Elementary Proofs for Convolution Identities of Abel and Hagen–Rothe

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Abstract

By means of series–rearrangements and finite differences, elementary proofs are presented for the well–known convolution identities of Abel and Hagen–Rothe.

1 Introduction

There are numerous identities in mathematical literature. Among them, Newton's binomial theorem is well–known

$$\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = (x+y)^n.$$

Abel found its following deep generalization (cf. Comtet [6, §3.1] for example)

$$\sum_{k=0}^{n}\frac{a}{a+bk}\frac{(a+bk)^k}{k!}\frac{(c-bk)^{n-k}}{(n-k)!} = \frac{(a+c)^n}{n!}. \tag{1}$$

Another binomial identity is the Chu–Vandermonde convolution formula

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

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It has been generalized by Hagen and Rothe to the following one (cf. Chu [3,4], Gould [8] and Graham *et al* [10, §5.4])

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k} = \binom{a+c}{n}. \tag{2}$$

These convolution identities are fundamental in enumerative combinatorics. The reader can refer to Strehl [15] for a historical note. The existing proofs for the identities of Abel and Hagen–Rothe can be summarized as follows:

- The classical Lagrange expansion formula: Riordan [13, §4.5].
- Gould–Hsu Inverse series relations: Chu and Hsu [1,5].
- Generating function method: Gould [8,9] (see Chu [2] also).
- The Cauchy residue method of integral representation: Egorychev [7, §2.1].
- Lattice path combinatorics: Mohanty [11, §4.2] and Narayana [12, Appendix].
- Riordan arrays (which can trace back to Lagrange expansion): Sprugnoli [14].

However to our knowledge, there does not seem to have appeared really elementary proofs for these identities in classical combinatorics, even though this has long been desirable. By utilizing the standard method of series–rearrangement that was systematically used by Wilf [16], this short paper will present elementary proofs for the convolution identities of Abel and Hagen–Rothe. It may be unexpected that these proofs are surprisingly simple, which depend upon the following almost trivial fact that the finite differences of a polynomial results in zero if the polynomial degree is less than the order of differences.

2 Proofs of the Abel Formulae

According to the binomial theorem, we have

$$(c-bk)^{n-k} = \sum_{i=k}^{n}(-1)^{i-k}\binom{n-k}{i-k}(a+c)^{n-i}(a+bk)^{i-k}.$$

Consider the following double sum

$$\begin{aligned} U := &\sum_{k=0}^{n} \frac{a}{a+bk}\frac{(a+bk)^k}{k!}\frac{(c-bk)^{n-k}}{(n-k)!} \\ =&\frac{a}{n!}\sum_{k=0}^{n}\binom{n}{k}(a+bk)^{k-1}\sum_{i=k}^{n}(-1)^{i-k}\binom{n-k}{i-k}(a+c)^{n-i}(a+bk)^{i-k}. \end{aligned}$$

Interchanging the summation order and observing that

$$\binom{n}{k}\binom{n-k}{i-k}=\binom{n}{i}\binom{i}{k}$$

we get the following expression

$$U=\frac{a}{n!}\sum_{i=0}^{n}\binom{n}{i}(a+c)^{n-i}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}(a+bk)^{i-1}.$$

When $i>0$, the inner sum with respect to k vanishes because it results in the ith differences of the polynomial $(a+bx)^{i-1}$ of degree $i-1$. Therefore we have found that $U=\frac{(a+c)^n}{n!}$, which confirms exactly (1).

3 Proofs of Hagen–Rothe Identities

Analogously we have from the Chu–Vandermonde convolution

$$\binom{c-bk}{n-k}=\sum_{i=k}^{n}\binom{a+c}{n-i}\binom{-a-bk}{i-k}.$$

Then consider another double sum

$$\begin{aligned}V:=&\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k}\\=&\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\sum_{i=k}^{n}\binom{a+c}{n-i}\binom{-a-bk}{i-k}.\end{aligned}$$

Interchanging the summation order and observing that

$$\frac{a}{a+bk}\binom{a+bk}{k}\binom{-a-bk}{i-k}=\frac{(-1)^{i-k}\,a}{a+bk-k+i}\binom{i}{k}\binom{a+bk-k+i}{i}$$

we get another double sum expression

$$V=\sum_{i=0}^{n}\binom{a+c}{n-i}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\frac{a}{a+bk-k+i}\binom{a+bk-k+i}{i}.$$

When $i>0$, the inner sum with respect to k becomes zero because it results again in the finite differences of a polynomial with the polynomial degree less than the difference order by one. Consequently we have shown that $V=\binom{a+c}{n}$, which is equivalent to (2).

For the identities displayed in (1) and (2), their linear combinations yield the following respective symmetric forms

$$\sum_{k=0}^{n}\frac{a}{a+bk}\frac{(a+bk)^k}{k!}\frac{c-bn}{c-bk}\frac{(c-bk)^{n-k}}{(n-k)!}=\frac{a+c-bn}{a+c}\frac{(a+c)^n}{n!},$$
$$\sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\frac{c-bn}{c-bk}\binom{c-bk}{n-k}=\frac{a+c-bn}{a+c}\binom{a+c}{n}.$$

The approach presented here can also be employed to prove them similarly. The details are left to the reader as exercises.

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