Another characterisation of planar graphs

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Submitted: Sep 13, 2009; Accepted: Mar 1, 2010; Published: Mar 8, 2010

Mathematics Subject Classification: 05C10

Abstract

A new characterisation of planar graphs is presented. It concerns the structure of the cocycle space of a graph, and is motivated by consideration of the dual of an elementary property enjoyed by sets of circuits in any graph.

1 Introduction

Several characterisations of planar graphs are known. (See [1–20].) We present a new one based on the structure of the cocycle space of a graph.

Let G be a graph with vertex set VG and edge set EG. If S and T are disjoint sets of vertices, we denote by $[S,T]$ the set of edges with one end in S and the other in T. For any $S \subseteq VG$ we write $\overline{S} = VG - S$, and we define $\partial S = [S, \overline{S}]$. This set is called a *cocycle*, the cocycle *determined* by S (or $\overline{S}$). A *bond* is a minimal non-empty cocycle. Thus a non-empty cocycle ∂S in a connected graph G is a bond if and only if $G[S]$ and $G[\overline{S}]$ are both connected. An *isthmus* is the unique member of a bond of cardinality 1.

Let A and B be distinct bonds in G. We say that two distinct edges of $B-A$ are *bound* (to each other) by A with respect to B if they join vertices in the same two components of $G-(A\cup B)$.

Now let A_1, A_2, A_3, B be distinct bonds in G, and let $a \in B - (A_1 \cup A_2 \cup A_3)$. For each $i \in \{1,2,3\}$ let S_i be the set of edges bound to a by A_i with respect to B. We say that a is *tied* by $\{A_1, A_2, A_3\}$ with respect to B if the following conditions hold:

*Supported by KVPY

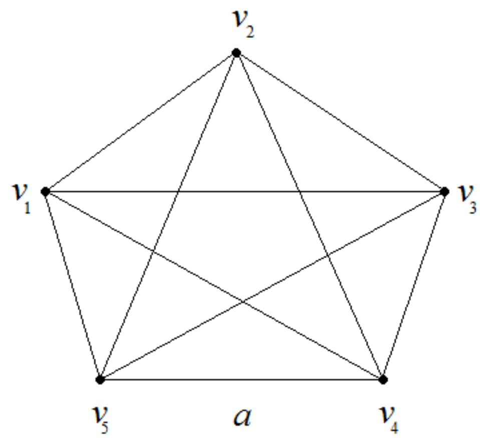

Figure 1: K_5

1. for each i there is a component of $G - (A_i \cup B)$ that contains an end of a and an end of an edge of $A_i \cap B$;

2. each of S_1, S_2, S_3 contains an edge that is in neither of the other two.

Edge a is *tied* if there exist bonds A_1, A_2, A_3, B such that a is tied by $\{A_1, A_2, A_3\}$ with respect to B. The aim of this paper is to prove the following theorem.

Theorem 1.1 *A graph is non-planar if and only if it has a tied edge.*

We define a *circuit* in a graph as the edge set of a minimal non-empty connected subgraph in which no edge is an isthmus. In its dual form, the condition stated in Theorem 1.1 is close in spirit to a theorem of Tutte [17] which asserts that a 3-connected graph is planar if and only if every edge is contained in just two induced non-separating circuits, but our result is motivated by consideration of the dual of an elementary observation about the circuits of any graph. (See Theorem 3.1 for the details.) It is similar in flavour to the dual of Fournier's characterisation [7]. However it differs from this dual, which requires three bonds that have at least an edge in common.

2 Every non-planar graph has a tied edge

We begin by showing that every edge of K_5 and $K_{3,3}$ is tied.

Lemma 2.1 *Every edge of K_5 is tied.*

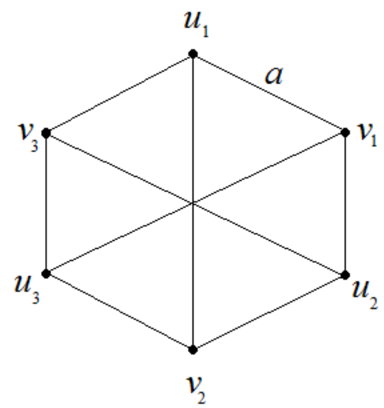

Figure 2: $K_{3,3}$

Proof: Let $VK_5 = \{v_1, v_2, v_3, v_4, v_5\}$, and for each i and $j \neq i$ let e_{ij} be the edge joining v_i to v_j. (See Figure 1.) Without loss of generality, let $a = e_{45}$. Let $Y = \{v_5\}$ and $X_i = \{v_i, v_4, v_5\}$ for each i. Define $B = \partial Y$ and $A_i = \partial X_i$ for each i. Then $X_i \cap Y = \{v_5\}$ and $X_i \cap \overline{Y} = \{v_i, v_4\}$ for each i. Thus $a \in \partial(X_i \cap Y) = \partial\{v_5\}$ for each i, $e_{25} \in \partial\{v_5\} \cap A_1 \cap B$ and $e_{15} \in \partial\{v_5\} \cap A_2 \cap A_3 \cap B$. We conclude that, for each i, $G[\{v_5\}]$ is a component of $G - (A_i \cup B)$ containing an end of a and of an edge of $A_i \cap B$. Moreover $S_i = \{e_{i5}\}$ for each i. It follows that a is tied by $\{A_1, A_2, A_3\}$ with respect to B. $\square$

Lemma 2.2 *Every edge of $K_{3,3}$ is tied.*

Proof: Let $VK_{3,3} = \{u_1, u_2, u_3, v_1, v_2, v_3\}$, where each of u_1, u_2, u_3 is adjacent to each of v_1, v_2, v_3, and for each i and j let e_{ij} be the edge joining u_i to v_j. (See Figure 2.) Let $a = e_{11}$, without losing generality. Let $Y = \{u_1, v_2\}$, $X_1 = \{u_1, v_1, u_2, v_3\}$, $X_2 = \{u_1, v_1, u_2, v_2\}$ and $X_3 = \{u_1, v_1, v_2, u_3\}$, and define $B = \partial Y$ and $A_i = \partial X_i$ for each i. Then $X_1 \cap Y = \{u_1\}$, $X_1 \cap \overline{Y} = \{v_1, u_2, v_3\}$, $X_2 \cap Y = X_3 \cap Y = \{u_1, v_2\}$, $X_2 \cap \overline{Y} = \{v_1, u_2\}$ and $X_3 \cap \overline{Y} = \{v_1, u_3\}$. Thus $a \in \partial(X_1 \cap \overline{Y})$, $e_{22} \in \partial(X_1 \cap \overline{Y}) \cap A_1 \cap B$, $a \in \partial(X_2 \cap Y) = \partial(X_3 \cap Y)$ and $e_{13} \in \partial(X_2 \cap Y) \cap A_2 \cap A_3 \cap B$, and so, for each i, there is a component L_i of $G - (A_i \cup B)$ such that $a \in \partial VL_i$ and $\partial VL_i \cap A_i \cap B \neq \emptyset$. Moreover $S_1 = \{e_{13}\}$, $S_2 = \{e_{22}\}$ and $S_3 = \{e_{32}\}$. We conclude that a is tied. $\square$

We complete the proof that every non-planar graph has a tied edge by applying the following characterisation of planar graphs, due to Wagner [17].

Theorem 2.3 *A graph is non-planar if and only if it contains a subgraph contractible to K_5 or $K_{3,3}$.*

In order to reverse the process of contracting an edge, we need the concept of splitting a vertex. Let e be an edge joining distinct vertices v_1 and v_2 in a graph G. Let G_e be the graph obtained from $G - \{e\}$ by identifying v_1 and v_2 to form a single vertex v. Then G_e is said to be obtained from G by *contracting* e, and the process of forming G_e from G is called a *contraction*. The graph G is said to be *contractible* to a graph H if H can be obtained from G by a sequence of contractions. On the other hand, G is said to be obtained from G_e by *splitting* v into v_1 and v_2. Observe that any bond of G_e is also a bond of G. The next lemma and its corollary follow immediately from this observation.

Lemma 2.4 *Let $A = \partial X$ and $B = \partial Y$ be bonds of a graph G, where X and Y are subsets of VG. Let $a \in B - A$, and let S be the set of edges bound to a by A with respect to B. Let G' be a graph obtained from G by splitting a vertex v into v_1 and v_2. For each $Z \in \{X, Y\}$, define $Z' = Z$ if $v \notin Z$, and let $Z' = (Z - \{v\}) \cup \{v_1, v_2\}$ otherwise. Define $A' = \partial X'$ and $B' = \partial Y'$, and let S' be the set of edges of G' bound to a by A' with respect to B'. Then $S' = S$.*

Corollary 2.5 *If an edge is tied in G, then it is tied in G'.*

It remains only to reverse the operation of forming a subgraph of a graph. This operation is effected by a sequence of deletions of edges and isolated vertices. We therefore need to consider the adjunction of an edge to a graph G, where the new edge joins either two vertices of G or a vertex of G to a new vertex.

Lemma 2.6 *Let $A = \partial X$ and $B = \partial Y$ be bonds of a graph G, where X and Y are subsets of VG, and suppose that $G[X \cap Y]$ and $G[X \cap \overline{Y}]$ are connected. Let $a \in B - A$, and let S be the set of edges bound to a by A with respect to B. Let G' be a graph obtained from G by the adjunction of an edge e joining two vertices of G or one vertex of G to a vertex not in G. For each $Z \in \{X, Y\}$, define $Z' = Z$ if e joins two vertices of G. If e joins a vertex $x \in VG$ to a vertex $y \notin VG$, then define $Z' = Z \cup \{y\}$ if $x \in Z$ and $Z' = Z$ otherwise. Define $A' = \partial X'$ and $B' = \partial Y'$, and let S' be the set of edges of G' bound to a by A' with respect to B'. Then $S' = S$ or $S' = S \cup \{e\}$.*

Proof: If $e \in [X \cap Y, X \cap \overline{Y}]$, then $S' = S \cup \{e\}$. In the remaining case, $S' = S$ since $G[X \cap Y]$ and $G[X \cap \overline{Y}]$ are connected. □

Corollary 2.7 *If an edge is tied in G, then it is tied in G'.*

Proof: Let a be an edge that is tied in G by a set $\{A_1, A_2, A_3\}$ of bonds with respect to a bond B. There exist sets Y, X_1, X_2, X_3 of vertices of G such that $B = \partial Y$, $A_i = \partial X_i$ for each i and $a \in [X_i \cap Y, X_i \cap \overline{Y}]$ for each i. Let $L_{i1}, L_{i2}, \ldots, L_{il_i}$ be the components of $G[X_i \cap Y]$, where a is incident on a vertex of L_{i1}. By replacing X_i with $X_i - \bigcup_{j=2}^{l_i} VL_{ij}$ if necessary, we adjust each A_i so that $G[X_i \cap Y]$ is connected for each i. Similarly we further adjust each A_i, if necessary, so that $G[X_i \cap \overline{Y}]$ is also connected for each i. Then $G'[X_i' \cap Y']$ and $G'[X_i' \cap \overline{Y'}]$ are connected as well, and at least one of them contains an end of an edge of $A_i' \cap B'$ since at least one of $G[X_i \cap Y]$ and $G[X_i \cap \overline{Y}]$ contains an end of an edge of $A_i \cap B$.

Finally, for each i, let S_i be the set of edges bound to a by A_i with respect to B. Each of S_1, S_2, S_3 has an edge not in either of the others. Using the lemma with A replaced successively by A_1, A_2, A_3, we find that $S_i' \cap EG = S_i$ for each i. Hence each of S_1', S_2', S_3' has an edge not in either of the others. Therefore a is tied in G'. $\square$

Thus the property of an edge being tied is impervious to splittings of vertices and adjunctions of edges. That every non-planar graph has a tied edge now follows from Theorem 2.3, Lemmas 2.1 and 2.2 and Corollaries 2.5 and 2.7.

3 No planar graph has a tied edge

Why can't a planar graph have a tied edge? The answer lies in the concept of duality. Let G be a planar graph and contemplate its dual G^* with respect to a specific embedding of G in the plane. Thus $EG^* = EG$ and the vertices of G^* are the faces of the embedding of $G - R$ obtained by deleting the set R of isthmuses from the embedding of G. Any line segment representing an edge e of G that is not an isthmus separates two distinct faces, and e is incident on those faces in G^*. If on the other hand e is an isthmus of G, then the interior of the line segment representing e is drawn in the interior of a face of the embedding of $G - R$, and e is a loop incident on that face in G^*. Note that G^* is necessarily connected. Each bond of G is a circuit of G^*, and each circuit of G is a bond of G^*. If a theorem holds for a given set T of edges of G, then the dual of that theorem holds for T in G^*.

Let B be a circuit in a graph G (not necessarily planar) and let A be another circuit that meets B. An A-*arc* of B is a minimal subset S of $B - A$ such that $S \cup A$ includes a circuit distinct from A. Thus it is a path included in $B - A$, of minimal length, that joins distinct vertices of $VA \cap VB$. (If C is a circuit in G, then we write $VC = VG[C]$.)

Now fix an edge $a \in B - A$. This edge belongs to a unique A-arc of B. This A-arc is called the *principal* A-arc $P(A)$ of B (with respect to a). The edges of $B - P(A)$ are said to be *subtended* by A (with respect to a). These edges form a path included in $B - \{a\}$. We state the following obvious result as a theorem for the sake of emphasis.

Theorem 3.1 *Let a be an edge of a circuit B in a graph G. Let A_1, A_2, A_3 be circuits in G that meet B but do not contain a. Suppose that there is an edge subtended by A_1 and A_3 but not A_2 and an edge subtended by A_2 and A_3 but not A_1. Then any edge subtended by A_1 and A_2 is also subtended by A_3.*

Corollary 3.2 *Let a be an edge of a circuit B in a graph G. Let A_1, A_2, A_3 be circuits in G that meet B but do not contain a. Then at least one of $P(A_1), P(A_2), P(A_3)$ is a subset of the union of the others.*

We now resuscitate the assumption that G be planar and return to the consideration of G^*, a graph in which A_1, A_2, A_3, B are bonds. Let A be any circuit of G that meets B but does not contain a. Then A is also a bond of G^*. Let $A = \partial X$ and $B = \partial Y$ for some subsets X and Y of VG^*. Thus $G^*[X], G^*[\overline{X}], G^*[Y], G^*[\overline{Y}]$ are all connected. Since $a \in B - A$, we may assume without loss of generality that a joins vertices in $G^*[X \cap Y]$ and $G^*[X \cap \overline{Y}]$. Let L and M be the components of $G^*[X \cap Y]$ and $G^*[X \cap \overline{Y}]$, respectively, that contain an end of a.

Lemma 3.3 *Cocycles ∂VL and ∂VM are bonds of G^*.*

Proof: We shall show only that ∂VL is a bond of G^*, as the proof that ∂VM is a bond is similar. For this purpose it is enough to show that $G^* - VL$ is connected.

Suppose first that $\overline{X} \cap Y = \emptyset$. As $G^*[Y]$ is connected, so is $G^*[X \cap Y]$, so that L is its only component. Hence $G^* - VL = G^*[\overline{Y}]$, which is also connected.

We may therefore suppose that $\overline{X} \cap Y \neq \emptyset$. If also $\overline{X} \cap \overline{Y} \neq \emptyset$, then $G^* - (X \cap Y)$ is connected since $G^*[\overline{X}]$ and $G^*[\overline{Y}]$ are connected. If on the other hand $\overline{X} \cap \overline{Y} = \emptyset$, then $G^*[\overline{X} \cap Y]$ and $G^*[X \cap \overline{Y}]$ are connected for the same reason, so that M is the only component of the latter graph. Once again $G^* - (X \cap Y)$ is connected, for $[VM, \overline{X} \cap Y] \neq \emptyset$ since A and B meet.

Thus $G^* - (X \cap Y)$ is connected in any case. As $G^*[Y]$ is connected, any component of $G^*[X \cap Y]$ must have a vertex that is adjacent to some vertex of $G^*[\overline{X} \cap Y]$. Therefore $G^* - VL$ is connected. □

Bonds ∂VL and ∂VM are distinct from A as $a \in (\partial VL \cap \partial VM) - A$. Moreover if $X \cap Y = VL$ then $[VL, VM]$ is a subset of $B - A$ such that $[VL, VM] \cup A$ includes the bond $\partial VM \neq A$. In fact it is a minimal subset S of $B - A$ such that $S \cup A$ includes a bond of G^* distinct from A. As $a \in [VL, VM]$, it follows that $[VL, VM] = P(A)$. We therefore perceive that if $X \cap Y = VL$ then $P(A)$ consists of a and the edges that are bound to a by A with respect to B. By a similar argument we find that the same conclusion holds if $X \cap \overline{Y} = VM$.

We now apply these observations to the bonds A_1, A_2, A_3 that meet B but do not contain a. Suppose that, for each $i \in \{1, 2, 3\}$, there is a component L_i of $G - (A_i \cup B)$ that contains an end of a and an end of an edge of $A_i \cap B$. Let S_i be the set of edges bound to a by A_i with respect to B. Let $A_i = \partial X_i$. Adjusting A_i if necessary, we may suppose, for each i, that $X_i \cap Y = VL_i$ or $X_i \cap \overline{Y} = VL_i$. Then Corollary 3.2 shows that at least one of S_1, S_2, S_3 is included in the union of the others. In other words, $\{A_1, A_2, A_3\}$ does not tie a with respect to B. As G^* may be any connected planar graph, no connected planar graph has an edge a and bonds A_1, A_2, A_3, B such that $\{A_1, A_2, A_3\}$ ties a with respect to B. By applying this result to each component of a planar graph, we deduce that no planar graph can have a tied edge.

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