Another abstraction of the Erdős-Szekeres Happy End Theorem

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Abstract

The Happy End Theorem of Erdős and Szekeres asserts that for every integer n greater than two there is an integer N such that every set of N points in general position in the plane includes the n vertices of a convex n-gon. We generalize this theorem in the framework of certain simple structures, which we call "happy end spaces".

In the winter of 1932/33, Esther Klein observed that

> from any set of five points in the plane of which no three lie on the same line it is always possible to select four points that are vertices of a convex polygon.

When she shared this news with a circle of her friends in Budapest, the following prospect of generalizing it emerged:

> Can we find for each integer n greater than two an integer $N(n)$ such that from any set of $N(n)$ points in the plane of which no three lie on the same line it is always possible to select n points that are vertices of a convex polygon?

Endre Makai proved that $N(5) = 9$ works here. A few weeks later, George Szekeres proved the existence of $N(n)$ for all n. His argument produced very large upper bounds for $N(n)$: for instance, it gave $N(5) \leqslant 2^{10000}$. Soon afterwards, Paul Erdős came up with a different proof, which led to much smaller values of $N(n)$:

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($\star$) *From any set of* $\binom{2n-4}{n-2}+1$ *points in the plane of which no three lie on the same line it is always possible to select* n *points that are vertices of a convex polygon.*

In December 1934, Erdős and Szekeres submitted for publication a manuscript containing both proofs; the paper [6] appeared in 1935. Esther Klein and George Szekeres got married on June 13, 1937 and Paul Erdős began referring to ($\star$) as *The Happy End Theorem.*

Abstractions of this theorem have been studied by Korte and Lovász [7] and by Morris and Soltan [8]. These deal with abstract convexity spaces satisfying the anti-exchange property and the simplex partition property, and having a finite Caratheodory number $c \geqslant 3$. See [8], [7] for the precise definitions and more details. A somewhat less abstract version is considered in [10], based on the order type of a configuration of points in the plane. This enables the authors to show, using an exhaustive computer search, that any configuration of 17 points in general position in the plane contains a convex 6-gon. Here we propose another abstraction: A *happy-end space* is a set S along with a function

$$f : S \times S \times S \to \{+, -\}$$

such that

$$f(x, y, z) = f(y, x, z) \quad \text{for all } x, y, z;$$

in this space, a subset C of S is called *convex* if, and only if, for every subset B of C such that $|B| > 2$ and for every point x of B, there is another y in B such that $f(x, y, z)$ is constant on $B - \{x, y\}$. (In this definition, the only values of $f(x, y, z)$ that matter are those where x, y, z are all distinct.) Every set S of points in the plane such that no three lie on the same line and no two have the same first coordinate defines a happy-end space by

$$f(x, y, z) = \begin{cases} +1 & \text{if point } z \text{ lies above the line } \overline{xy}, \\ -1 & \text{if point } z \text{ lies below the line } \overline{xy}; \end{cases}$$

in this space, a set is convex if and only if it consists of vertices of a convex polygon.

In this abstract setting, the Happy End Theorem generalizes as follows:

Theorem 1. *For every positive integer* n *there is a positive integer* N *such that every happy-end space on* N *points contains a convex set of* n *points.*

Proof. Following Erdős and Rado [4, 5], we let $a \to (b)^k_r$ denote the statement that

> whenever the k-point subsets of an a-point set are coloured by r colours, there is a b-point set whose k-point subsets are all of the same colour.

Frank Ramsey [9] proved that

> for every choice of positive integers b, k, r, there is an integer a such that $a \to (b)^k_r$.

We claim that if N satisfies $N \to (n)^3_8$ then it also satisfies the conclusion of Theorem 1.

To justify this claim, consider an arbitrary happy-end space on N points, impose a linear order $\prec$ on its underlying set S and, for each set T of three points in S, write

$$g(T) = (f(v,w,u), f(u,w,v), f(u,v,w)), \text{ where } T = \{u,v,w\} \text{ and } u \prec v \prec w.$$

Since $N \to (n)^3_8$, there are a set C of n points in S and a vector (s_1, s_2, s_3) in $\{+1,-1\}^3$ such that $g(T) = (s_1, s_2, s_3)$ for every three-point subset T of C; this means that

$$f(x,y,z) = \begin{cases} s_1 & \text{if } z \prec x \prec y \text{ or } z \prec y \prec x, \\ s_2 & \text{if } x \prec z \prec y \text{ or } y \prec z \prec x, \\ s_3 & \text{if } x \prec y \prec z \text{ or } y \prec x \prec z. \end{cases}$$

We are going to show that C is convex. For this purpose, consider an arbitrary subset B of C such that $|B| > 2$ and enumerate its points as $u_1, u_2, \ldots, u_k$ in such a way that $u_1 \prec u_2 \prec \ldots \prec u_k$. Given any u_i in B, we will find another u_j in B such that $f(u_i, u_j, z)$ is constant on $B - \{u_i, u_j\}$.

Case 1: $s_1 = s_2$. In this case, we may set $j = k$ unless $i = k$, in which case any j smaller than k will do. It is not difficult to check that with this choice of j, $f(u_i, u_j, u_r) = s_1$ $(= s_2)$ for all $u_r \in B - \{u_i, u_j\}$.

Case 2: $s_2 = s_3$. In this case, we may set $j = 1$ unless $i = 1$, in which case any j greater than 1 will do. With this choice, $f(u_i, u_j, u_r) = s_2$ $(= s_3)$ for all $u_r \in B - \{u_i, u_j\}$.

Case 3: $s_1 \neq s_2$ and $s_2 \neq s_3$. In this case, $s_1 = s_3$; we may set $j = i + 1$ unless $i = k$ and we may set $j = i - 1$ unless $i = 1$. This ensures that $f(u_i, u_j, u_r) = s_1$ $(= s_3)$ for all $u_r \in B - \{u_i, u_j\}$. □

We have found the proof of Theorem 1 during the 7th International Colloquium on Graph Theory in Hyères in September 2005. Another proof of the theorem was found, independently, by Pierre Duchet during the same conference [3]. His proof also uses Ramsey's theorem, but it gives a far larger upper bound for N.

Let $N_0(n)$ denote the smallest integer N such that from any set of N points in the plane of which no three lie on the same line it is always possible to select n points that are vertices of a convex polygon. As shown in [6], $N_0(n) \geqslant 2^{n-2} + 1$ for every $n \geqslant 3$, and the authors of [6] conjectured that this is tight for all n.

The conjecture is open for all $n > 6$. Regarding upper bounds, the Happy End Theorem asserts that

$$N_0(n) \leqslant \binom{2n-4}{n-2} + 1;$$

in 1998, Chung and Graham [2] improved this upper bound to

$$N_0(n) \leqslant \binom{2n-4}{n-2};$$

subsequently, Tóth and Valtr [11] reduced it to roughly its half,

$$N_0(n) \leqslant \binom{2n-5}{n-2} + 1.$$

Chung and Graham [1] have offered \$100 for the first proof that $N_0(n) = O(c^n)$ for some constant c smaller than 4.

Our second theorem shows that even the original Erdős-Szekeres upper bound on $N_0(n)$ grows much too slowly with n to be applicable in the more general setting of happy-end spaces.

Theorem 2. *For every positive integer n there is a happy-end space on $2^{\Omega(n^2)}$ points that contains no convex set of n points.*

It is worth noting that the upper bound that follows from the proof of Theorem 1 and the known bounds for hypergraph Ramsey numbers is doubly exponential in n, hence there is still a substantial gap between the upper and lower bounds proved here.

Proof. Given a positive integer n, set

$$N = \left\lfloor \left(\frac{\sqrt{e}}{2\sqrt{n}} \cdot 2^{n/6} \right)^n \right\rfloor .$$

We will prove that in a happy-end space taken uniformly at random from all possible happy-end spaces on N points, the expected number of convex sets of size n is less than 1. For this purpose, we will bound from above the probability p_n that a prescribed set of size n in this happy-end space is convex.

If a set C of size n is convex, then, for each of its points x_1,

(i) the $n-1$ points of $C - \{x_1\}$ can be enumerated as $x_2, x_3, \ldots, x_n$ in such a way that, for each $j = 2, 3, \ldots, n-2$, the $n-j$ values of $f(x_1, x_j, x_k)$ with $k = j+1, j+2, \ldots n$ are identical,

(ii) $C - \{x_1\}$ is convex.

The probability of (i) is at most $(n-1)! \prod_{j=2}^{n-2} (\frac{1}{2})^{n-j-1}$; since events (i) and (ii) are independent, we conclude that

$$p_n \leqslant \left((n-1)! \prod_{j=2}^{n-2} \left(\frac{1}{2} \right)^{n-j-1} \right) p_{n-1} = \left((n-1)! \left(\frac{1}{2} \right)^{(n-2)(n-3)/2} \right) p_{n-1}.$$

Since $p_3 = 1$ and $m! < (m/e)^m$ and $(n-1)(n-2)(n-3) > n^3 - 6n^2$, it follows that

$$p_n \leqslant \left(\prod_{m=3}^{n-1} m! \right) \left(\frac{1}{2} \right)^{(n-1)(n-2)(n-3)/6} < \left(\frac{2\sqrt{n}}{\sqrt{e}} \left(\frac{1}{2} \right)^{n/6} \right)^{n^2} \leqslant \frac{1}{N^n}.$$

Finally, we note that the expected number of convex sets of size n equals $\binom{N}{n} p_n$ and that $\binom{N}{n} = o(N^n)$. □

Let $N_1(n)$ denote the smallest integer N that works in Theorem 1: in this notation, Theorem 2 states that $N_1(n) = 2^{\Omega(n^2)}$. By definition, $N_0(n) \leqslant N_1(n)$ for all n; trivially, $N_0(3) = N_1(3) = 3$; Esther Klein's observation was that $N_0(4) = 5$. In closing, we note that

$$8 \leqslant N_1(4) \leqslant 11.$$

To see that $N_1(4) \leqslant 11$, consider an arbitrary happy-end space (S, f) such that $|S| = 11$. Choose two distinct points u, v of S. By the pigeon-hole principle, there is a 5-point subset R of $S - \{u, v\}$ such that $f(u, v, t)$ is constant as t ranges through R. Choose a point w in $S - (\{u, v\} \cup R)$. By the pigeon-hole principle again, there are first a 3-point subset Q of R such that $f(u, w, t)$ is constant as t ranges through Q, then distinct points x, y in Q such that $f(v, w, x) = f(v, w, y)$, and finally distinct points z, z' in $\{u, v, w\}$ such that $f(x, y, z) = f(x, y, z')$. It is now not too difficult to check that the set $\{x, y, z, z'\}$ is convex. Indeed, by the construction above $f(u, v, x) = f(u, v, y)$, $f(u, w, x) = f(u, w, y)$ and $f(v, w, x) = f(v, w, y)$. By symmetry, we may assume without loss of generality that z, z' above are u, v, and thus $f(x, y, u) = f(x, y, v)$. To show that the set $S = \{x, y, u, v\}$ is convex we have to check that for every element $a \in S$ there is another element $b \in S$, so that $f(a, b, c)$ is constant on the two elements $c \in S - \{a, b\}$. Indeed, for $a = x$ take $b = y$ and vice versa, and for $a = u$, take $b = v$ and vice versa, as needed.

To see that $N_1(4) \geqslant 8$, consider the happy-end space $(\{0, 1, \ldots, 6\}, f)$ with the values of $f(0, y, z)$ given by the table

	$z=1$	$z=2$	$z=3$	$z=4$	$z=5$	$z=6$
$f(0,1,z) =$		$+$	$-$	$-$	$+$	$+$
$f(0,2,z) =$	$-$		$+$	$+$	$+$	$-$
$f(0,3,z) =$	$-$	$+$		$+$	$-$	$+$
$f(0,4,z) =$	$+$	$-$	$+$		$-$	$+$
$f(0,5,z) =$	$+$	$+$	$+$	$-$		$-$
$f(0,6,z) =$	$+$	$-$	$-$	$+$	$+$	

and the remaining values of $f(x, y, z)$ determined by the circular symmetry $f(x, y, z) = f(x+1, y+1, z+1)$ with the increment modulo 7. (In particular, $f(4, 0, z) = f(0, 3, z+3)$, $f(5, 0, z) = f(0, 2, z+2)$, $f(6, 0, z) = f(0, 1, z+1)$, and so the last three rows of the table are redundant. We include them for the ease of checking the argument that follows.) A four-point set R is not convex if and only if it has an element x such that, for each way of enumerating the remaing three elements of R as y, z, z', we have $f(x, y, z) \neq f(x, y, z')$. Let us call such an x *an interior point of* R. In our happy-end space, point 0 is the interior point of the five sets

$$\{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{0, 1, 4, 5\}, \{0, 2, 4, 6\}, \{0, 3, 5, 6\};$$

the circular symmetry guarantees that each of $1, 2, \ldots, 6$ is the interior point of an additional five four-points sets, and so each of the 35 four-point subsets of $\{0, 1, \ldots, 6\}$ has an interior point.

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