H})$ of a connected hypergraph $\mathcal{H}$ is a tree.*

Proof. If $\mathcal{H}$ is a graph, we have nothing to show.

If $\mathcal{H}$ is not a graph, we show that $bc(\mathcal{H})\cong bc([\mathcal{H}]_2)$. Then Lemma 1.2 completes the proof.

We have to verify that $e,f\in\mathcal{E}$ are in the same block of $\mathcal{H}$ if and only if $e',f'\in\mathcal{E}_2$ are in the same block of $[\mathcal{H}]_2$ for all $e'\subseteq e$, $f'\subseteq f$. This implies also that the common vertices of the blocks of $\mathcal{H}$ and $[\mathcal{H}]_2$ coincide.

Let $e'\subseteq e$, $f'\subseteq f$, $e'\neq f'$ be in the same block of $[\mathcal{H}]_2$. Then $[\mathcal{H}]_2$ contains a cycle $v_1,e'_1,\ldots,e',\ldots,f',\ldots,e'_m,v_{m+1}$, $v_i\neq v_j$, $1\leq i<j<m$, $v_1=v_{m+1}$. We replace every edge $x'\in[\mathcal{E}]_2$ in this cycle by the corresponding edge $x\in\mathcal{E},x'\subseteq x$. The result is a cycle in $\mathcal{H}$ which contains e,f.

Conversely, let $e'\subseteq e$, $f'\subseteq f$, where e,f are in the same block of $\mathcal{H}$. Then there exists a cyclic chain $u_1,e_1,\ldots,e_n,u_{n+1}$, $u_i\neq u_j$, $1\leq i<j<n$, $u_1=u_{n+1}$, where w.l.o.g. $e_k=e$, $e_l=f$ with $1\leq k<l\leq n$. Replace e_i by the 2-edge $\{u_i,u_{i+1}\}$, $i=1,\ldots,n$. If $e'=\{u_i,u_{i+1}\}$ and $f'=\{u_j,u_{j+1}\}$, we are finished. Assume that $e'=\{u,v\}$, $u,v\in e$, with $\{u,v\}\neq\{u_i,u_{i+1}\}$ for all $i=1,\ldots,n$. Then the cycle $u,\{u,v\},v,\{v,u_i\},u_i,\{u_i,u_{i+1}\},u_{i+1}\,\{u_{i+1},u\},u$ exists because each substituted 2-edge

exists by the definition of $[\mathcal{H}]_2$. It follows that e', $\{u_i, u_{i+1}\}$ and $\{u_j, u_{j+1}\}$ are in the same block of $[\mathcal{H}]_2$. We apply the same argument to f' to complete the proof. $\square$

Proof of Theorem 2.2. We use induction on the number b of blocks.

If $b = 1$, then $\mathcal{H}$ is either a bridge-block or consists of an elementary hypercycle. The evaluation of (2.2) yields either (1.1) or (2.1).

If $b \geq 2$, $bc(\mathcal{H})$ is a tree by Lemma 2.1. Therefore, we can split $\mathcal{H} = \mathcal{Y} \cup_1 \mathcal{Z}$, where $\mathcal{Y}, \mathcal{Z}$ are hypercacti. Obviously, the hypercycles and bridge-blocks of $\mathcal{H}$ are divided in those of $\mathcal{Y}$ and $\mathcal{Z}$, i.e. $\mathcal{F}_\mathcal{Y} = \mathcal{F} \cap \mathcal{E}_\mathcal{Y}$ and $\mathcal{F}_\mathcal{Z} = \mathcal{F} \cap \mathcal{E}_\mathcal{Z}$, where $\mathcal{E}_\mathcal{Y}, \mathcal{E}_\mathcal{Z}$ are the edge sets of $\mathcal{Y}, \mathcal{Z}$. Hence we can use the induction hypothesis and (1.11).

$$
\begin{aligned}
P(\mathcal{H},\lambda) &= \frac{1}{\lambda} P(\mathcal{Y},\lambda) P(\mathcal{Z},\lambda) \\
&= \frac{1}{\lambda} \left(\frac{1}{\lambda^{|I_\mathcal{Y}|-1}} \prod_{x \in \mathcal{E}_\mathcal{Y} \setminus \mathcal{F}_\mathcal{Y}} (\lambda^{r(x)-1} - 1) \prod_{i \in I_\mathcal{Y}} \left[\prod_{x \in \mathcal{F}_i} (\lambda^{r(x)-1} - 1) + (-1)^{p_i} (\lambda - 1) \right] \right) \\
&\qquad \left(\frac{1}{\lambda^{|I_\mathcal{Z}|-1}} \prod_{x \in \mathcal{E}_\mathcal{Z} \setminus \mathcal{F}_\mathcal{Z}} (\lambda^{r(x)-1} - 1) \prod_{i \in I_\mathcal{Z}} \left[\prod_{x \in \mathcal{F}_i} (\lambda^{r(x)-1} - 1) + (-1)^{p_i} (\lambda - 1) \right] \right) \\
&= \frac{1}{\lambda^{|I_\mathcal{Y}|+|I_\mathcal{Z}|-1}} \prod_{x \in (\mathcal{E}_\mathcal{Y} \setminus \mathcal{F}_\mathcal{Y}) \cup (\mathcal{E}_\mathcal{Z} \setminus \mathcal{F}_\mathcal{Z})} (\lambda^{r(x)-1} - 1) \\
&\qquad \times \prod_{i \in I_\mathcal{Y} \cup I_\mathcal{Z}} \left[\prod_{x \in \mathcal{F}_i} (\lambda^{r(x)-1} - 1) + (-1)^{p_i} (\lambda - 1) \right] \\
&= \frac{1}{\lambda^{|I|-1}} \prod_{x \in \mathcal{E} \setminus \mathcal{F}} (\lambda^{r(x)-1} - 1) \prod_{i \in I} \left[\prod_{x \in \mathcal{F}_i} (\lambda^{r(x)-1} - 1) + (-1)^{p_i} (\lambda - 1) \right]
\end{aligned}
$$

$\square$

Proof of Theorem 2.3. Assume first that the sunflower hypergraph $\mathcal{S}$ has only one petal, i.e. $\mathcal{S}$ consists of one edge of size $q + 1 \leq r \leq R$. Then by (2.4)

$$
P(\mathcal{S},\lambda) = \lambda \left[\lambda^{r-1} - \lambda^{r-q} + (\lambda^{r-q} - 1) \right] = \lambda(\lambda^{r-1} - 1) \tag{2.11}
$$

For the remaining cases, we use induction on $n - q$. The case $n - q = 1$ was just verified.

Let $u \in Y$, Y be a petal of $\mathcal{S}$ and v be a seed. Add the edge $k = \{u, v\}$ to $\mathcal{S}$. Then the edge $e = X \mathbin{\dot\cup} Y$ becomes chromatically inactive. We consider two cases.

Case 1: The petal Y can be chosen to have size 1.

Then $\mathcal{S}+k \cong K_2 \cup_1 \mathcal{U}$, where $\mathcal{U}$ is the sunflower hypergraph induced by $\mathcal{E} \setminus e$, with $e = X \mathbin{\dot\cup} Y$. We contract k and drop all chromatically inactive edges. We receive the

Sperner hypergraph $\mathcal{S}.k = \overline{K}_{\sum_{x\in\mathcal{E}\setminus e}(r(x)-q)} \uplus \mathcal{H}_{\{X\}}$ because e shrinks to X. By Lemma 1.4 and (2.10)

$$\begin{aligned}
P(\mathcal{S},\lambda) &= (\lambda-1)\lambda\left[\lambda^{n-2}-\lambda^{n-q-1}+\prod_{x\in\mathcal{E}\setminus e}(\lambda^{r(x)-q}-1)\right]+\lambda(\lambda^{q-1}-1)\lambda^{\sum_{x\in\mathcal{E}\setminus e}(r(x)-q)}\\
&\quad \textit{by induction hypothesis}\\
&= \lambda\left[(\lambda-1)\lambda^{n-2}-(\lambda-1)\lambda^{n-q-1}+(\lambda-1)\prod_{x\in\mathcal{E}\setminus e}(\lambda^{r(x)-q}-1)+(\lambda^{q-1}-1)\lambda^{n-q-1}\right]\\
&\quad \textit{because} \sum_{x\in\mathcal{E}\setminus e}(r(x)-q)=n-q-1\\
&= \lambda\left[\lambda^{n-1}-\lambda^{n-q}+\prod_{x\in\mathcal{E}}(\lambda^{r(x)-q}-1)\right]\\
&\quad \textit{because } \lambda^{r(e)-q}=\lambda
\end{aligned}$$

Case 2: All petals, especially Y, have size greater 1.

Then $\mathcal{S}+k \cong \overline{K}_{r(e)-q-1} \uplus (K_2 \cup_1 \mathcal{U})$, where $\mathcal{U}$ is the sunflower hypergraph induced by $\mathcal{E}\setminus e$, having $n-r(e)+q$ vertices. $\mathcal{S}.k$ is the sunflower hypergraph of order $n-1$ which is induced by $\mathcal{E}\setminus e\cup e'$, where $e' = X \uplus Y'$, $Y' = Y\setminus\{u\}$ is a petal. All other petals remain chromatically active in $\mathcal{S}.k$. Thus,

$$\begin{aligned}
P(\mathcal{S},\lambda) &= \lambda(\lambda-1)\lambda^{r(e)-q-1}\left[\lambda^{n-r(e)+q-1}-\lambda^{n-r(e)-1}+\prod_{x\in\mathcal{E}\setminus e}(\lambda^{r(x)-q}-1)\right]\\
&\quad +\lambda\left[\lambda^{n-2}-\lambda^{n-q-1}+(\lambda^{r(e')-q}-1)\prod_{x\in\mathcal{E}\setminus e'}(\lambda^{r(x)-q}-1)\right]\\
&\quad \textit{by induction hypothesis}\\
&= \lambda\Bigg[\lambda^{n-1}-\lambda^{n-q}-\lambda^{n-2}+\lambda^{n-q-1}+(\lambda-1)\lambda^{r(e)-q-1}\prod_{x\in\mathcal{E}\setminus e}(\lambda^{r(x)-q}-1)\\
&\quad +\lambda^{n-2}-\lambda^{n-q-1}+(\lambda^{r(e)-q-1}-1)\prod_{x\in\mathcal{E}\setminus e}(\lambda^{r(x)-q}-1)\Bigg]\\
&= \lambda\left[\lambda^{n-1}-\lambda^{n-q}+\prod_{x\in\mathcal{E}}(\lambda^{r(x)-q}-1)\right]
\end{aligned}$$

□

3 Chromaticity of hypertrees

The fact that trees are chromatically closed within the class of graphs can be extended to the case of r-uniform hypertrees, $r \geq 2$, by use of the following lemma due to Tomescu [16] in combination with Theorem 1.2 and (1.1).

Lemma 3.1 ([16, Lemma 3.1]). *If simple r-uniform hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are chromatically equivalent and $\mathcal{H}$ is linear then $\mathcal{G}$ is linear too.*

Theorem 3.1. *The class of r-uniform hypertrees is chromatically closed within the class of r-uniform hypergraphs, where $r \geq 2$.*

Borowiecki/Łazuka already mentioned in [6], without giving concrete examples, that the class of r-uniform hypertrees might not be chromatically closed in general. The following theorem shows that this is indeed true except for a few cases.

Theorem 3.2. *The class $\mathfrak{T}$ of hypertrees with m edges, where m_r edges have size r, $r \geq 2$, is chromatically closed if and only if $m \leq 4$, $m_2 \geq m - 1$.*

To prove this, we use some lemmas concerning the coefficients of the chromatic polynomial of a hypergraph $\mathcal{H}$ of order n expressed in the standard form

$$P(\mathcal{H}, \lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i} \tag{3.1}$$

Borowiecki/Łazuka [6] showed

Lemma 3.2 ([6, Lemma 1]). *Let $\mathcal{H}$ be a hypergraph of order n and the chromatic polynomial expressed by (3.1). If $a_{n-1} \neq 0$ then $\mathcal{H}$ is connected.*

Dohmen [7] showed

Lemma 3.3 ([7, Theorem 1.4.1]). *Let $\mathcal{H}$ be a hypergraph of order n having m_r edges of minimal size r, where $2 \leq r \leq n$ and the chromatic polynomial expressed by (3.1). Then $a_k = 0$, $k = 1, \ldots, r-2$ and $a_{r-1} = -m_r$.*

Proof of Theorem 3.2. We show first that the class of all hypertrees is chromatically closed if $m \leq 4$, $m_2 \geq m - 1$. It suffices to consider only hypertrees having exactly four edges by the following reason. If a hypertree $\mathcal{T}$ with $m \leq 3$ edges would be chromatically equivalent to hypergraph $\mathcal{H}$ which is not a tree then $\mathcal{H} \cup_1 \mathcal{S}_{(4-m)2}$ would be chromatically equivalent to a hypertree with four edges.

Assume there exists a Sperner hypergraph $\mathcal{H}$ which is chromatically equivalent to a hypertree with four edges and at most one r-edge, $r \geq 3$. Obviously, $\mathcal{H}$ is connected by Lemma 3.2 and if $\mathcal{H}$ has the same number of k-edges as $\mathcal{T}$ then it is hypertree. We therefore inspect the number m_k of k-edges of $\mathcal{H}$, $k = 2, \ldots, r+3$.

Clearly $m_{r+3} = 0$, because no chromatically active r+3-edge can exist. Furthermore Lemma 3.3 implies that $\mathcal{H}$ has the same number of 2-edges as $\mathcal{T}$, i.e. $m_2 = 3$, if $r \geq 3$, and $m_2 = 4$, if $r = 2$.

To verify the remaining cases m_k, $2 < k \leq r+2$, observe that if $m_k \neq 0$ then $\mathcal{H}$ contains a spanning hypergraph with one k-edge and all 2-edges. This hypergraph is either a forest or one of the hypergraphs $\mathcal{H}_i, i = 1, \ldots, 6$ in Figure 1.

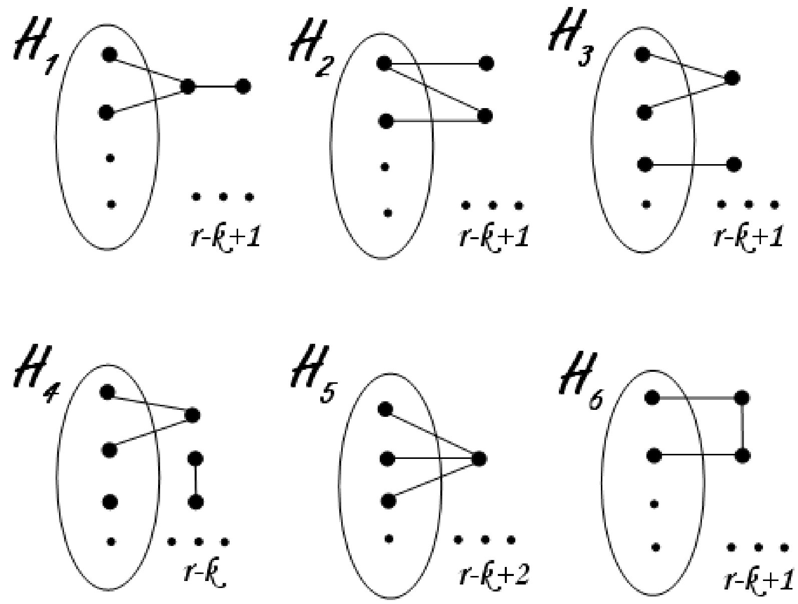

Figure 1

First $m_{r+2} = 0$. Otherwise, assume there exists an $r{+}2$-edge. Since $\lambda - 2 \nmid P(\mathcal{H}, \lambda)$ we conclude that $K_3 \nsubseteq \mathcal{H}$. The fact that $\mathcal{H}$ is Sperner implies that $\mathcal{H} \cong \mathcal{H}_5$, where no isolated vertices exist.

We delete/contract the $r{+}2$-edge. By (1.10)

$$\begin{aligned} P(\mathcal{H}, \lambda) &= \lambda^r(\lambda-1)^3 - \lambda(\lambda-1) = \lambda(\lambda^{r-1}-1)(\lambda-1)^3 + \lambda(\lambda-1)^3 - \lambda(\lambda-1) \\ &\neq P(\mathcal{T}, \lambda) \end{aligned}$$

Next, we show that $m_k = 0$, $2 < k < r$. This is done by comparing $P(\mathcal{H}, \lambda)$ and $P(\mathcal{T}, \lambda)$ for $\lambda \in \mathbb{N}$.

Assume that $\mathcal{H}$ contains a spanning hyperforest $\mathcal{F}$ with all the 2-edges and one k-edge, $2 < k \leq r$. By (1.8) we have

$$\begin{aligned} P(\mathcal{H}, \lambda) \leq P(\mathcal{F}, \lambda) &= \lambda^{r-k+1}(\lambda^{k-1}-1)(\lambda-1)^3 \\ &= \lambda(\lambda^{r-1}-1)(\lambda-1)^3 - \lambda(\lambda^{r-k}-1)(\lambda-1)^3 \leq P(\mathcal{T}, \lambda) \end{aligned}$$

Only in case $k = r$ equality holds, i.e. $\mathcal{H} \approx \mathcal{F} \approx \mathcal{T}$.

Assume next that $\mathcal{H}_i \subseteq \mathcal{H}$ for some $i = 1, \ldots, 6$ and $k \leq r$.

If $\mathcal{H}_i \subseteq \mathcal{H}$ for $i = 1, \ldots, 4$, we apply (2.1) and (1.11).

$$\begin{aligned} P(\mathcal{H}, \lambda) \leq P(\mathcal{H}_i, \lambda) &= \lambda^{r-k+1}(\lambda - 1)\left[(\lambda^{k-1} - 1)(\lambda - 1)^2 - (\lambda - 1)\right] \\ &= \lambda(\lambda^{r-1} - 1)(\lambda - 1)^3 - \lambda(\lambda - 1)^2(\lambda^{r-k+1} - \lambda + 1) \\ &< P(\mathcal{T}, \lambda), \text{ because of } k \leq r. \end{aligned}$$

If $\mathcal{H}_5 \subseteq \mathcal{H}$ we delete/contract the k-edge and apply (1.10), (1.11) and (2.1).

$$\begin{aligned} P(\mathcal{H}, \lambda) \leq P(\mathcal{H}_5, \lambda) &= \lambda^r(\lambda - 1)^3 - \lambda^{r-k+3}(\lambda - 1) \\ &= \lambda(\lambda^{r-1} - 1)(\lambda - 1)^3 - \lambda(\lambda - 1)\left[\lambda^{r-k+2} - \lambda^2 + 2\lambda - 1\right] \\ &< P(\mathcal{T}, \lambda), \text{ because of } k \leq r. \end{aligned}$$

If $\mathcal{H}_6 \subseteq \mathcal{H}$ and $k < r$, we apply (1.11) and (2.1).

$$\begin{aligned} P(\mathcal{H}, \lambda) \leq P(\mathcal{H}_6, \lambda) &= \lambda^{r-k+1}\left[(\lambda^{k-1} - 1)(\lambda - 1)^3 + (\lambda - 1)\right] \\ &= \lambda(\lambda^{r-1} - 1)(\lambda - 1)^3 - \lambda(\lambda - 1)(\lambda^{r-k+2} - 2\lambda^{r-k+1} - \lambda^2 + 2\lambda - 1) \\ &< P(\mathcal{T}, \lambda), \text{ because of } k < r. \end{aligned}$$

Consider $\mathcal{H}_6 \subseteq \mathcal{H}$ and $k = r$. $\mathcal{H} \cong \mathcal{H}_6$ is impossible because (1.11) and (2.1) imply

$$P(\mathcal{H}_6, \lambda) = \lambda\left[(\lambda^{r-1} - 1)(\lambda - 1)^3 + (\lambda - 1)\right] > P(\mathcal{T}, \lambda)$$

Therefore $\mathcal{H}$ must contain additional edges, each of size *r* or size *r+1*. If we delete these edges in an arbitrary sequence until $\mathcal{H}_6$ remains, the order of the hypergraphs resulting from the contraction is always at least *3*. Applying (1.10) repeatedly subtracts from $P(\mathcal{H}_6, \lambda)$ a polynomial of at least degree 3. Hence $P(\mathcal{H}, \lambda) < P(\mathcal{T}, \lambda)$.

In summary, we get that $m_k = 0$ for $2 < k < r$ and that if $\mathcal{H}$ contains an r-edge then $\mathcal{H}$ is a tree.

It remains to exclude the case that a hypergraph containing only *r+1*-edges besides the 2-edges is chromatically equivalent to $\mathcal{T}$. Obviously, $\mathcal{H}$ cannot contain a subhypergraph isomorphic to $\mathcal{H}_4$.

If $\mathcal{H} \cong \mathcal{H}_i$, $i = 1, \ldots, 3$, we apply (2.1) and (1.11)

$$\begin{aligned} P(\mathcal{H}_i, \lambda) &= (\lambda - 1)\left[(\lambda^r - 1)(\lambda - 1)^2 - (\lambda - 1)\right] \\ &= \lambda(\lambda^{r-1} - 1)(\lambda - 1)^3 + \lambda(\lambda - 1)^2(\lambda - 2) > P(\mathcal{T}, \lambda) \end{aligned}$$

If $\mathcal{H} \cong \mathcal{H}_5$, we apply (1.10), (2.1) and (1.11)

$$\begin{aligned} P(\mathcal{H}_5, \lambda) &= \lambda^r(\lambda - 1)^3 - \lambda^2(\lambda - 1) \\ &= \lambda(\lambda^{r-1} - 1)(\lambda - 1)^3 + \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 1) > P(\mathcal{T}, \lambda) \end{aligned}$$

If $\mathcal{H} \cong \mathcal{H}_6$, we apply (2.1)

$$\begin{aligned} P(\mathcal{H}_6, \lambda) &= (\lambda^r - 1)(\lambda - 1)^3 + (\lambda - 1) \\ &= \lambda(\lambda^{r-1} - 1)(\lambda - 1)^3 + \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3) > P(\mathcal{T}, \lambda) \end{aligned}$$

Thus, $\mathcal{H}$ contains additional edges each of size *r+1* because $P(\mathcal{H}_i, \lambda) > P(\mathcal{T}, \lambda)$, for $i = 1, \ldots, 3, i = 5, 6$. If we delete these edges in an arbitrary sequence until $\mathcal{H}_i$ remains, the order of the hypergraphs resulting by the contraction is always equal 3. Applying (1.10) repeatedly subtracts from $P(\mathcal{H}_i, \lambda)$ a polynomial of degree 3. Therefore $P(\mathcal{H}, \lambda) > P(\mathcal{T}, \lambda)$ in each case.

Conversely, if $m \geq 5$ or $m_2 < m - 1$, we can construct a chromatically equivalent hypergraph which is not a hypertree.

Case (1): $\mathcal{H}$ contains two edges of size greater 2.

We can assume that the starting point of our construction is a hyperstar, i.e. all edges have one vertex u in common.

In case of $\mathcal{H} \cong \mathcal{S}_{r,s}$, $r, s \geq 3$, create $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, with $\mathcal{V}_1 = \mathcal{V} \setminus \{v_e, v_f\} \cup \{p, q\}$, $p, q \notin \mathcal{V}$ and with $\mathcal{E}_1 = \mathcal{E} \setminus \{e, f\} \cup \{e_1, f_1\}$, where $e_1 = e \setminus \{v_e\} \cup \{p\}$, $f_1 = f \setminus \{v_f\} \cup \{p\}$. Observe that $e_1 \not\subseteq f_1$, $f_1 \not\subseteq e_1$, i.e. e_1, f_1 are chromatically active (see Figure 2).

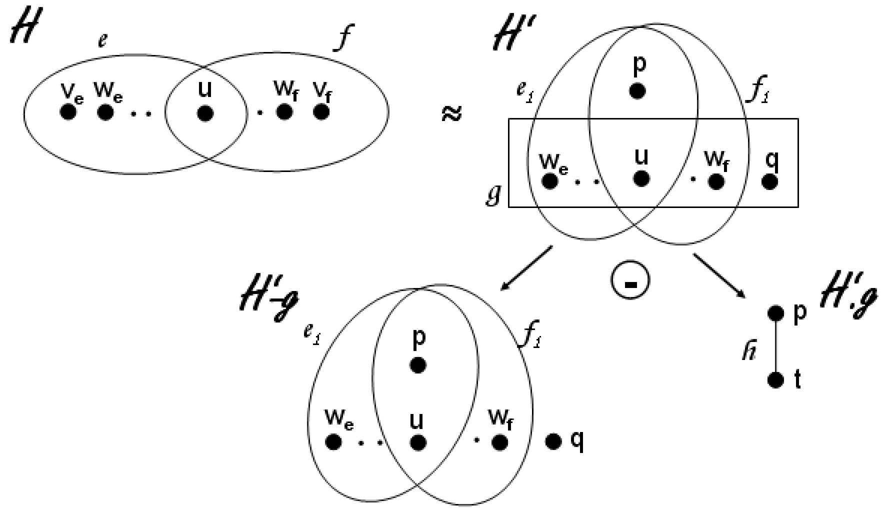

Figure 2

Let $\mathcal{H}' = \mathcal{H}_1 + g$, where $g = e_1 \cup f_e \setminus \{p\} \cup \{q\}$ (see Figure 2). Then $\mathcal{H}' - g \cong K_1 \cup \mathcal{C}_{r,s}$ and $\mathcal{H}'.g \cong K_2$.

We apply (1.10)

$$P(\mathcal{H}', \lambda) = \lambda \left[(\lambda^{r-1} - 1)(\lambda^{s-1} - 1) + (\lambda - 1) \right] - \lambda(\lambda - 1) = \lambda(\lambda^{r-1} - 1)(\lambda^{s-1} - 1)$$

If the hyperstar $\mathcal{H}$ has $m > 2$ edges, we take $\mathcal{H}'' \cong \mathcal{H}' \cup_1 \mathcal{S}$, where $\mathcal{S}$ is the hyperstar defined by the remaining edges. Applying (1.11) to $\mathcal{H}''$ completes the proof of this case.

Case (2): If $m \geq 5$, it remains only to consider the cases $m_2 \geq m-1$.

Let $m = 5$. We can assume that $\mathcal{H}$ is of the form given in Figure 3, because (1.8) is independent of the block arrangement of the hypertree. Note that the edge e might be a 2-edge. Then change $\mathcal{H}$ to $K_1 \uplus \left(K_2 \cup_1 \mathcal{C}_{(3)2,r}\right)$.

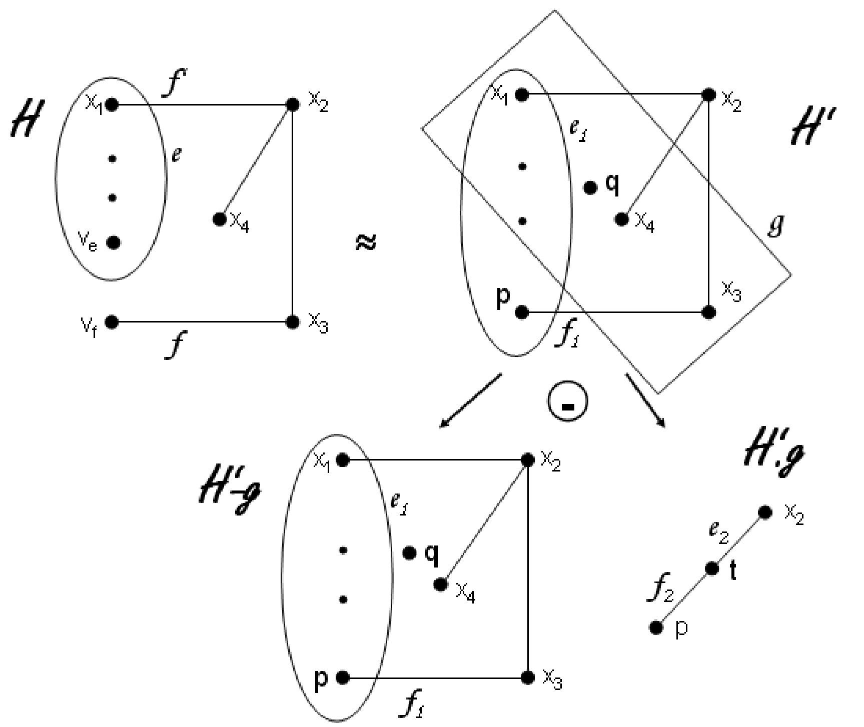

Figure 3

Adding the edge $g = \mathcal{V} \setminus \{p, x_2\}$ yields $\mathcal{H}'$. Deleting the edge g yields $\mathcal{H}' - g \cong K_1 \uplus \left(K_2 \cup_1 \mathcal{C}_{(3)2,r}\right)$. Contracting the edge g yields $\mathcal{H}'.g \cong \mathcal{S}_{2,2}$.

We apply (1.10)

$$\begin{aligned} P(\mathcal{H}', \lambda) &= \lambda(\lambda-1)\left[(\lambda-1)^3(\lambda^{r-1}-1) + \lambda - 1\right] - \lambda(\lambda-1)^2 \\ &= \lambda(\lambda-1)^4(\lambda^{r-1}-1) + \lambda(\lambda-1)^2 - \lambda(\lambda-1)^2 = \lambda(\lambda-1)^4(\lambda^{r-1}-1) \end{aligned}$$

If $m > 5$, take $\mathcal{H}'' \cong \mathcal{H}' \cup_1 \mathcal{S}_{(m-5)2}$. Use of (1.11) completes the proof. □

Corollary 3.1. *The class of trees with order n is chromatically closed if and only if $n \leq 5$.*

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