A q-analogue of Graham, Hoffman and Hosoya's Theorem

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Abstract

Graham, Hoffman and Hosoya gave a very nice formula about the determinant of the distance matrix D_G of a graph G in terms of the distance matrix of its blocks. We generalize this result to a q-analogue of D_G. Our generalization yields results about the equality of the determinant of the mod-2 (and in general mod-k) distance matrix (i.e. each entry of the distance matrix is taken modulo 2 or k) of some graphs. The mod-2 case can be interpreted as a determinant equality result for the *adjacency matrix* of some graphs.

1 Introduction

Graham and Pollak (see [3]) considered the distance matrix $D_T = (d_{u,v})$ of a tree $T = (V, E)$. For $u, v \in V$, its distance $d_{u,v}$ is the length of a shortest (in this case unique) path between u and v in T and since any tree is connected, all entries $d_{u,v}$ are finite. Let D_T be the distance matrix of T with $|V| = n$. They showed a surprising result that $\det(D_T) = (-1)^{n-1}(n-1)2^{n-2}$. Thus, the determinant of D_T only depends on n, the number of vertices of T and is independent of T's structure.

Graham, Hoffman and Hosoya [2] proved a very attractive theorem about the determinant of the distance matrix D_G of a strongly connected digraph G as a function of the distance matrix of its *2-connected blocks* (also called blocks). Denote the sum of the cofactors of a matrix A as $\mathsf{cofsum}(A)$. Graham, Hoffman and Hosoya (see [2]) showed the following.

Theorem 1 *If G is a strongly connected digraph with 2-connected blocks $G_1, G_2, \ldots, G_r$, then* $\mathsf{cofsum}(D_G) = \prod_{i=1}^r \mathsf{cofsum}(D_{G_i})$ *and* $\det(D_G) = \sum_{i=1}^r det(D_{G_i}) \prod_{j\neq i} \mathsf{cofsum}(D_{G_j})$.

Since all the $(n-1)$ blocks of any tree T on n vertices are K_2's, we can recover Graham and Pollak's result from Theorem 1. Yan and Yeh [5] showed a similar "tree structure independent" result for the problem of counting the number of signed permutations with a fixed number k as the *Spearman measure* where distances are induced from an underlying tree T.

Bapat et al [1] obtained a q-analogue of Graham and Pollak's result and Sivasubramanian [4] obtained a q-analogue of Theorem 1 for the case when all the blocks of a graph are triangles. In this present work, we show a q-analogue of Theorem 1.

1.1 The q-analogue

For a strongly connected digraph $G = (V, E)$, the q-analogue of its distance matrix qD_G is obtained from its distance matrix D_G by replacing all positive entries i by $[i]_q = 1+q+\cdots+q^{i-1}$ where q is an indeterminate and $[0]_q = 0$. Let the distance between vertices u and v in G be denoted as $d_{u,v}$ and let the cofactor matrix (see Section 2 for definitions) of qD_G be $\mathsf{qCOF}_G = (c_{u,v})$. Let the rowsum of qCOF_G corresponding to row v be rsum_v. Given $\mathbf{w} \in V$, consider the weighted cofactor sum defined as $\mathsf{qcofsum}_G^{\mathbf{w}} = \sum_{v \in G} q^{d_{v,\mathbf{w}}} \mathsf{rsum}_v$. We note that setting $q = 1$ gives $\mathsf{qcofsum}_G^{\mathbf{w}} = \sum_{u,v} c_{u,v}$ which is the sum of the cofactors as used in [2] and that this sum is independent of $\mathbf{w}$. In Lemma 3, we show that $\mathsf{qcofsum}_G^{\mathbf{w}}$ is independent of $\mathbf{w}$ (and hence can be denoted as $\mathsf{qcofsum}_G$). In Subsection 3.1, we prove the following q-analogue of Graham, Hoffman and Hosoya's result.

Theorem 2 *Let G be a strongly connected digraph with distance matrix D_G. Let the q-analogue of D_G be qD_G and let G have blocks $G_1, G_2, \ldots, G_r$. For each $1 \leqslant i \leqslant r$, let the distance matrix of G_i and its q-analogue be D_{G_i} and qD_{G_i} respectively. Then,*

1. $\mathsf{qcofsum}_G = \prod_{i=1}^{r} \mathsf{qcofsum}_{G_i}$
2. $\det(qD_G) = \sum_{i=1}^{r} \det(qD_{G_i}) \prod_{j \neq i} \mathsf{qcofsum}_{G_j}$.

Thus, we show a polynomial generalisation of Graham, Hoffman and Hosoya's Theorem. We also prove a similar polynomial generalisation - when two $n \times n$ matrices M_1, M_2 have the same determinant, then replacing all the entries of both matrices by twice (or any scalar times) its original value clearly still gives two different matrices (say M_1', M_2') also with the same determinant value. For distance matrices, we show in Subsection 3.3 that replacing each entry by a "two-times" polynomial (and more generally by a "k-times" polynomial, where k is a positive integer) again gives identical determinant values as polynomials.

Consider the mod-2 distance matrix of a graph, where only the parity of each entry of the distance matrix is used. We show that if two graphs G_1, G_2 have an identical multiset of isomorphic blocks, then the mod-2 distance matrices of G_1 and G_2 have the same determinant value, independent of the tree-like connection of their blocks. This shows that the *adjacency matrix* of several graphs have the same determinant value.

More generally for a positive integer $k \geqslant 3$, we first replace all the distance matrix entries by its mod-k values. In the resulting matrix, if we change all entries i (for $0 \leqslant i < k$) to $1 + \zeta + \zeta^2 + \cdots + \zeta^{i-1}$, where ζ is a primitive k-th root of unity, then the determinant of this (complex) matrix is again independent of the tree structure on the blocks of G. Subsection 3.2 contains these results.

2 Preliminaries

In this section, we note a few linear algebraic preliminaries that we will need for the proof of Theorem 2. All our vectors will be column vectors and given an $n \times p$ matrix A, we denote its transpose by A^t. For a square matrix A, $\det(A)$ denotes its determinant.

Given an $n \times n$ matrix A, its row and column indices begin with 1 and we denote its i-th row (for $1 \leqslant i \leqslant n$) by Row_i and its j-th column (for $1 \leqslant j \leqslant n$) by Col_j. It is convenient for determinant calculations to represent some combinations of elementary row and column operations on A by multiplications of the following $n \times n$ matrices:

$$R = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n & 0 & \cdots & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & \beta_2 & \cdots & \beta_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

It follows that RAC is the result of the following elementary row and column operations on A performed in any order: $\mathsf{Row}_i := \mathsf{Row}_i + \alpha_i \mathsf{Row}_1$ and $\mathsf{Col}_i := \mathsf{Col}_i + \beta_i \mathsf{Col}_1$ for $2 \leqslant i \leqslant n$.

Given an $n \times n$ matrix A and $n \times 1$ vectors ρ and τ, we will need to find $\det(A + x\rho\tau^t)$ where x is a fresh variable, not occurring in A, τ or ρ. We will restrict attention to vectors ρ, τ where both $\rho_1 \neq 0$ and $\tau_1 \neq 0$. Let $\mathsf{cA} = (A_{i,j})$ be the cofactor matrix of A with $A_{i,j}$ for $1 \leqslant i, j \leqslant n$ denoting the cofactor at position (i, j). Specifically, $A_{i,j}$ is $(-1)^{i+j}$ times the determinant of the submatrix of A obtained by deleting Row_i and Col_j. Lastly, define $C_{\rho,\tau}(\mathsf{cA}) = \rho^t \mathsf{cA} \tau$.

Lemma 1 *The coefficient of x in $\det(A + x\rho\tau^t)$ is $C_{\rho,\tau}(\mathsf{cA})$*

Proof: The coefficient of x in $\det(A + x\rho\tau^t)$ is $\sum_{i,j} \rho_i \tau_j A_{i,j}$. (This follows by observing that the only way to get an x in the determinant expansion is to choose $x\rho_i\tau_j$ from the i-th row and j-th column and non-x terms from other rows and columns.) ■

Let $\tilde{A}$ be obtained from an $n \times n$ matrix A by performing $\mathsf{Row}_i := \mathsf{Row}_i - \frac{\rho_i}{\rho_1}\mathsf{Row}_1$ for $2 \leqslant i \leqslant n$ and then performing $\mathsf{Col}_i := \mathsf{Col}_i - \frac{\tau_i}{\tau_1}\mathsf{Col}_1$. Let

$$R = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\rho_2}{\rho_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\rho_n}{\rho_1} & 0 & \cdots & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & -\frac{\tau_2}{\tau_1} & \cdots & -\frac{\tau_n}{\tau_1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Celarly, $\tilde{A} = RAC$. We will use the matrices R and C again in this work and though they depend on the vectors ρ and τ, instead of using a more correct subscripted notation R_ρ and C_τ, we will define vectors ρ and τ and only then use R, C. In our proof of Theorem 2, we will apply this notation to cases with $A = qD_G$ and with A being each of two principal submatrices of qD_G with only index 1 in common; vertex 1 will be the separator between one block and the rest of the graph G. In each of these three cases, the vertices of the appropriate subgraph of G will be labelled by the indices of A, R, C, cA, ρ and τ and these indices are used in the multiplications defining $C_{\rho,\tau}(\mathsf{cA}) = \rho^t \mathsf{cA} \tau$ and $\tilde{M} = RMC$ (for $M = A$ and others). The common vertex has index 1. In all cases, the cofactor of $\tilde{A}$ at position $(1, 1)$ is denoted by $\tilde{A}_{1,1}$.

Lemma 2 $\rho_1\tau_1\widetilde{A}_{1,1} = C_{\rho,\tau}(\mathsf{cA})$.

Proof: Since R and C have determinant 1, $\det(A + x\rho\tau^t) = \det(R(A + x\rho\tau^t)C) = \det(RAC + M) = \det(\tilde{A} + M)$, where

$$M = \begin{pmatrix} x\rho_1\tau_1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Therefore, the coefficient of x in $\det(A+x\rho\tau^t)$ is $\rho_1\tau_1\widetilde{A}_{1,1}$. The proof is complete by combining with Lemma 1. ■

3 The q-analogue

3.1 Proofs of results

With the notation of Section 1, we begin with the Lemma below.

Lemma 3 *For vertices* $u_1, u_2 \in G$, $u_1 \neq u_2$, $\mathsf{qcofsum}_G^{u_1} = \mathsf{qcofsum}_G^{u_2}$. *Thus,* $\mathsf{qcofsum}_G^{v}$ *is independent of the vertex* v. *Further, for all* $u \in G$, $\mathsf{qcofsum}_G^{u} = (q-1)\det(qD_G) + \mathsf{cofsum}(qD_G)$, *where* $\mathsf{cofsum}(qD_G) = \sum_{u,v} c_{u,v}$ *is the sum of the cofactors of* qD_G.

Proof: We recall that qD_G is the q-analogue of the distance matrix $D_G = (d_{u,v})$ of G and $\mathsf{qCOF}_G = (c_{u,v})$ is the cofactor matrix of qD_G. For two vertices $u, v \in G$, $d_{u,v}$ is the distance between them and $[d_{u,v}]_q = 1 + q + q^2 + \cdots + q^{d_{u,v}-1}$. Let rsum_v be the row-sum of qCOF_G corresponding to row v and for a vertex u, $\mathsf{qcofsum}_G^u = \sum_v q^{d_{v,u}}\mathsf{rsum}_v$

Elementary properties of the determinant and the adjugate imply for all vertices $u \in G$, $\det(qD_G) = \sum_{v\in G}[d_{v,u}]_q \cdot c_{v,u} = \sum_{v\in G}[d_{v,u}]_q \cdot \mathsf{rsum}_v$. Thus,

$$\begin{aligned}(q-1)\det(qD_G) &= \sum_{v\in G}(q-1)[d_{v,u}]_q \cdot \mathsf{rsum}_v \\ &= \sum_{v\in G}(q^{d_{v,u}} - 1) \cdot \mathsf{rsum}_v \\ &= \mathsf{qcofsum}_G^u - \mathsf{cofsum}(qD_G)\end{aligned}$$

This completes the proof. ■

For simplicity, $d_{i,j}$ denotes d_{v_i,v_j} for vertices v_i, v_j in any graph and sometimes, the index i will be identified with vertex v_i. Lemma 3 can be stated in the following alternate way. For a strongly connected digraph G, let $\mathsf{ED}_G = (e_{u,v})$ be its *exponential distance matrix* defined as $e_{u,v} = q^{d_{u,v}}$ where $d_{u,v}$ is the distance between u and v, q is an indeterminate and $q^0 = 1$.

Corollary 1 *Consider the matrix* $M_G = \mathsf{ED}_G^t \cdot \mathsf{qCOF}_G$. *The all-ones vector* $\mathbb{1}$, *of dimension* $|V(G)| \times 1$ *is an eigenvector of* M_G *corresponding to eigenvalue* $\mathsf{qcofsum}_G$.

Proof: Let RS be the $|V(G)| \times 1$ vector with $\mathsf{RS}_v = \mathsf{rsum}_v$. Clearly, $\mathsf{qCOF}_G \cdot \mathbb{1} = \mathsf{RS}$ and $(\mathsf{ED}_G^t \cdot \mathsf{RS})_v = \sum_u q^{d_{u,v}} \mathsf{rsum}_u = \mathsf{qcofsum}_G$. The proof follows. ∎

We note the following lemma similar to the lemma in [2]. We recall the q-weighted cofactor sum with respect to column j is $\mathsf{qcofsum}_G^j = \sum_{1 \leqslant i \leqslant n} q^{d_{i,j}} \mathsf{rsum}_i$. Since by Lemma 3, cofsum_G^j is independent of j, we fix $j = 1$ and write $\mathsf{cofsum}_G = \mathsf{cofsum}_G^j$. We will use Lemma 2 with

$$A = qD_G, \rho^t = [1, q^{d_{2,1}}, q^{d_{3,1}}, \ldots, q^{d_{n,1}}] \text{ and } \tau^t = \mathbb{1}. \tag{1}$$

These values for the ρ_i's and the τ_i's define the matrices R, C and thus $\widetilde{qD_G}$. It is simple to see from the definition that $\mathsf{qcofsum}_G = \mathsf{qcofsum}_G^1 = C_{\rho,\tau}(\mathsf{qCOF}_G)$, where we recall $C_{\rho,\tau}(\mathsf{qCOF}_G) = \rho^t(\mathsf{qCOF}_G)\tau$. The following lemma gives the cofactor of $q\tilde{D}_G$ at position $(1,1)$.

Lemma 4 *With the above notation,* $C_{\rho,\tau}(\mathsf{qCOF}_G) = \widetilde{(qD_G)}_{1,1}$.

Proof: Follows from Lemma 2 by noting $\rho_1 = \tau_1 = 1$. ∎

Proof: (Of Theorem 2) Pairs of distinct blocks have at most one vertex in common; the common vertex joining two adjacent blocks is called a cut-vertex. Among the blocks of G, let H be a block which has only one cut-vertex. We call such blocks as leaf-blocks. Clearly, leaf-blocks exist and let H be a leaf block connected to the rest of G along a cut-vertex. Let us label the vertices so that this cut-vertex is labelled by 1, so when v_i denotes a vertex of H and u_j denotes a vertex of G', $v_1 = u_1 = 1$ denotes this cut-vertex in G. We recall the cofactor matrix $\mathsf{qCOF}_H = (c_{u,v}^H)$ of qD_H, and the q-weighted cofactor sum $\mathsf{qcofsum}_H$ defined above.

Let $|H| = k$ and $V(H) = \{1, v_2, \ldots, v_k\}$. We recall $G' = G - (H - \{1\})$, and if $|G'| = r$, let $V(G') = \{1, u_2, \ldots, u_r\}$. Let us introduce the following notation. Row vector $\overline{[a]_q} = ([a_2]_q, \ldots, [a_k]_q)$, row vector $\overline{[f]_q} = ([f_2]_q, \ldots, [f_r]_q)$, column vector $\overline{[b]_q} = ([b_2]_q, \ldots, [b_k]_q)^t$ and column vector $\overline{[g]_q} = ([g_2]_q, \ldots, [g_r]_q)^t$. We also use $(M(i,j))$ to denote the matrix with entries $M(i,j)$ and various ranges of indices. We now verify that given the following block decompositions

$$qD_H = \begin{pmatrix} 0 & \overline{[a]_q} \\ \overline{[b]_q} & P \end{pmatrix} \text{ and } qD_{G'} = \begin{pmatrix} 0 & \overline{[f]_q} \\ \overline{[g]_q} & Q \end{pmatrix}$$

we can express

$$qD(G) = \left(\begin{array}{c|c|c} 0 & \overline{[a]_q} & \overline{[f]_q} \\ \overline{[b]_q} & P & ([b_i]_q + q^{b_i}[f_j]_q) \\ \overline{[g]_q} & ([g_i]_q + q^{g_i}[a_j]_q) & Q \end{array}\right)$$

We must verify that $[d_{i,j}]_q = [b_i]_q + q^{b_i}[f_j]_q$ when v_i, $i \neq 1$ is a vertex of H and v_j, $j \neq 1$ is a vertex of G'. Consider such a pair of vertices. Since v_1 is a cut-vertex separating H and G', the distances satisfy $d_{i,j} = d_{i,1} + d_{1,j}$ It follows from the fact that $[n+m]_q = [n]_q + q^n[m]_q$ that $[d_{i,j}]_q = [d_{i,1}]_q + q^{d_{i,1}}[d_{1,j}]_q$. However, by the block decomposition of qD_H, $[d_{i,1}]_q = [b_i]_q$; and by the block decomposition of $qD_{G'}$, $[d_{1,j}]_q = [f_j]_q$. We verify in the same manner that $[d_{i,j}]_q = [d_{i,1}]_q + q^{g_i}[a_j]_q$ when $i \neq 1$ labels a vertex of G' and $j \neq 1$ labels a vertex of H.

As operation $\sim$ preserves determinant, and by definition of $\widetilde{(qD_{G'})}_{1,1}$ and $\widetilde{(qD_H)}_{1,1}$, we have

$$\begin{aligned}\det(qD_G) &= \det(R \cdot qD_G \cdot C) = \det \left(\begin{array}{c|c|c} 0 & \overline{[a]_q} & \overline{[f]_q} \\ \overline{[b]_q} & P - ([b_i]_q + q^{b_i}[a_j]_q) & 0 \\ \overline{[g]_q} & 0 & Q - ([g_i]_q + q^{g_i}[f_j]_q) \end{array} \right) \\ &= \det \left(\begin{array}{c|c} 0 & \overline{[a]_q} \\ \overline{[b]_q} & P - ([b_i]_q + q^{b_i}[a_j]_q) \end{array} \right) \cdot \det(Q - ([g_i]_q + q^{g_i}[f_j]_q)) \\ &\quad + \det \left(\begin{array}{c|c} 0 & \overline{[f]_q} \\ \overline{[g]_q} & Q - ([g_i]_q + q^{g_i}[f_j]_q) \end{array} \right) \cdot \det(P - ([b_i]_q + q^{b_i}[a_j]_q)) \\ &= \det(\widetilde{qD_H}) \cdot \widetilde{(qD_{G'})}_{1,1} + \det(\widetilde{qD_{G'}}) \cdot \widetilde{(qD_H)}_{1,1} \\ &= \det(qD_H) \cdot \mathsf{qcofsum}_{qD_{G'}} + \det(qD_{G'}) \cdot \mathsf{qcofsum}_{qD_H}\end{aligned}$$

where the last line follows from Lemma 4, with the observation that ρ, τ restricted to the vertices of H, G' are as in Equation 1, with the dimensions of the restrictions of ρ, τ matching that of either $A = qD_H$ or $A = qD_{G'}$. Using Lemma 4 again, we note that

$$\begin{aligned}\mathsf{qcofsum}_{qD_G} &= \det \left(\begin{array}{c|c} P - ([b_i]_q + q^{b_i}[a_j]_q) & 0 \\ 0 & Q - ([g_i]_q + q^{g_i}[f_j]_q) \end{array} \right) \\ &= \det(P - ([b_i]_q + q^{b_i}[a_j]_q)) \cdot \det(Q - ([g_i]_q + q^{g_i}[f_j]_q)) \\ &= \widetilde{(qD_H)}_{1,1} \cdot \widetilde{(qD_{G'})}_{1,1} \\ &= \mathsf{qcofsum}_{qD_H} \cdot \mathsf{qcofsum}_{qD_{G'}}\end{aligned}$$

The proof is complete. ∎

We apply Theorem 2 to obtain a few known corollaries and some new ones as well. When $G = T$ is a tree, each block G_i is an edge (i.e. a K_2). It is simple to note that $\mathsf{qcofsum}_{G_i} = -(1+q)$ and $\det(D_{G_i}) = -1$. Thus, we get a q-analogue of Graham, Hoffman and Hosoya's result first observed by Bapat et. al [1, Corollary 5.2].

Corollary 2 (*Corollary 5.2, [1]*) *When G is a tree on n vertices, then* $\det(qD_G) = (-1)^{n-1}(n-1)(1+q)^{n-2}$.

When each block of G, is a 3-clique(i.e. a K_3), we get

$$D_{G_i} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

thus $\mathsf{qcofsum}_{G_i} = (1+2q)$ and $\det(D_{G_i}) = 2$. From this, we recover the following result of Sivasubramanian [4]. More generally, when each block of G is an r-clique (ie K_r), then $D_{G_i} = J - I$, where J is the matrix of all ones and I is the identity matrix, both of dimension $r \times r$. It is simple to check that $\mathsf{qcofsum}_{G_i} = (-1)^{r-1}[1 + (r-1)q]$ and $\det(D_{G_i}) = (-1)^{r-1}(r-1)$.

Corollary 3 *Let G have k blocks all of which are r-cliques (thus, G has $n = (r-1)k + 1$ vertices).*

- *When $r = 3$, $\det(qD_G) = 2k(1+2q)^{k-1}$. ([4, Corollary 3].)*
- *More generally for any r, $\det(qD_G) = (-1)^{n-1}[(r-1)\cdot k][1 + (r-1)q]^{k-1}$.*

3.2 Mod k distances, setting values to q

In this subsection, by setting values to q, we get a few pleasing corollaries about some modifications of the distance matrix of graphs, some of which seem non obvious.

If we set $q = -1$, then it is easy to check that for odd i, $[i]_q = 1$ and for even i, $[i]_q = 0$. Let G be a connected graph with distance matrix D_G and let qD_G be the q-analogue of D_G. If we set $q = -1$ in all entries of qD_G, this operation corresponds to considering the distance matrix D_G with all entries modulo 2.

Theorem 3 *Let G and H be graphs with an identical multiset of isomorphic blocks (they may differ in the tree structure of the connection among these blocks). Let D'_G and D'_H be the mod-2 distance matrices (where all distances are all considered modulo 2) of G and H respectively. Then $\det(D'_G) = \det(D'_H)$.*

Proof: Follows from Theorem 2 by setting $q = -1$. ∎

Corollary 4 *Let G be a tree and let D'_G be its mod-2 distance matrix where all distances are all considered modulo 2. Then D'_G is singular (ie $\det(D'_G) = 0$).*

We get the following pleasant mod-2 analogue of Corollary 3 for which simple proofs would be interesting.

Corollary 5 *Let G be a graph with k blocks, all of which are r-cliques (ie K_r's), and let D'_G be its mod-2 distance matrix (i.e. where each entry is considered modulo 2).*

- *If $r = 3$, $\det(D'_G) = 2k(-1)^{k-1}$.*
- *For a general r, $\det(D'_G) = (r-1)k(-r)^{n+k-2}$.*

Remark 1 *Theorem 3 answers the following question. Akin to determinant of the distance matrices of some graphs being equal, are there graphs such that the determinant of their adjacency matrices are identical? Since a mod-2 distance matrix has 0-1 entries, Theorem 3 gives families of graphs whose adjacency matrices have the same determinant. It would be interesting to see if there is some structure or some description of all or even a subset of the graphs which arise in this mod-2 manner from the distance matrix of graphs having an identical multiset of isomorphic blocks.*

Just as we set the value $q = -1$, we set other values to q and get further corollaries. The following corollary was suggested by the referee. For a positive integer k, let ζ be a primitive k-th root of unity. Clearly setting $q = \zeta$ corresponds to the following operation: replace each positive entry i in the distance matrix of G by $1 + \zeta + \cdots + \zeta^{(i \mod k)-1}$. Setting $q = -1$ corresponds to this operation with $k = 2$. Thus, we get the following.

Corollary 6 *Let G and H be graphs with an identical multiset of isomorphic blocks (they may differ in the tree structure of the connection among these blocks). For any fixed positive integer k, let ζ be a primitive k-th root of unity. Let D'_G and D'_H be the mod-k distance matrices of G and H respectively, where all positive distances i are replaced by $1 + \zeta + \cdots + \zeta^{i-1}$. Then* $\det(D'_G) = \det(D'_H)$.

3.3 $[kd]_q$-analogues

In this subsection, for any positive integer k, we consider kD_q analogues of D, where we replace positive integers i in D by $[ki]_q = 1 + q + q^2 + \cdots + q^{ki-1}$. Thus, we replace all entries $[i]_q$ in qD_G by $[ki]_q$ to get kD_q. It is easy to see that $[ki]_q = (1 + q^i + q^{2i} + \cdots + q^{(k-1)i})[i]_q$. Thus, if we define $[k]_{q^i}$ analogously as $1 + q^i + q^{2i} + \cdots + q^{(k-1)i}$, we get $[ki]_q = [k]_{q^i}[i]_q$. It can be checked that with weights $q^{k \cdot d_{u,v}}$ multiplying rsum_v, we get $\mathsf{qcofsum}^u_{kG}$, independent of vertex u. The proofs of all Lemmata and Theorem 2 in Subsection 3.1 go through as before. We omit the details and state the following result for trees in the case $k = 2$.

Corollary 7 *Let T be a tree on n vertices and let D be its distance matrix. Let $2D_q$ be the polynomial matrix obtained from D by replacing all entries i by $[2i]_q = 1+q+q^2+\cdots+q^{2i-1}$. Then,* $\det(2D_q) = (-1)^{n-1}(n-1)(1+q)^n(1+q^2)^{n-2}$.

Proof: Follows by observing that for $H = K_2$, $\det(2H_q) = -(1+q)^2$ and that $\mathsf{qcofsum}_{2H_q} = -(1+q^2)(1+q)$ ∎

We end with a question. Just as multiplying all entries of an $n \times n$ matrix by a factor α results in multiplication of its determinant by α^n, multiplying just the elements of a subset S with $|S| = k$ of the rows by α results in multiplication of its determinant by α^k. It would be interesting to see if for some distinct trees T_1, T_2, some subsets S_1, S_2 with $|S_1| = |S_2|$ exist such that the q-analogue of just the rows of S_i in T_i can be multiplied to get identical polynomials for the determinant of the distance matrix.

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