Bijection between bigrassmannian permutations maximal below a permutation and its essential set

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Abstract

Bigrassmannian permutations are known as permutations which have precisely one left descent and one right descent. They play an important role in the study of Bruhat order. Fulton introduced the essential set of a permutation and studied its combinatorics. As a consequence of his work, it turns out that the essential set of bigrassmannian permutations consists of precisely one element. In this article, we generalize this observation for essential sets of arbitrary permutations. Our main theorem says that there exists a bijection between bigrassmanian permutations maximal below a permutation and its essential set. For the proof, we make use of two equivalent characterizations of bigrassmannian permutations by Lascoux-Schützenberger and Reading.

1 Introduction

Bigrassmannian elements play an important role in study of the Bruhat order on Coxeter groups. They are known as elements which have precisely one left descent and one right descent. In particular, in the symmetric group (type A), bigrassmannian permutations have two other equivalent characterizations, one as join-irreducible permutations and one as monotone triangles with some minimal condition (we will see detail of these in Fact 2.6). Here let us recall the definitions of join and join-irreducibility from poset theory.

Definition 1.1. Let $(P, \leqslant)$ be a finite poset and $Q \subseteq P$. Then consider the set

$$\{x \in P \mid x \geqslant y \text{ for all } y \in Q\}.$$

If this set has a unique minimal element, we call it the *join* of Q denoted by $\vee Q$. Define the *meet* of Q ($\wedge Q$) order dually. P is said to be a *lattice* if $\vee Q$ and $\wedge Q$ exist for all Q. We say that $x \in P$ is *join-irreducible* if whenever $x = \vee Q$, then $x \in Q$.

It is important to note that in a finite poset P, we can write each $x \in P$ as the join of some subsets of join-irreducible elements of P [7, Proposition 9]. Note also that if $\vee Q$ exists, then $\vee Q = \vee \operatorname{Max}(Q)$ where Max means the set of maximal elements of Q.

Unfortunately, the symmetric group S_n with Bruhat order is not a lattice. However, as already mentioned, a permutation is bigrassmannian if and only if it is join-irreducible. Consequently, each $x \in S_n$ is the join of some subsets of bigrassmannian permutations. More precisely, define

$$B(x) = \{w \in S_n \mid w \leqslant x \text{ and } w \text{ is bigrassmannian}\}.$$

Then we obtain $x = \vee B(x)$ in S_n. However, it is not easy to see $B(x)$ (and $\operatorname{Max} B(x)$) from the usual definition of Bruhat order. Instead we will make use of the set of monotone triangles (say $L(S_n)$) because there is a natural identification of permutations with monotone triangles. It is also helpful that $L(S_n)$ has the partial order which coincides with Bruhat order all over S_n. Furthermore, it is much easier to say which monotone triangle is larger or smaller (or incomparable). It helps us find $B(x)$ and $\operatorname{Max} B(x)$. We will discuss detail in Section 2.

Fulton [5] introduced the *essential set* of a permutation as the set of southeast corners of the diagram of the permutations. As the name "essential" suggests, it is a combinatorial object which completely determines a permutation. Eriksson-Linusson studied its combinatorics for 321-avoiding, vexillary permutations and some enumeration [3, 4]. There is a less-known but interesting property of bigrassmannians described in terms of essential sets (Section 3).

Fact 1.2. If w is bigrassmannian, then its essential set consists of the only one element.

As far as the author is aware, it appears only as a consequence of [5, Proposition 9.18]. In fact, the converse is also true, which we will obtain as a consequence of the main theorem in Section 4. Notice that we can rephrase "w is bigrassmannian" as "$\operatorname{Max} B(w)$ consists of the only one element (w itself)". From this point of view, the main theorem gives more general aspect to Fact 1.2 for even arbitrary permutations.

Theorem. *Let $x \in S_n$ and $\operatorname{Max} B(x)$ the set of bigrassmannian permutations maximal weakly below x in Bruhat order, and $\operatorname{Ess}(x)$ the essential set of x. Then there exists a bijection between $\operatorname{Max} B(x)$ and $\operatorname{Ess}(x)$.*

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2 Bigrassmannian permutations and monotone triangles

Throughout this article, we treat the Bruhat order on the symmetric groups as the suborder induced by the set of all monotone triangles (for the usual definition of the Bruhat order in context of Coxeter groups, see [2, Chapter 2]). This section discusses relation between this suborder and bigrassmannian (join-irreducible) permutations.

Definition 2.1. For $w \in S_n$, define the set of *left* and *right descents* as

$$D_L(w) = \{1 \leqslant i \leqslant n-1 \mid w^{-1}(i) > w^{-1}(i+1)\},$$
$$D_R(w) = \{1 \leqslant i \leqslant n-1 \mid w(i) > w(i+1)\}.$$

We say that w is *bigrassmannian* if $\#D_L(w) = \#D_R(w) = 1$. Define

$$B(w) = \{x \in S_n \mid x \leqslant w \text{ and } x \text{ is bigrassmannian}\}.$$

Example 2.2. 34512 is bigrassmannian, but 42513 is not.

As mentioned in introduction, there is an equivalent characterization of bigrassmannian permutations in terms of join-irreducicble monotone triangles. We recall the definition of monotone triangles following [7].

Definition 2.3. A *monotone triangle* x of order n is an $n(n-1)/2$-tuple $(x_{ab} \mid 1 \leqslant b \leqslant a \leqslant n-1)$ such that $1 \leqslant x_{ab} \leqslant n, x_{ab} < x_{a,b+1}, x_{ab} \geqslant x_{a+1,b}$ and $x_{ab} \leqslant x_{a+1,b+1}$ for all a, b. Regard a permutation $x \in S_n$ as a monotone triangle of order n as follows: For each $1 \leqslant a \leqslant n-1$, let $x_{a1}, x_{a2}, \ldots, x_{aa}$ be positive integers such that $\{x(1), x(2), \ldots, x(a)\} = \{x_{a1}, x_{a2}, \ldots, x_{aa}\}, x_{ab} < x_{a,b+1}$ for all $1 \leqslant b \leqslant a-1$. Then $x = (x_{ab})$ is a monotone triangle. Denote by $L(S_n)$ the set of all monotone triangles of order n. We introduce the partial order on $L(S_n)$ as $x \leqslant y \iff x_{ab} \leqslant y_{ab}$ for all a, b.

Sometimes it is convenient to define $x_{nb} = b$ for all x since n-th row is always $1 < 2 < \cdots < n$, but we usually omit it.

Example 2.4. As a monotone triangle of order 5, we have

$$34512 = \begin{matrix} 3 & & & \\ 3 & 4 & & \\ 3 & 4 & 5 & \\ 1 & 3 & 4 & 5 \end{matrix} \quad \text{and} \quad 42513 = \begin{matrix} 4 & & & \\ 2 & 4 & & \\ 2 & 4 & 5 & \\ 1 & 2 & 4 & 5. \end{matrix}$$

Remark 2.5.

(1) In fact, $L(S_n)$ is a smallest lattice containing S_n (MacNeille completion of S_n). Moreover, for $x \in L(S_n)$, x is join-irreducible in S_n if and only if so is x in $L(S_n)$. See [7, Section 6].

(2) In particular, for permutations $x, y \in S_n$, we have $x \leqslant y$ in Bruhat order if and only if $x \leqslant y$ as monotone triangles (called *tableaux criterion* [1]).

Now we observe equivalent characterizations of bigrassmannian permutations (as mentioned in introduction) and important consequences.

Fact 2.6. For $w \in S_n$, the following are equivalent:

(1) w is bigrassmannian.

(2) w is join-irreducible.

(3) There exist positive integers (a, b, c) such that $1 \leqslant b \leqslant a \leqslant n-1$, $b+1 \leqslant c \leqslant n-a+b$ and $w = J_{abc}$ where J_{abc} is the componentwise smallest monotone triangle such that a, b entry is $\geqslant c$.

Proof. See [6, Théorème 4.4] for (1) $\iff$ (2) and [7, Section 8] for (2) $\iff$ (3). $\square$

Thanks to Remark 2.5 and Fact 2.6, for $x \in L(S_n)$, the symbol $B(x)$ still makes sense as the set of join-irreducible elements weakly below x in $L(S_n)$. Note that J_{abc} is the monotone triangle satisfying the following: For $x \in L(S_n)$, we have

$$c \leqslant x_{ab} \iff J_{abc} \leqslant x \iff J_{abc} \in B(x)$$

because of minimality of J_{abc}. This observation is useful to identify $J_{abx_{ab}} \in B(x)$ with the entry x_{ab} showing up in the monotone triangle x.

Next we must understand how to choose maximal elements from $B(x)$. The following proposition describes all covering relations among bigrassmannian permutations.

Proposition 2.7. *For each J_{abc}, define*

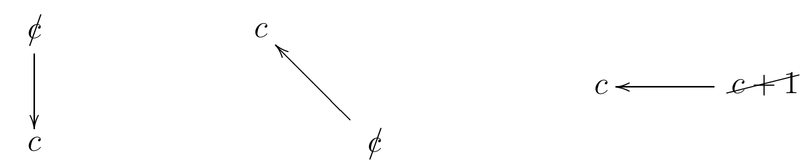
$$\begin{aligned}
C_1(J_{abc}) &= J_{a,b-1,c-1} \text{ if } b \geqslant 2,\\
C_2(J_{abc}) &= J_{a+1,b,c} \quad \text{ if } c \leqslant n-a+b-1,\\
C_3(J_{abc}) &= J_{a-1,b-1,c} \text{ if } b \geqslant 2,\\
C_4(J_{abc}) &= J_{a,b,c+1} \quad \text{ if } c \leqslant n-a+b-1.
\end{aligned}$$

In the poset of all bigrassmannian permutations in S_n, J_{abc} is covered by $C_i(J_{abc})$ if and only if $C_i(J_{abc})$ is a valid monotone triangle. More specifically, (i) *J_{abc} is covered by $C_1(J_{abc})$ and $C_3(J_{abc})$ if and only if $b \geqslant 2$.* (ii) *J_{abc} is covered by $C_2(J_{abc})$ and $C_4(J_{abc})$ if and only if $c \leqslant n-a+b-1$.* (iii) *Furthermore, no other elements cover J_{abc} (and hence at most four elements cover J_{abc}).*

Proof. Reading described all covering relations in bigrassmannian permutations by certain triples $\{(i, j, k)\}$ which are another (equivalent) expressions of $\{J_{abc}\}$ with the relation $a = j-i+k, b = j-i+1$ and $c = j+1$. For detail, see [7, p.91-94]. $\square$

In particular, the covering relation of type C_4 tells us that it is easy to compare two bigrassmannian permutations at the same position (a, b) as $J_{abc} < J_{abd} \iff c < d$. Hence in order to choose maximal elements from $B(x)$ (here x need not be a permutation. It may be a general monotone triangle), it is enough to determine maximal elements of $\{J_{abx_{ab}} \mid 1 \leqslant b \leqslant a \leqslant n-1\}$. By Proposition 2.7 and the above observation on type C_4, we only need to check whether $C_i(J_{abx_{ab}})(i = 1, 2, 3)$ is in $B(x)$ or not. The procedure of finding $\operatorname{Max} B(x)$ is as follows: First, write entries of monotone triangle of x. Second, cross out all entries such that $x_{ab} = b$. Third, following three types of covering relations C_1, C_2, C_3, cross out non-maximal elements accordingly under the identification $x_{ab} \longleftrightarrow J_{abx_{ab}}$ as in Figure 1. For example, if there exists c such that $x_{ab} = c = x_{a+1,b}$, then draw an arrow from the upper c to the lower c to mean $J_{abc} \lhd J_{a+1,b,c}$. Then cross out the upper c. In the same way, check all other possible covering relations. Survived entries (circled in Figure 2) correspond to maximal elements of $B(x)$.

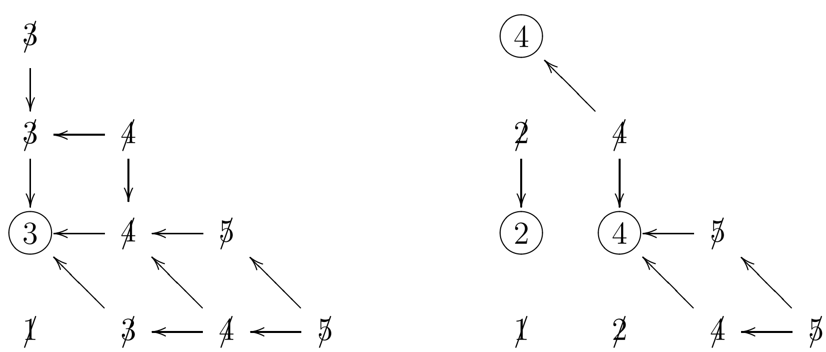
Figure 1: Crossing out non-maximal $J_{abx_{ab}}$



$J_{abc} \lhd J_{a+1,b,c}$ $\quad$ $J_{a+1,b+1,c} \lhd J_{abc}$ $\quad$ $J_{a,b+1,c+1} \lhd J_{a,b,c}$

Example 2.8. Figure 2 below illustrates $\operatorname{Max} B(34512) = \{J_{313}\}$ (bigrassmannian) and $\operatorname{Max} B(42513) = \{J_{114}, J_{312}, J_{324}\}$.

Figure 2: Choosing maximal elements from $B(x)$



3 Bigrassmannian permutations and essential set

This section discusses another property of bigrassmannian permutaitons described in terms of the essential set. Their essential sets consist of only one element.

Definition 3.1. Let $w \in S_n$.

(1) The *diagram* of w is $D(w) = \{(i,j) \mid 1 \leqslant i, j \leqslant n-1, i < w^{-1}(j), j < w(i)\}$.

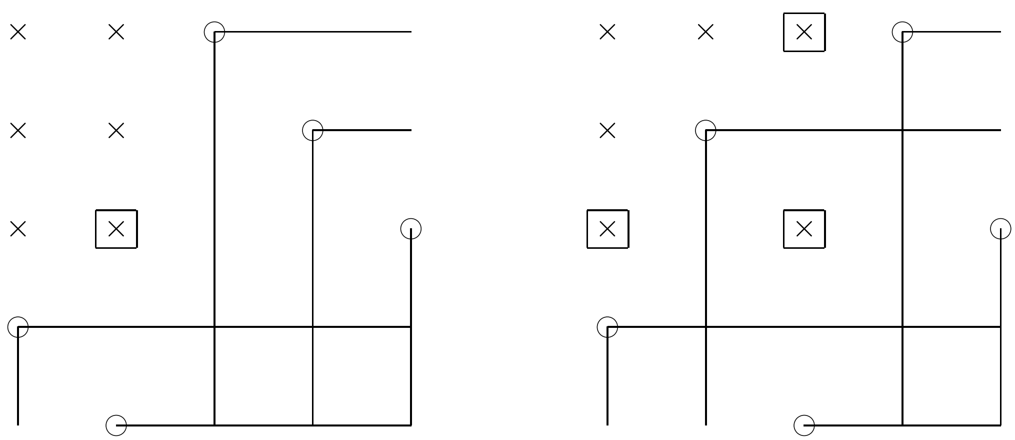
(2) The *essential set* of w (a subset of $D(w)$) is

$$\mathrm{Ess}(w) = \{(i,j) \mid i < w^{-1}(j), j < w(i), w(i+1) \leqslant j, w^{-1}(j+1) \leqslant i\}.$$

We draw a picture of $D(w)$ and $\mathrm{Ess}(w)$ as follows: take an $n \times n$ matrix. Plot $(i, w(i))\,(1 \leqslant i \leqslant n)$ in the matrix (indicated by $\bigcirc$ as in Figure 3). Then kill all cells right or below of these. The survived cells ($\times$) are elements of $D(w)$. The set of all points ($\boxed{\times}$) at southeast corner of $D(w)$ is $\mathrm{Ess}(w)$.

Example 3.2. Figure 3 below shows that $\mathrm{Ess}(34512) = \{(3,2)\}$ (bigrassmannian) and $\mathrm{Ess}(42513) = \{(1,3),(3,1),(3,3)\}$.

Figure 3: Ess(34512) and Ess(42513)



Fact 3.3. Let $w \in S_n$. If w is bigrassmannian, then $\#\mathrm{Ess}(w) = 1$.

Proof. See [5, Proposition 9.18]. $\square$

In other words, if $\#\mathrm{Max}\, B(w) = 1$, then $\#\mathrm{Ess}(w) = 1$ as discussed after Fact 1.2. We will see that the converse is also true as a cosequence of the main theorem.

4 Main theorem

This section gives a proof of main theorem.

Theorem. *For all $x \in S_n$, there exists a bijection between* $\mathrm{Max}\, B(x)$ *and* $\mathrm{Ess}(x)$.

Note that we may assume that x is not the identity permutation since

$$\operatorname{Max} B(x) = \varnothing \iff x \text{ is the identity permutation } \iff \operatorname{Ess}(x) = \varnothing.$$

The proof of Theorem follows from the following two lemmas.

Lemma 4.1. *Let $x \in S_n$. For each $1 \leqslant b \leqslant a \leqslant n-1$, x_{ab} is equal to either $x_{a+1,b}$ or $x_{a+1,b+1}$. Moreover,*

$$\begin{array}{lllll} x_{ab} = x_{a+1,b} & \iff & x(a+1) > x_{ab} & \iff & x(a+1) \geqslant x_{a+1,b+1}, \\ x_{ab} = x_{a+1,b+1} & \iff & x(a+1) < x_{ab} & \iff & x(a+1) \leqslant x_{a+1,b}. \end{array}$$

Proof. Clear by the construction of monotone triangles from permutations (see Definition 2.3). ☐

Lemma 4.2. *Let $x \in S_n$. For $1 \leqslant a, c \leqslant n-1$, set*

$$b = b(a,c) = \#\{a' \mid 1 \leqslant a' \leqslant a \text{ and } x(a') \leqslant c+1\}.$$

Then $(a,c) \in \operatorname{Ess}(x) \iff J_{a,b,c+1} \in \operatorname{Max} B(x)$.

Before the proof, we make several comments. Let $y = J_{a,b,c+1}$. By Proposition 2.7, y is covered by at most four elements in the set of bigrassmannian permutations. Therefore in order to show $y \in \operatorname{Max} B(x)$, we need to verify all of the following five statements:

$$\begin{array}{lll} \text{(a) } C_1(y) = J_{a,b-1,c} \not\leqslant x & \text{or equivalently} & x_{a,b-1} \leqslant c-1 \\ \text{(b) } C_2(y) = J_{a+1,b,c+1} \not\leqslant x & \text{or equivalently} & x_{a+1,b} \leqslant c \\ \text{(c) } C_3(y) = J_{a-1,b-1,c+1} \not\leqslant x & \text{or equivalently} & x_{a-1,b-1} \leqslant c \\ \text{(d1) } C_4(y) = J_{a,b,c+2} \not\leqslant x & \text{or equivalently} & x_{ab} \leqslant c+1 \\ \text{(d2) } y \in B(x) & \text{or equivalently} & x_{ab} \geqslant c+1 \end{array}$$

Note that if some of $C_i(y)$ do not exist, then $C_i(y) \not\leqslant x$ is vacuously true. Hence to prove Lemma 4.2 (and Theorem), it is enough to show that for each $1 \leqslant a, c \leqslant n-1$, the statements (i)-(iv) below are equivalent to the statements (a)-(d) below:

$$\left.\begin{array}{l} \text{(i) } a < x^{-1}(c) \\ \text{(ii) } c < x(a) \\ \text{(iii) } x(a+1) \leqslant c \\ \text{(iv) } x^{-1}(c+1) \leqslant a \end{array}\right. \iff \left.\begin{array}{l} \text{(a) } x_{a,b-1} \leqslant c-1 \\ \text{(b) } x_{a+1,b} \leqslant c \\ \text{(c) } x_{a-1,b-1} \leqslant c \\ \text{(d) } x_{ab} = c+1 \end{array}\right.$$

where $b = b(a,c)$ is as in the statement of Lemma 4.2.
The key of the proof is to consider repeated entries in the monotone triangles of permutations. In the proof below, for convenience, by $x[a]$ we will mean entries of a-th row of x

$$x[a] = \{x_{a1}, x_{a2}, \ldots, x_{aa}\} = \{x(1), x(2), \ldots, x(a)\}.$$

Proof. ($\Longrightarrow$): Suppose (i)-(iv). First, (iv) implies $c+1 \in x[a]$. Then by the very definition of b, we have $x_{ab} = c+1$. Second, (i) implies that $c \notin x[a]$. That is, c does not appear in $x[a]$ while $c+1$ appears in $x[a]$ as $x_{ab} = c+1$ as just seen. Hence $x_{a,b-1} \leqslant c-1$. Third, to see (b), recall from Lemma 4.1 that x_{ab} is equal to $x_{a+1,b}$ or $x_{a+1,b+1}$. Now (iii) tells us that $x(a+1) \leqslant c < x_{ab}$, and hence we must have (the second case of Lemma 4.1) $c+1 = x_{ab} = x_{a+1,b+1}$. Thus $x_{a+1,b} \leqslant c$. Fourth, apply Lemma 4.1 to $x_{a-1,b-1}$. Since $x(a) > c$, i.e., $x(a) \geqslant c+1 = x_{ab}$, we have the first case $x_{a-1,b-1} = x_{a,b-1}$. Since we know $x_{ab} = c+1$, we obtain $x_{a,b-1} \leqslant c$.
($\Longleftarrow$): Suppose (a)-(d). First, (iv) follows immediately from $x_{ab} = c+1$, i.e., $c+1 \in x[a]$. Second, $x_{a,b-1} \leqslant c-1$ and $x_{ab} = c+1$ imply that $c \notin x[a]$. Thus $a < x^{-1}(c)$. Third, since $x_{a-1,b-1} \leqslant c-1$ and $x_{ab} = c+1$, we see $x_{a-1,b-1} \neq x_{ab}$. Then we have the first case of Lemma 4.1 $x_{a-1,b-1} = x_{a,b-1}$. Thus $x(a) \geqslant x_{ab} = c+1 > c$. Fourth, in the same way, since $x_{ab} = c+1$ and $x_{a+1,b} \leqslant c$, we have $x_{ab} \neq x_{a+1,b}$. The first case of Lemma 4.1 implies that $x(a+1) < x_{ab} = c+1$, i.e., $x(a+1) \leqslant c$.

We completed the proof of Lemma 4.2 and Theorem. $\square$

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