Loose Hamilton Cycles in Random 3-Uniform Hypergraphs

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Abstract

In the random hypergraph $H = H_{n,p;3}$ each possible triple appears independently with probability p. A *loose* Hamilton cycle can be described as a sequence of edges $\{x_i, y_i, x_{i+1}\}$ for $i = 1, 2, \ldots, n/2$ where $x_1, x_2, \ldots, x_{n/2}, y_1, y_2, \ldots, y_{n/2}$ are all distinct. We prove that there exists an absolute constant $K > 0$ such that if $p \geqslant \frac{K \log n}{n^2}$ then

$$\lim_{\substack{n \to \infty \\ 4|n}} \mathbf{Pr}(H_{n,p;3} \text{ contains a loose Hamilton cycle}) = 1.$$

1 Introduction

The threshold for the existence of Hamilton cycles in the random graph $G_{n,p}$ has been known for many years, see [7], [1] and [3]. There have been many generalisations of these results over the years and the problem is well understood. It is natural to try to extend these results to Hypergraphs and this has proven to be difficult. The famous Pósa lemma fails to provide any comfort and we must seek new tools. In the graphical case, Hamilton cycles and perfect matchings go together and our approach will be to build on the deep and difficult result of Johansson, Kahn and Vu [6], as well as what we have learned from the graphical case.

A k-uniform Hypergraph is a pair $H = (V, E)$ where $E \subseteq \binom{V}{k}$. We say that a k-uniform sub-hypergraph C of H is a Hamilton cycle of type ℓ, for some $1 \leqslant \ell \leqslant k$, if there exists a cyclic ordering of the vertices V such that every edge consists of k consecutive vertices

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and for every pair of consecutive edges E_{i-1}, E_i in C (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. When $\ell = k - 1$ we say that C is a *loose* Hamilton cycle and in this paper we will restrict our attention to loose Hamilton cycles in the random 3-uniform hypergraph $H = H_{n,p;3}$. In this hypergraph, $V = [n]$ and each of the $\binom{n}{3}$ possible edges (triples) appears independently with probability p. While n needs to be even for H to contain a loose Hamilton cycle, we need to go one step further and assume that n is a multiple of 4. Extensions to other k, ℓ and $n = 2 \mod 4$ pose problems. We will prove the following theorem:

Theorem 1 *There exists an absolute constant $K > 0$ such that if $p \geqslant \frac{K \log n}{n^2}$ then*

$$\lim_{\substack{n \to \infty \\ 4 | n}} \mathbf{Pr}(H_{n,p;3} \text{ contains a loose Hamilton cycle}) = 1.$$

Thus $\frac{\log n}{n^2}$ is the threshold for the existence of loose Hamilton cycles, at least for n a multiple of 4. This is because if $p \leqslant \frac{(1-\epsilon) \log n}{2n^2}$ and $\epsilon > 0$ is constant, then **whp**[1] $H_{n,p;3}$ contains isolated vertices.

The proof of Theorem 1 will follow fairly easily from the following three theorems.

We start with a special case of the theorem of [6]: Let X and Y be a disjoint sets. Let $\Omega = \binom{X}{2} \times Y$. Let $\Gamma = \Gamma(X, Y, p)$ be the random 3-uniform hypergraph where each triple in Ω is independently included with probability p. Assuming that $|X| = 2|Y| = 2m$, a perfect matching of Γ is a set of m triples (x_{2i-1}, y_i, x_{2i}), $i = 1, 2, \ldots, m$ such that $X = \{x_1, \ldots, x_{2m}\}$ and $Y = \{y_1, \ldots, y_m\}$.

Theorem 2 *[6]*
There exists an absolute constant $K > 0$ such that if $p \geqslant \frac{K \log n}{n^2}$ then **whp** Γ *contains a perfect matching.*

This version is not actually proved in [6], but can be obtained by straightforward changes to their proof.

Our next theorem concerns *rainbow* Hamilton cycles in random regular graphs. If we edge colour a graph then a set S of edges is rainbow if all edges in S are a different colour. Janson and Wormald [5] proved the following: Let G_{2r} be a random $2r$-regular multi-graph on vertex set $[n]$. The distribution is not uniform, it is the one induced by the configuration model, see e.g. Bollobás [2]. We can condition on there being no loops.

Theorem 3 *If the edges of G_{2r} are coloured randomly with n colours so that each colour is used exactly r times, $r \geqslant 4$, then* **whp** *it contains a rainbow Hamilton cycle.*

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[1] An event $\mathcal{E}_n$ occurs *with high probability*, or **whp** for brevity, if $\lim_{n \to \infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

(This of course implies the result for random $2r$-regular graphs).

We partition $[n = 4m]$ into $X = [2m]$ and $\bar{X} = [2m+1, n]$. The (multi-)graph G^* has vertex set X and an edge (x, x') of colour y if (x, y, x') is an edge of H. If G^* contains a rainbow Hamilton cycle, then H contains a loose Hamilton cycle. We will use Theorem 2 to show that **whp** G^* contains an edge coloured graph that is close to satisfying the conditions of Theorem 3.

There is a minor technical point in that we can only use Theorem 2 to prove the existence of a randomly coloured (multi-)graph Γ_{2r} that is the union of $2r$ independent matchings. Fortunately,

Theorem 4 Γ_{2r} *is* contiguous *to* G_{2r}

By this we mean that if $\mathcal{P}_n$ is some sequence of (multi-)graph properties, then

$$\Gamma_{2r} \in \mathcal{P}_n \text{ \textbf{whp}} \iff G_{2r} \in \mathcal{P}_n \text{ \textbf{whp}}. \tag{1}$$

Theorem 4 is proved in Janson [4] (Theorem 11) and in Molloy, Robalewska-Szalat, Robinson and Wormald [8].

2 Proof of Theorem 1

We begin by letting $\mathcal{Y}$ be a set of size $2rm$ consisting of $r = O(1)$ copies $y_1, y_2, \ldots, y_r$ of each $y \in \bar{X}$. We will later fix r at 4, but we leave it unspecified for now. Next let $Y_1, Y_2, \ldots, Y_{2r}$ be a uniformly random partition of $\mathcal{Y}$ into $2r$ sets of size m.

Define p_1 by $p = 1 - (1 - p_1)^{2r}$. With this choice, we can generate $H_{n,p;3}$ as the union of $2r$ independent copies of $H_{n,p_1;3}$. Similarly, define p_2 by $p_1 = 1 - (1 - p_2)^r$.

Viewing $H_{n,p_1;3}$ as the union of r independent copies $H_1, H_2, \ldots, H_r$ of $H_{n,p_2;3}$ we can couple $\Gamma(X, Y_j, p_1)$ with a subgraph of $H_{n,p_1;3}$ by placing (x, y, x') in $E(H_i)$ whenever $(x, y_i, x') \in E(\Gamma(X, Y_j, p_1))$. It follows from Theorem 2 that **whp** $\Gamma(X, Y_j, p_1)$ contains a perfect matching M_j. (We need the split into r copies of $H_{n,p_2;3}$ to allow a "colour" to appear several times in a matching).

Now each perfect matching M_j gives rise to an edge-coloured perfect matching M_j^* of G^* where (x, y_i, x') gives rise to an edge (x, x') of colour y. By symmetry, these matchings are uniformly random and they are independent by construction. Also the edges have been randomly coloured so that each colour appears exactly r times. Indeed to achieve such a random colouring we can take any partition of the edge set of $M_1^* \cup M_2^* \cup \cdots \cup M_{2r}^*$ into $2r$ sets $S_1, S_2, \ldots, S_{2r}$ of size m and then colour the edges by using random bijections from $Y_j \rightarrow S_j$ for $j = 1, 2, \ldots, 2r$.

We apply Theorems 3 and 4 to finish the proof. For a $2r$-regular graph G let Ω_G denote the set of equitable edge colourings of G. By equitable, we mean that each colour is used r

times. Suppose that σ is chosen uniformly from Ω_G and $\pi_G = \mathbf{Pr}(\mathcal{R})$ where $\mathcal{R}$ is the event that there is **no** rainbow Hamilton cycle. Theorem 3 can be expressed as follows: Let $\mathcal{G}_{2r}$ denote the set of $2r$-regular loopless multi-graphs with vertex set $[n]$ and configuration distribution κ_G. Then,

$$\sum_{G \in \mathcal{G}_{2r}} \kappa_G \pi_G \leqslant \frac{1}{\omega} \tag{2}$$

where $\omega \to \infty$ as $n \to \infty$. The event $\mathcal{P}_n$ of (1) can now be defined:

$$\mathcal{P}_n = \left\{ \pi_G \leqslant \frac{1}{\omega^{1/2}} \right\}.$$

Think of π_G as a random variable for G chosen from $\mathcal{G}_{2r}$. Then (2) states that $\mathbf{E}(\pi_G) \leqslant 1/\omega$. The Markov inequality then implies that $\mathbf{Pr}(\pi_G \geqslant 1/\omega^{1/2}) \leqslant 1/\omega^{1/2}$ and so $\mathbf{Pr}(\mathcal{P}_n) \geqslant 1 - 1/\omega^{1/2}$ and this completes the proof of Theorem 1.

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