A simple bijection between binary trees and colored ternary trees

Yidong Sun

Department of Mathematics,
Dalian Maritime University, 116026 Dalian, P.R. China

sydmath@yahoo.com.cn

Submitted: Feb 25, 2009; Accepted: Mar 28, 2010; Published: Apr 5, 2010
Mathematics Subject Classification: 05C05, 05A19

Abstract

In this short note, we first present a simple bijection between binary trees and colored ternary trees and then derive a new identity related to generalized Catalan numbers.

Keywords: Binary tree; Ternary tree; Generalized Catalan number.

1 Introduction

Recently, Mansour and the author [2] obtained an identity involving 2-Catalan numbers $C_{n,2} = \frac{1}{2n+1}\binom{2n+1}{n}$ and 3-Catalan numbers $C_{n,3} = \frac{1}{3n+1}\binom{3n+1}{n}$, i.e.,

$$\sum_{p=0}^{[n/2]} \frac{1}{3p+1}\binom{3p+1}{p}\binom{n+p}{3p} \quad = \quad \frac{1}{2n+1}\binom{2n+1}{n}. \tag{1.1}$$

In this short note, we first present a simple bijection between complete binary trees and colored complete ternary trees and then derive the following generalized identity,

$$\sum_{p=0}^{[n/2]} \frac{m}{3p+m}\binom{3p+m}{p}\binom{n+p+m-1}{n-2p} \quad = \quad \frac{m}{2n+m}\binom{2n+m}{n}. \tag{1.2}$$

2 A bijective algorithm for binary and ternary trees

A *colored ternary trees* is a complete ternary tree such that all its vertices are signed a nonnegative integer called *color number*. Let $\mathbf{T}_{n,p}$ denote the set of colored ternary trees

T with p internal vertices such that the sum of all the color numbers of T is $n-2p$. Define $\mathbf{T}_n = \bigcup_{p=0}^{[n/2]} \mathbf{T}_{n,p}$. Let $\mathbf{B}_n$ denote the set of complete binary trees with n internal vertices. For any $B \in \mathbf{B}_n$, let $P = v_1 v_2 \cdots v_k$ be a path of *length* k of B (viewed from the root of B). P is called a *R-path*, if (1) v_i is the right child of v_{i-1} for $2 \leqslant i \leqslant k$ and (2) the left child of v_i is a leaf for $1 \leqslant i \leqslant k$. In addition, P is called a *maximal R-path* if there exists no vertex u such that uP or Pu forms a R-path. P is called an *L-path*, if $k \geqslant 2$ and v_i is the left child of v_{i-1} for $2 \leqslant i \leqslant k$. P is called a *maximal L-path* if there exists no vertex u such that uP or Pu forms an L-path. Clearly, a leaf can never be R-path or L-path.

Note that the definition of L-path is different from that of R-path. Hence, if P is a maximal R-path, then (1) the right child u of v_k must either be a leaf or the left child of u is not a leaf; (2) v_1 must either be a left child of its father (if exists) or the father of v_1 has a left child which is not a leaf. If P is a maximal L-path, then (1) v_k must be a leaf which is also a left child of v_{k-1}; (2) v_1 must be the right child of its father (if exists).

Theorem 2.1 *There exists a simple bijection ϕ between $\mathbf{B}_n$ and $\mathbf{T}_n$.*

Proof. We first give the procedure to construct a complete binary tree from a colored complete ternary tree.

Step 1. For each vertex v of $T \in \mathbf{T}_n$ with color number $c_v = k$, remove the color number and add an R-path $P = v_1 v_2 \cdots v_k$ of length k to v such that v is a right child of v_k and v_1 is a child of the father (if exists) of v, and then annex a left leaf to v_i for $1 \leqslant i \leqslant k$. See Figure 1(a) for example.

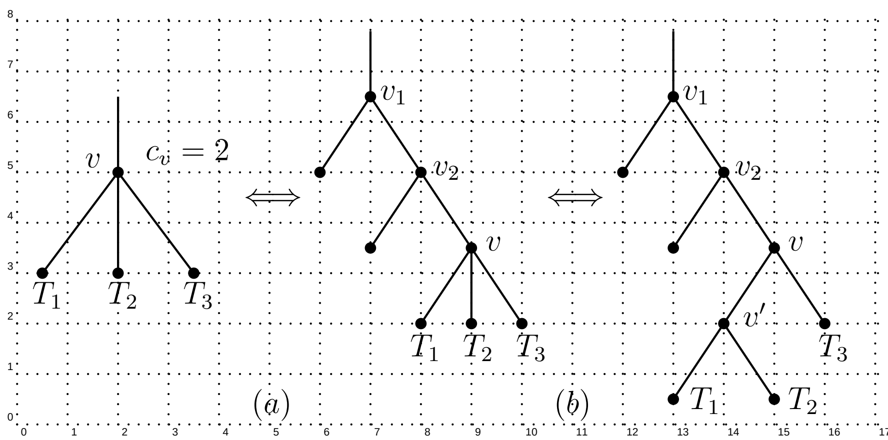

Figure 1:

Step 2. Let T^* be the tree obtained from T by Step 1. For any internal vertex v of T^* which has out-degree 3, let T_1, T_2 and T_3 be the three subtrees of v. Remove the subtrees T_1 and T_2 , annex a left child v' to v and take T_1 and T_2 as the left and right subtrees of v' respectively. See Figure 1(b) for example.

It is clear that any $T \in \mathbf{T}_n$, after Step 1 and 2, generates a binary tree $B \in \mathbf{B}_n$. Conversely, we can obtain a colored ternary tree from a complete binary tree as follows.

Step 3. Choose any maximal L-path of $B \in \mathbf{B}_n$ of length k (according to its definition, $k \geqslant 2$), say $P = v_1 v_2 \cdots v_k$, then each v_{2i-1} absorbs its left child v_{2i} for $1 \leqslant i \leqslant [k/2]$. This operation guarantees the resulting vertices v_{2i-1} are of out-degree 3 for $1 \leqslant i \leqslant [k/2]$ and v_k is always a leaf if k is odd. See Figure 2(a) for example.

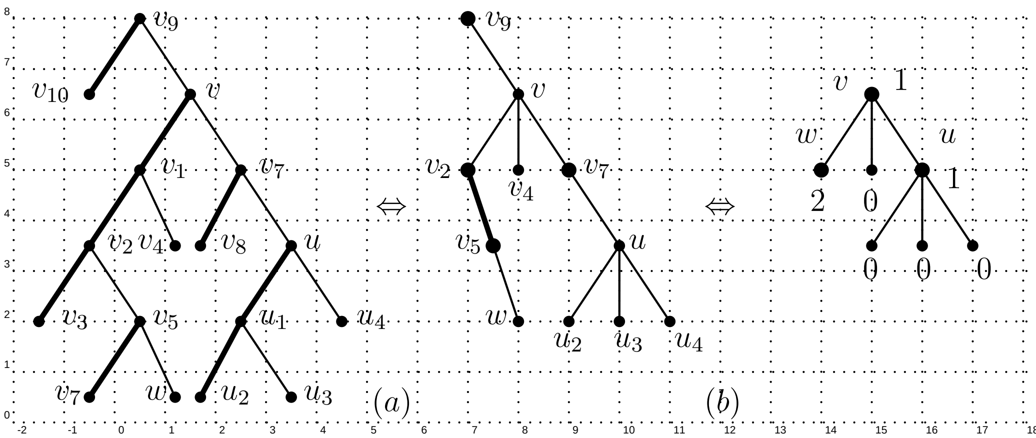

Figure 2:

Step 4. Choose any maximal R-path of T' derived from B by Step 3 (note that any maximal R-path is not changed after this operation), say $Q = u_1 u_2 \cdots u_k$, let u be the right child of u_k, then u absorbs all the vertices $u_1, u_2, \ldots, u_k$ and assign the color number $c_u = k$ to u. Any remaining leaf is assigned a 0 at the end of the process. See Figure 2(b) for example. Hence we get a colored ternary tree. ∎

Given a complete ternary tree T with p internal vertices, there are a total number of $3p+1$ vertices, choose $n-2p$ vertices with repetition allowed and define the color number of a vertex to be the number of times that vertex is chosen. Then there are $\binom{n+p}{n-2p}$ colored ternary trees in $\mathbf{T}_n$ generated by T. Note that $\frac{1}{3p+1}\binom{3p+1}{p}$ and $\frac{1}{2n+1}\binom{2n+1}{n}$ count the number of complete ternary trees with p internal vertices and complete binary trees with n internal vertices respectively [3]. Then the bijection ϕ immediately leads to (1.1).

To prove (1.2), consider the forest of colored ternary trees $F = (T_1, T_2, \ldots, T_m)$ with $T_i \in \mathbf{T}_{n_i}$ and $n_1 + n_2 + \cdots + n_m = n$, define $\phi(F) = (\phi(T_1), \phi(T_2), \ldots, \phi(T_m))$, then it is clear that ϕ is a bijection between forests of colored ternary trees and forests of complete binary trees. Note that there are totally $m+3p$ vertices in a forest F of complete ternary trees with m components and p internal vertices, so there are $\binom{m+n+p-1}{n-2p}$ forests of colored ternary trees with m components, p internal vertices and the sum of color numbers equal to $n-2p$. It is clear from [3] that $\frac{m}{3p+m}\binom{3p+m}{p}$ counts the number of forests of complete ternary trees with p internal vertices and m components, and that $\frac{m}{2n+m}\binom{2n+m}{n}$ counts the number forests of complete binary trees with n internal vertices and m components. Then the above bijection ϕ immediately leads to (1.2).

Remark: A similar type of bijection is presented by Edelman [1] in terms of non-crossing partitions.

3 Further comments

It is well known [3] that the k-Catalan number $C_{n,k} = \frac{1}{kn+1}\binom{kn+1}{n}$ counts the number of complete k-ary trees with n internal vertices, whose generating function $C_k(x)$ satisfies

$$C_k(x) = 1 + xC_k(x)^k.$$

Let $G(x) = \frac{1}{1-x}C_3(\frac{x^2}{(1-x)^3})$, then one can deduce that

$$\begin{aligned} G(x) &= \frac{1}{1-x}C_3(\frac{x^2}{(1-x)^3}) \\ &= \frac{1}{1-x}(1+\frac{x^2}{(1-x)^3}C_3(\frac{x^2}{(1-x)^3})^3) \\ &= \frac{1}{1-x}(1+x^2G(x)^3), \end{aligned}$$

which generates that $G(x) = C_2(x)$, the generating function for 2-Catalan numbers.

By the Lagrange inversion formula, we have

$$\begin{aligned} C_3(x)^m &= \sum_{p\geqslant 0}\frac{m}{3p+m}\binom{3p+m}{p}x^p, \\ C_2(x)^m &= \sum_{n\geqslant 0}\frac{m}{2n+m}\binom{2n+m}{n}x^n. \end{aligned}$$

Then

$$\begin{aligned} G(x)^m &= \sum_{p\geqslant 0}\frac{m}{3p+m}\binom{3p+m}{p}\frac{x^{2p}}{(1-x)^{3p+m}} \\ &= \sum_{n\geqslant 0}x^n\sum_{p=0}^{[n/2]}\frac{m}{3p+m}\binom{3p+m}{p}\binom{n+p+m-1}{n-2p}. \end{aligned}$$

Comparing the coefficient of x^n in $C_2(x)^m$ and $G(x)^m$, one obtains (1.2).

Similarly, let $F(x) = \frac{1}{1-x}C_k(\frac{x^{k-1}}{(1-x)^k})$, then $F(x) = \frac{1+xF(x)}{1-x^{k-1}F(x)^{k-1}}$, using the Lagrange inversion formula for the case $k=5$, one has

$$\begin{aligned} &\sum_{p=0}^{[n/4]}\frac{m}{5p+m}\binom{5p+m}{p}\binom{n+p+m-1}{n-4p} \qquad (3.1) \\ &= \sum_{p=0}^{[n/2]}(-1)^p\frac{m}{m+n}\binom{m+n+p-1}{p}\binom{m+2n-2p-1}{n-2p}, \end{aligned}$$

which, in the case $m=1$, leads to

$$\sum_{p=0}^{[n/4]} \frac{1}{4p+1}\binom{5p}{p}\binom{n+p}{5p} \quad = \quad \sum_{p=0}^{[n/2]}(-1)^p \frac{1}{n+1}\binom{n+p}{n}\binom{2n-2p}{n}. \qquad (3.2)$$

One may ask to give a combinatorial proof of (3.1) or (3.2). Later, based on the idea of our bijection, Yan [4] provided nice proofs for them.

Acknowledgements

The author is grateful to the anonymous referees for the helpful suggestions and comments. The work was supported by The National Science Foundation of China (Grant No. 10801020 and 70971014).

References

[1] P. H. Edelman, Mutichains, non-crossing partitions and trees, *Discrete Mathematics*, Volume 40, (1982), 171-179.

[2] T. Mansour and Y. Sun, Bell polynomials and k-generalized Dyck paths, *Discrete Applied Mathematics*, Volume 156(12), (2008), 2279-2292.

[3] R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1999.

[4] S. H. F. Yan, Bijective proofs of identities from colored binary trees, *The Electronic Journal of Combinatorics*, Volume 15(1), (2008), #N20.