Combinatorial proof of a curious q-binomial coefficient identity

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Abstract

Using the Algorithm Z developed by Zeilberger, we give a combinatorial proof of the following q-binomial coefficient identity

$$\sum_{k=0}^{m}(-1)^{m-k}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n+k\\a\end{bmatrix}(-xq^a;q)_{n+k-a}q^{\binom{k+1}{2}-mk+\binom{a}{2}}$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m+k\\a\end{bmatrix}x^{m+k-a}q^{mn+\binom{k}{2}},$$

which was obtained by Hou and Zeng [European J. Combin. 28 (2007), 214–227].

1 Introduction

Binomial coefficient identities continue to attract the interests of combinatorists and computer scientists. As shown in [7, p. 218], differentiating the simple identity

$$\sum_{k\leqslant m}\binom{m+r}{k}x^k y^{m-k}=\sum_{k\leqslant m}\binom{-r}{k}(-x)^k(x+y)^{m-k}$$

n times with respect to y, and then replacing k by $m-n-k$, we immediately get the *curious* binomial coefficient identity:

$$\sum_{k\geqslant 0}\binom{m+r}{m-n-k}\binom{n+k}{n}x^{m-n-k}y^k=\sum_{k\geqslant 0}\binom{-r}{m-n-k}\binom{n+k}{n}(-x)^{m-n-k}(x+y)^k. \tag{1}$$

Identity (1) has been rediscovered by several authors in the last years. Indeed, Simons [13] reproved the following special case of (1):

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k. \tag{2}$$

Several different proofs of (2) were soon given by Hirschhorn [8], Chapman [4], Prodinger [11], and Wang and Sun [15]. As a key lemma in [14, Lemma 3.1], Sun proved the following identity:

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k}{a}(1+x)^{n+k-a}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{a}x^{m+k-a}. \tag{3}$$

Finally, by using the method of Prodinger [11], Munarini [10] generalized (2) to

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(1+x)^k=\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k. \tag{4}$$

The identities (1), (3) and (4) are obviously equivalent. Recently, an elegant combinatorial proof of (4) was given by Shattuck [12], and a little complicated combinatorial proof of (2) was provided by Chen and Pang [5].

On the other hand, as a q-analogue of Sun's identity (3), Hou and Zeng [9, (20)] proved the following q-identity:

$$\begin{aligned}&\sum_{k=0}^{m}(-1)^{m-k}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n+k\\r\end{bmatrix}(-xq^r;q)_{n+k-r}q^{\binom{k+1}{2}-mk+\binom{r}{2}}\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m+k\\r\end{bmatrix}x^{m+k-r}q^{mn+\binom{k}{2}},\end{aligned} \tag{5}$$

where the *q-shifted factorial* is defined by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ and the *q-binomial coefficient* $\begin{bmatrix}\alpha\\k\end{bmatrix}$ is defined as

$$\begin{bmatrix}\alpha\\k\end{bmatrix}=\begin{cases}\dfrac{(q^{\alpha-k+1};q)_k}{(q;q)_k}, & \text{if } k\geqslant 0,\\ 0, & \text{if } k<0.\end{cases}$$

Note that, rewriting (5) as

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}\beta-\alpha+n\\n-k\end{bmatrix}\begin{bmatrix}\beta+k\\k\end{bmatrix}q^{\binom{n-k}{2}-\binom{n}{2}}(-xq^{\beta};q)_k\\&=\sum_{k=0}^{n}\begin{bmatrix}\alpha\\n-k\end{bmatrix}\begin{bmatrix}\beta+k\\k\end{bmatrix}q^{\binom{n-k+1}{2}-(n-k)\alpha+n\beta}x^k,\end{aligned}$$

we obtain a q-analogue of (4).

In this paper, motivated by the two aforementioned combinatorial proofs for $q=1$, we propose a combinatorial proof of (5) within the framework of partition theory by applying an algorithm due to Zeilberger [3].

2 The interpretation of (5) in partitions

A *partition* λ is defined as a finite sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ in decreasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$. Each nonzero λ_i is called a part of λ. The number and sum of parts of λ are denoted by $\ell(\lambda)$ and $|\lambda|$, respectively.

Recall [1, Theorem 3.1] that

$$\begin{bmatrix} n+k \\ r \end{bmatrix} = \sum_{\substack{\ell(\lambda)\leqslant r \\ \lambda_1 \leqslant n+k-r}} q^{|\lambda|}. \tag{6}$$

Therefore

$$\begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k+1}{2}-mk} = q^{\binom{k+1}{2}-mk} \sum_{\substack{\ell(\lambda)\leqslant k \\ \lambda_1 \leqslant m-k}} q^{|\lambda|} = \sum_{m-1\geqslant \mu_1 > \cdots > \mu_k \geqslant 0} q^{-|\mu|},$$

where $\mu_i = m - i - \lambda_{k-i+1}$ $(1 \leqslant i \leqslant k)$. Moreover, the coefficient of x^s in $(-xq^r; q)_{n+k-r}$ is equal to

$$\sum_{n+k-1\geqslant \lambda_1 > \cdots > \lambda_s \geqslant r} q^{|\lambda|} = q^{\binom{s}{2}+rs} \sum_{\substack{\ell(\nu)\leqslant s \\ \nu_1 \leqslant n+k-r-s}} q^{|\nu|},$$

where $\nu_i = \lambda_i - r - s + i$ $(0 \leqslant i \leqslant s)$. It follows that the coefficient of x^s in the left-hand side of (5) is given by

$$q^{\binom{r}{2}+\binom{s}{2}+rs} \sum_{k=0}^{m} (-1)^{m-k} \sum_{m-1\geqslant \mu_1 > \cdots > \mu_k \geqslant 0} \sum_{\substack{\ell(\lambda)\leqslant r \\ \lambda_1 \leqslant n+k-r}} \sum_{\substack{\ell(\nu)\leqslant s \\ \nu_1 \leqslant n+k-r-s}} q^{|\lambda|+|\nu|-|\mu|}. \tag{7}$$

Now we need to prove the following relation

$$\sum_{\substack{\ell(\lambda)\leqslant r \\ \lambda_1 \leqslant n+k-r}} \sum_{\substack{\ell(\nu)\leqslant s \\ \nu_1 \leqslant n+k-r-s}} q^{|\lambda|+|\nu|} = \sum_{\substack{\ell(\lambda)\leqslant r+s \\ \lambda_1 \leqslant n+k-r-s}} \sum_{\substack{\ell(\nu)\leqslant r \\ \nu_1 \leqslant s}} q^{|\lambda|+|\nu|}. \tag{8}$$

In view of (6), the last identity is equivalent to

$$\begin{bmatrix} n+k \\ r \end{bmatrix} \begin{bmatrix} n+k-r \\ s \end{bmatrix} = \begin{bmatrix} n+k \\ r+s \end{bmatrix} \begin{bmatrix} r+s \\ r \end{bmatrix}. \tag{9}$$

Zeilberger [3] gave a bijective proof of (9) using the partition interpretation (8). This bijection is then called *Algorithm Z* (see also [2]). For reader's convenience, we include a brief description of this algorithm. Note that Fu [6] also used this algorithm in her recent study of the Lebesgue identity.

3 Algorithm Z

For simplicity, performing parameter replacements $n+k-r-s \to t$ and $\nu \to \mu$, we can rewrite (8) as follows:

$$\sum_{\substack{\ell(\lambda)\leqslant r\\ \lambda_1\leqslant s+t}} \sum_{\substack{\ell(\mu)\leqslant s\\ \mu_1\leqslant t}} q^{|\lambda|+|\mu|} = \sum_{\substack{\ell(\lambda)\leqslant r+s\\ \lambda_1\leqslant t}} \sum_{\substack{\ell(\mu)\leqslant r\\ \mu_1\leqslant s}} q^{|\lambda|+|\mu|}.$$

The Algorithm Z constructs a bijection between pairs of partitions (λ,μ) and (λ',μ') with zeros permitted, satisfying

(i) λ has $r+s$ parts, all $\leqslant t$,
(ii) μ has r parts, all $\leqslant s$,
(iii) λ' has s parts, all $\leqslant t$,
(iv) μ' has r parts, all $\leqslant s+t$,
(v) $|\lambda|+|\mu| = |\lambda'|+|\mu'|$.

Here is a brief description of this algorithm. Let $\lambda = (\lambda_1,\dots,\lambda_{r+s})$ and $\mu = (\mu_1,\dots,\mu_r)$ be two partitions with $\lambda_1 \leqslant t$ and $\mu_1 \leqslant s$. For $1 \leqslant i \leqslant r$, place μ_i under $\lambda_{s-\mu_i+i}$. Note that $1 \leqslant s-\mu_i+i \leqslant r+s$ and if $i \neq j$ then $s-\mu_i+i \neq s-\mu_j+j$. The parts from λ with nothing below form a new partition λ'. It is clear that λ' has s parts, all less than or equal to t. Each of the other parts from λ is added to the parts from μ which lies below it, yielding a part in μ'. Note that μ' has r parts, all less than or equal to $s+t$.

For instance, let $r=6$, $s=4$, $t=10$, and let $\lambda = (9,8,7,7,6,6,6,4,2,0)$ and $\mu = (4,2,2,1,1,0)$, then $\lambda' = (8,7,6,2)$ and $\mu' = (13,9,8,7,5,0)$.

		8	7			6			2		λ'
λ	9	8	7	7	6	6	6	4	2	0	
μ	4			2	2		1	1		0	
	13			9	8		7	5		0	μ'

The algorithm is clearly reversible. Let $\lambda' = (a_1,\dots,a_s)$ and $\mu' = (b_1,\dots,b_r)$. If $b_1 \leqslant a_s$, then $\lambda = (a_1,\dots,a_s,b_1,\dots,b_r)$ and $\mu = (0,\dots,0)$. Otherwise, for any $b_k > a_s$, we take the smallest $i_k \geqslant 1$ such that $b_k - i_k \leqslant a_{s-i_k}$ $(a_0 = +\infty)$ and $b_k - i_k$ becomes a part of λ and i_k becomes a positive part of μ.

4 The proof of (5)

By the inverse of Algorithm Z, the relation (8) holds and therefore (7) may be rewritten as

$$q^{\binom{r+s}{2}} \sum_{k=0}^{m} (-1)^{m-k} \sum_{m-1\geqslant\mu_1>\cdots>\mu_k\geqslant 0} \sum_{\substack{\ell(\lambda)\leqslant r+s\\ \lambda_1\leqslant n+k-r-s}} \sum_{\substack{\ell(\nu)\leqslant r\\ \nu_1\leqslant s}} q^{|\lambda|+|\nu|-|\mu|}. \tag{10}$$

For any pair $(\mu;\lambda) = (\mu_1,\ldots,\mu_k;\lambda_1,\ldots,\lambda_{r+s})$ such that $m-1 \geqslant \mu_1 > \cdots > \mu_k \geqslant 0$ and $n+k-r-s \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{r+s} \geqslant 0$, we construct a new pair $(\mu';\lambda')$ as follows:

- If $\mu_k > 0$ or $\mu = \emptyset$, then $\mu' = (\mu_1,\ldots,\mu_k,0)$ and $\lambda' = \lambda$;
- If $\mu_k = 0$ and $\lambda_1 < n+k-r-s$, then $\mu' = (\mu_1,\ldots,\mu_{k-1})$ and $\lambda' = \lambda$;
- If $\mu_k = 0$ and $\lambda_1 = n+k-r-s$, we choose the largest i and j such that $\mu_{k+1-i} = i-1$ and $\lambda_j = \lambda_1$. If $i \leqslant j$ and $i \leqslant m-1$, then let

  $$\mu' = (\mu_1,\ldots,\mu_{k-i},i,\mu_{k+1-i},\ldots,\mu_k) \quad \text{and} \quad \lambda' = (\lambda_1+1,\ldots,\lambda_i+1,\lambda_{i+1},\ldots,\lambda_{r+s}).$$

  If $i > j$, then let

  $$\mu' = (\mu_1,\ldots,\mu_{k-j-1},\mu_{k+1-j},\ldots,\mu_k) \quad \text{and} \quad \lambda' = (\lambda_1-1,\ldots,\lambda_j-1,\lambda_{j+1},\ldots,\lambda_{r+s}).$$

Note that $|\lambda| - |\mu| = |\lambda'| - |\mu'|$ and the lengths of μ and μ' differ by 1. It is easy to see that the mapping $(\mu;\lambda) \mapsto (\mu';\lambda')$ is a *weight-preserving-sign-reversing involution*. Only the pairs $(\mu;\lambda)$ such that $\mu = (m-1,m-2,\ldots,1,0)$, $r+s \geqslant m$ and $\lambda_1 = \cdots = \lambda_m = n+m-r-s$ will survive. That is to say, the expression (10) is equal to 0 if $r+s \leqslant m-1$, and

$$q^{\binom{r+s}{2}} \sum_{\substack{\ell(\lambda)\leqslant r+s-m \\ \lambda_1 \leqslant n+m-r-s}} \sum_{\substack{\ell(\nu)\leqslant r \\ \nu_1\leqslant s}} q^{|\lambda|+m(n+m-r-s)+|\nu|-\binom{m}{2}} \qquad \text{if } r+s \geqslant m, \tag{11}$$

namely

$$\begin{bmatrix} n \\ r+s-m \end{bmatrix} \begin{bmatrix} r+s \\ r \end{bmatrix} q^{mn+\binom{r+s-m}{2}},$$

which is the coefficient of x^s in the right-hand side of (5). This completes the proof.

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