Mutually Disjoint Steiner Systems $S(5, 8, 24)$ and 5-$(24, 12, 48)$ Designs

Makoto Araya
Department of Computer Science
Shizuoka University
Hamamatsu 432–8011, Japan
araya@inf.shizuoka.ac.jp

Masaaki Harada
Department of Mathematical Sciences
Yamagata University
Yamagata 990–8560, Japan and
PRESTO, Japan Science and Technology Agency
Kawaguchi, Saitama 332–0012, Japan
mharada@sci.kj.yamagata-u.ac.jp

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Abstract

We demonstrate that there are at least 50 mutually disjoint Steiner systems $S(5, 8, 24)$ and there are at least 35 mutually disjoint 5-$(24, 12, 48)$ designs. The latter result provides the existence of a simple 5-$(24, 12, 6m)$ design for $m = 24, 32, 40, 48, 56, 64, 72, 80, 112, 120, 128, 136, 144, 152, 160, 168, 200, 208, 216, 224, 232, 240, 248$ and 256.

1 Introduction

A t-(v, k, λ) *design* D is a pair of a set X of v *points* and a collection $\mathcal{B}$ of k-subsets of X called *blocks* such that every t-subset of X is contained in exactly λ blocks. We often denote the design D by $(X, \mathcal{B})$. A design with no repeated block is called *simple*. All designs in this note are simple. A *Steiner system* $S(t, k, v)$ is a t-(v, k, λ) design with $\lambda = 1$. Two t-(v, k, λ) designs with the same point set are said to be *disjoint* if they have no blocks in common. Two t-(v, k, λ) designs are *isomorphic* if there is a bijection between their point sets that maps the blocks of the first design into the blocks of the second design. An automorphism of a t-(v, k, λ) design D is any isomorphism of the design with itself and the set consisting of all automorphisms of D is called the *automorphism group* $\mathrm{Aut}(D)$ of D.

The well-known Steiner system $S(5, 8, 24)$ and a 5-$(24, 12, 48)$ design are constructed by taking as blocks the supports of codewords of weights 8 and 12 in the extended Golay $[24, 12, 8]$ code, respectively. It is well known that there is a unique Steiner system $S(5, 8, 24)$ up to isomorphism [8], and there is a unique 5-$(24, 12, 48)$ design having even block intersection numbers [7]. By finding permutations on 24 points such that all images of a Steiner system $S(5,8,24)$ under these permutations are mutually disjoint, Kramer and Magliveras [6] found nine mutually disjoint Steiner systems $S(5,8,24)$. Then Araya [1] found 15 mutually disjoint Steiner systems $S(5,8,24)$. Recently Jimbo and Shiromoto [4] have found 22 mutually disjoint Steiner systems $S(5,8,24)$ and two disjoint 5-$(24,12,48)$ designs.

Our computer search has found more mutually disjoint Steiner systems $S(5,8,24)$ and 5-$(24,12,48)$ designs.

Proposition 1. *There are at least* 50 *mutually disjoint Steiner systems* $S(5,8,24)$*. There are at least* 35 *mutually disjoint* 5-$(24,12,48)$ *designs.*

Let $(X,\mathcal{B}_1),(X,\mathcal{B}_2),\ldots,(X,\mathcal{B}_{35})$ be 35 mutually disjoint 5-$(24,12,48)$ designs. Then for any non-empty subset $S \subset \{1,2,\ldots,35\}$, $(X,\cup_{i\in S}\mathcal{B}_i)$ is a simple 5-$(24,12,48|S|)$ design. Hence this provides the existence of the following designs. We remark that if a 5-$(24,12,\lambda)$ design exists then λ is divisible by 6.

Corollary 2. *There is a simple* 5-$(24,12,6m)$ *design for*

$$m = 24,32,40,48,56,64,72,80,112,120,128,136,\\ 144,152,160,168,200,208,216,224,232,240,248 \text{ and } 256.$$

For the above m, 5-$(24,12,6m)$ designs are constructed for the first time (see Table 4.46 in [5]). In addition, we have verified that there is a 5-$(24,12,48s)$ design with a trivial automorphism group for $s = 2,3,\ldots,35$.

2 Preliminaries

To give description of mutually disjoint 5-designs, we first define the extended Golay $[24,12,8]$ code G_{24} as the code with generator matrix

$$\left(\begin{array}{ccc} & & 1\\ I_{12} & A & \vdots\\ & & 1\\ & 1\ \cdots\ 1 & 0 \end{array}\right),$$

where A is the circulant matrix with first row $(1,1,0,1,1,1,0,0,0,1,0)$ and I_{12} is the identity matrix of order 12. The Steiner system $S(5,8,24)$ and the 5-$(24,12,48)$ design are constructed by taking as blocks the supports of codewords of weights 8 and 12 in G_{24}, respectively. We denote these 5-designs by $D_8 = (X_{24},\mathcal{B}_1)$ and $D_{12} = (X_{24},\mathcal{B}_2)$, respectively, where $X_{24} = \{1,2,\ldots,24\}$ (see [4]).

Let σ be a permutation on 24 points X_{24}. For $i = 1$ and 2, $\mathcal{B}_i^\sigma$ denotes $\{B^\sigma \mid B \in \mathcal{B}_i\}$ where B^σ denotes the image of a block B under σ. Similar to [1] and [6], in this note, we find permutations σ such that $(X_{24},\mathcal{B}_i^\sigma)$ are mutually disjoint. It is well known that both automorphism groups $\mathrm{Aut}(D_8)$ and $\mathrm{Aut}(D_{12})$ are the Mathieu group M_{24}. If α and β are in the same right coset for M_{24} in the symmetric group S_{24} on X_{24} then $\mathcal{B}_i^\alpha = \mathcal{B}_i^\beta$. Hence we only consider right coset representatives of M_{24} in S_{24}. One can calculate right coset representatives by using the method in [3] as follows. For disjoint subsets Δ and Δ' of X_{24}, we define a subset of S_{24}:

$$\begin{aligned}&\mathrm{Select}(\Delta, \Delta') = \{id\} \cup \\ &\qquad (\cup_{i=1}^{k} \{(\gamma_1, \delta_1)(\gamma_2, \delta_2) \cdots (\gamma_i, \delta_i) \mid \gamma_1 < \cdots < \gamma_i \in \Delta, \delta_1 < \cdots < \delta_i \in \Delta'\}),\end{aligned}$$

where $k = \min\{|\Delta|, |\Delta'|\}$ and id is the identity permutation. Let $\mathrm{Sym}(\Omega)$ denote the symmetric group on a set Ω. Then it follows from [3, Section 4] that $H^{(7)} U_7 U_6 U_5 \cdots U_1$ is the set of all right coset representatives of M_{24} in S_{24} where

$$\begin{aligned}U_i &= \{id\} \ (i = 1, 2, \ldots, 5), \\ U_6 &= \mathrm{Select}(\{6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 23, 24\}, \{10, 19, 22\}), \\ U_7 &= \mathrm{Select}(\{7, 13, 21\}, \{8, 9, 11, 12, 14, 15, 16, 17, 18, 20, 23, 24\}), \\ H^{(7)} &= \mathrm{Sym}(\{13, 21\}) \times \mathrm{Sym}(\{8, 9, 11, 12, 14, 15, 16, 17, 18, 20, 23, 24\}) \\ &\quad \times \mathrm{Sym}(\{10, 19, 22\}).\end{aligned}$$

We note that $|U_6| = 969$ and $|U_7| = 455$.

3 Description of mutually disjoint 5-designs

3.1 Mutually disjoint Steiner systems $S(5, 8, 24)$

We define the following set of 22 permutations:

$$G_1 = \{\sigma^i \tau^j \mid i = 0, 1, \ldots, 10, j = 0, 1\}$$

where

$$\begin{aligned}\sigma &= (13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23), \\ \tau &= (1, 13)(2, 14)(3, 15)(4, 16)(5, 17)(6, 18)(7, 19)(8, 20)(9, 21)(10, 22)(11, 23).\end{aligned}$$

Recently Jimbo and Shiromoto [4] showed that the set $\{(X_{24}, \mathcal{B}_1^\sigma) \mid \sigma \in G_1\}$ gives 22 mutually disjoint Steiner systems $S(5, 8, 24)$.

Let H_1 be the set of all right coset representatives α of M_{24} in S_{24} satisfying the condition that $\{(X_{24}, \mathcal{B}_1^\sigma) \mid \sigma \in G_1 \cup \{\alpha\}\}$ gives 23 mutually disjoint Steiner systems $S(5, 8, 24)$. Then we define the simple undirected graph Γ_1, whose set of vertices is the set $G_1 \cup H_1$ and two vertices α and β are adjacent if $\mathcal{B}_1^\alpha$ and $\mathcal{B}_1^\beta$ are disjoint. Clearly a t-clique in Γ_1 gives t mutually disjoint Steiner systems $S(5, 8, 24)$. It seems infeasible to construct the graph Γ_1 by computer. However, we found a subgraph Γ_1' of Γ_1 such that Γ_1' contains a 50-clique by considering subsets of the set $H^{(7)} U_7 U_6 U_5 \cdots U_1$ of all right coset

Table 1: Permutations P_1

Permutations
$\alpha_1 = (6, 19, 10, 22)(7, 14, 15)(8, 17, 24, 11, 23, 13, 12)(9, 16, 21)$
$\alpha_2 = (7, 15, 24, 13, 23, 12, 10, 9, 20, 8, 16, 14, 21, 22, 19, 11)$
$\alpha_3 = (7, 19, 24, 11, 16, 12, 15, 23, 8, 14, 10, 22, 9)(13, 21, 17)$
$\alpha_4 = (7, 22, 11, 15, 23, 8, 13, 16, 12, 14, 9, 10, 19)(18, 21)$
$\alpha_5 = (6, 19, 11, 7, 22, 23, 10)(8, 16, 21, 14, 18, 13, 17, 9, 12, 15)$
$\alpha_6 = (7, 8, 22, 23, 19, 15, 21, 17, 14)(9, 10, 24, 11, 12)$
$\alpha_7 = (7, 17, 15, 8, 9, 20, 14)(10, 23, 22, 11, 24, 16, 21, 12)(18, 19)$
$\alpha_8 = (8, 20, 14, 23, 13)(9, 12, 11, 17)(10, 24, 22, 16, 15, 19)$
$\alpha_9 = (7, 17, 24, 9)(8, 15, 20, 22, 23, 19, 12, 18, 11, 21, 14, 13, 16)$
$\alpha_{10} = (7, 15, 17, 19, 14, 21, 13, 23, 12, 10, 9, 16, 24, 8)(18, 22)$
$\alpha_{11} = (6, 19, 7, 20, 24, 21, 16, 17, 15, 9, 12, 14, 23, 13, 11, 8, 22, 18, 10)$
$\alpha_{12} = (7, 18, 24, 9, 15, 8, 11, 10, 14, 22, 16, 12)(13, 19, 17, 21, 23)$
$\alpha_{13} = (7, 11, 15, 12, 13, 21, 14, 10, 19, 18, 22, 8, 20, 9)(16, 24)$
$\alpha_{14} = (8, 9, 12, 24, 11, 16, 14, 22, 23, 19, 15, 10, 20, 21)(13, 17)$
$\alpha_{15} = (7, 10, 23, 8)(9, 15, 18, 21, 13)(11, 14, 12, 19)(16, 24, 22)$
$\alpha_{16} = (8, 13, 17, 22, 9, 18)(11, 24, 21, 16, 12, 19)(14, 15, 20)$
$\alpha_{17} = (7, 14, 11)(8, 20, 21, 24, 22, 16, 10, 9)(12, 23)(13, 18, 19)(15, 17)$
$\alpha_{18} = (7, 19, 20, 22, 14, 13, 11, 24, 8, 16, 9, 15, 23, 10)$
$\alpha_{19} = (7, 16, 19, 9, 18, 8, 20, 24, 14, 12, 22, 21, 17, 13, 11, 15)$
$\alpha_{20} = (7, 15, 17, 10, 18, 9, 19, 24, 16)(8, 14, 20, 21, 11)$
$\alpha_{21} = (7, 14, 18, 13, 12, 22, 17, 9, 8, 10, 11)(15, 19, 23, 21, 24, 16, 20)$
$\alpha_{22} = (6, 19, 7, 22, 17, 10)(8, 9, 24, 14, 11, 16, 12, 15, 13)(20, 21)$
$\alpha_{23} = (8, 14, 21, 10, 15, 22, 19, 17, 12, 18, 11)(9, 20)(16, 24)$
$\alpha_{24} = (7, 23, 15, 21, 24, 13, 9)(8, 10, 16, 22)(11, 18, 14)(12, 17)$
$\alpha_{25} = (7, 17, 22, 23, 13, 24, 16, 11, 9, 15, 20, 14, 19, 12, 21, 10, 8)$
$\alpha_{26} = (7, 23, 9, 17, 12, 10, 15, 16, 14, 8, 19, 22, 21, 18, 13, 11)$
$\alpha_{27} = (7, 18, 12, 9, 17, 21, 11, 23, 13, 10, 24, 16, 15)(8, 19)(20, 22)$
$\alpha_{28} = (7, 9, 15, 22, 10, 11)(8, 19, 13, 24, 21, 16, 12, 20, 14)$

representatives. This computation for finding cliques was performed using Magma [2]. The set $\{(X_{24}, \mathcal{B}_1^{\sigma}) \mid \sigma \in G_1 \cup P_1\}$ gives corresponding 50 mutually disjoint Steiner systems $S(5, 8, 24)$ where P_1 is listed in Table 1. Moreover we have verified by Magma [2] that the simple 5-$(24, 8, s + 22)$ design $(X_{24}, \cup_{\sigma \in Y} \mathcal{B}_1^{\sigma})$ has a trivial automorphism group where $Y = G_1 \cup \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ for $s = 1, 2, \dots, 28$.

3.2 Mutually disjoint 5-(24, 12, 48) designs

For the 5-$(24, 12, 48)$ design D_{12}, by a back-tracking algorithm, we found the set $G_2 = \{\beta_1, \beta_2, \dots, \beta_{20}\}$ of 20 permutations on 24 points satisfying the condition that $\{(X_{24}, \mathcal{B}_2^{\sigma}) \mid \sigma \in G_2\}$ gives 20 mutually disjoint 5-$(24, 12, 48)$ designs where β_1 is the identity permutation *id*. This was done by considering some subsets of the set $H^{(7)} U_7 U_6 U_5 \cdots U_1$ of all right coset representatives of M_{24} in S_{24}.

Let H_2 be the set of all right coset representatives β of M_{24} in S_{24} satisfying the condition that $\{(X_{24}, \mathcal{B}_2^\sigma) \mid \sigma \in G_2 \cup \{\beta\}\}$ gives 21 mutually disjoint 5-(24, 12, 48) designs. Similar to Γ_1, we define the simple undirected graph Γ_2 where $G_2 \cup H_2$ is the set of vertices. In this case, we found a subgraph Γ_2' of Γ_2 such that Γ_2' contains a 35-clique by considering subsets of the set $H^{(7)}U_7U_6U_5 \cdots U_1$. We list in Table 2 the set P_2 of 35 permutations corresponding to the 35-clique in Γ_2', where the set $\{(X_{24}, \mathcal{B}_2^\sigma) \mid \sigma \in P_2\}$ gives 35 mutually disjoint 5-(24, 12, 48) designs. As described in Section 1, a simple 5-(24, 12, 48s) design can be constructed from the 35 mutually disjoint 5-(24, 12, 48) designs for $s = 2, 3, \ldots, 35$. Moreover we have verified by MAGMA [2] that the 5-(24, 12, 48s) design $(X_{24}, \cup_{i=1}^{s} \mathcal{B}_2^{\beta_i})$ has a trivial automorphism group for $s = 2, 3, \ldots, 35$.

Table 2: Permutations P_2

Permutations
$\beta_1 = id$ (the identity permutation)
$\beta_2 = (7, 19, 21, 12, 22, 16, 10, 8)(11, 13)$
$\beta_3 = (7, 8, 22, 11, 21, 10)(9, 13)(17, 19)$
$\beta_4 = (6, 22, 20, 10)(7, 8)(9, 13)(11, 19)(16, 21)$
$\beta_5 = (7, 8)(10, 11, 13, 22, 17, 21)(18, 19)$
$\beta_6 = (7, 8, 22, 13, 14, 10)(9, 19)(15, 21)$
$\beta_7 = (7, 9, 22, 23, 10)(11, 13)(16, 21)(19, 20)$
$\beta_8 = (7, 19, 18, 22, 24, 10)(9, 13)(14, 21)$
$\beta_9 = (10, 12, 19, 20, 13, 22, 15)(21, 24)$
$\beta_{10} = (6, 22, 16, 19, 15, 10)(13, 14)(21, 24)$
$\beta_{11} = (9, 22, 24, 19, 12, 10)(13, 16)(20, 21)$
$\beta_{12} = (10, 17, 13, 15, 21, 19, 23, 22)$
$\beta_{13} = (7, 11, 19, 14, 22, 24, 13, 15, 21, 10)$
$\beta_{14} = (6, 10)(11, 22, 21, 23, 13, 12, 19)$
$\beta_{15} = (7, 9, 24, 8)(10, 16, 21, 17, 19, 13, 22, 23)$
$\beta_{16} = (7, 15, 19)(8, 22, 21, 23, 13, 20, 10, 9, 24)$
$\beta_{17} = (6, 22, 8, 7, 9, 24, 19, 16, 10)(13, 14, 21)$
$\beta_{18} = (8, 11, 22, 20, 19, 21, 10, 24)(13, 17)$
$\beta_{19} = (6, 22, 20, 10)(7, 15)(8, 18, 13, 16, 21, 24)(9, 19)$
$\beta_{20} = (7, 19, 9, 22, 13, 17, 21, 8, 23, 24, 10)$
$\beta_{21} = (7, 22, 20, 19, 17, 10)(8, 12, 24, 13)(9, 11, 21)$
$\beta_{22} = (7, 22, 9, 14)(8, 18, 24, 19, 20, 10)(13, 17, 21)$
$\beta_{23} = (7, 17, 24, 22, 10, 9, 21, 12, 13, 20, 8)$
$\beta_{24} = (7, 8, 20, 22, 19, 24, 14, 12, 13, 11, 9)(10, 15)(18, 21)$
$\beta_{25} = (7, 23, 10, 12)(8, 14, 13, 22, 11, 9, 15, 17, 24)(19, 20, 21)$
$\beta_{26} = (7, 20, 19, 8, 15, 9)(10, 12, 17, 13, 22)(11, 23, 14, 16, 21)$
$\beta_{27} = (7, 15, 12)(8, 10, 17, 9, 24, 16, 14, 11, 22)$
$\beta_{28} = (7, 12, 18, 24, 13, 16, 21, 9, 20, 8, 19)(10, 11)(14, 15, 23, 22)$
$\beta_{29} = (7, 17, 8, 12)(9, 11, 16, 19, 15, 14)(21, 22, 23)$
$\beta_{30} = (6, 10)(7, 16, 14, 13, 21, 18, 19, 9, 8)(11, 12, 20, 22)(15, 24)$
$\beta_{31} = (7, 10, 22, 17, 19, 8)(11, 16, 12, 13, 20)(21, 24)$
$\beta_{32} = (7, 11, 22, 21, 8)(9, 18, 10, 19, 14, 16, 12, 24)(13, 15)$
$\beta_{33} = (8, 13, 21)(9, 12, 20, 11, 22, 18, 19, 24, 14, 10, 16)$
$\beta_{34} = (7, 12, 14, 10, 22, 24, 15, 16, 21, 9, 8, 23, 19)$
$\beta_{35} = (8, 14, 24, 10, 9, 16, 19, 11, 17, 12)(13, 22)(15, 21)$

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