A New Approach to the Dyson Coefficients

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Abstract

In this paper, we introduce a direct method to evaluate the Dyson coefficients.

1 Introduction

In 1962, Dyson [2] conjectured the following constant term identity.

Theorem 1.1 (Dyson's Conjecture)**.** *For nonnegative integers* $a_1, a_2, \ldots, a_n$*,*

$$\operatorname*{CT}_{\mathbf{x}} D_n(\mathbf{x}, \mathbf{a}) = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!\, a_2! \cdots a_n!},$$

where $\mathrm{CT}_{\mathbf{x}} f(\mathbf{x})$ *denotes the constant term and*

$$D_n(\mathbf{x}, \mathbf{a}) := \prod_{1 \leqslant i \neq j \leqslant n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}. \qquad (\textit{Dyson product})$$

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Dyson's conjecture was proved independently by Gunson [5] and Wilson [11]. In 1970, a brief and elegant proof was published by Good [4]. Later Zeilberger [13] gave a combinatorial proof.

The q-analog of Theorem 1.1 was conjectured by Andrews [1] in 1975, and was first proved, combinatorially, by Zeilberger and Bressoud [14]. Recently, Gessel and Xin [3] gave a different proof by using properties of formal Laurent series.

In recent years, there has been increasing interest in evaluating the coefficients of monomials $M := \prod_{i=1}^{n} x_i^{b_i}$, where $\sum_{i=1}^{n} b_i = 0$, in the Dyson product. Based on Good's proof, Kadell [6] gave three non-constant term coefficients. Sills and Zeilberger [10] described an algorithm that automatically conjectures and proves closed-form expressions. Later, Sills [9] extended Good's idea and obtained the closed-form expressions for M being $\frac{x_s}{x_r}$, $\frac{x_s x_t}{x_r^2}$, $\frac{x_t x_u}{x_r x_s}$, respectively. By virtue of Zeilberger and Sills' Maple package `GoodDyson`, Lv, Xin and Zhou [7] found two closed-form expressions for M that has a square in the numerator. Moreover, by generalizing Gessel-Xin's method [3] for proving the Zeilberger-Bressoud q-Dyson Theorem, Lv, Xin and Zhou [8] established a family of q-Dyson style constant term identities.

In this note, we propose a direct calculation approach to evaluating the coefficients in the Dyson product, and illustrate this approach through the case of $M = x_r^2/x_s^2$. The applications of our method to other cases like $M = \frac{x_r^2}{x_s x_t}$, $M = \frac{x_r}{x_s}$ are analogous, and thus omitted. More explicitly, we will show that our approach leads to the following theorem.

Theorem 1.2 (Theorem 1.2 [7]). *Let r and s be distinct integers with $1 \leqslant r, s \leqslant n$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_s^2}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = \frac{a_r}{(1+a^{(r)})(2+a^{(r)})} \left[(a_r - 1) - \sum_{\substack{i=1 \\ i \neq r,s}}^{n} \frac{a_i(1+a)}{(1+a^{(r)} - a_i)} \right] C_n(\mathbf{a}), \tag{1.1}$$

where $a := a_1 + a_2 + \cdots + a_n$, $a^{(j)} := a - a_j$ and $C_n(\mathbf{a}) := \frac{(a_1+a_2+\cdots+a_n)!}{a_1!\, a_2! \cdots a_n!}$.

2 A New Approach to Theorem 1.2

In this section, we will deduce the coefficient for $M = \frac{x_r^2}{x_s^2}$. By induction on n, we have the following identity,

$$\sum_{k=2}^{n} \frac{(m+k-1)!}{(k-2)!} = \frac{(m+n)!}{(m+2)(n-2)!}, m, n \in \mathbb{N}. \tag{2.1}$$

Let

$$\Delta(x_1, x_2, \ldots, x_n) := \prod_{i<j} (x_i - x_j) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

be the Vandermonde determinant in $x_1, x_2, \ldots, x_n$. Then [12] presents the following result.

Lemma 2.1 (Lemma 1-2.12, [12]). *For each $i = 1, 2, \ldots, n$, if $f(x_i) \in \mathbb{C}((x_i))$, then we have*

$$\partial_{x_2}\partial_{x_3}\cdots\partial_{x_n} f(x_1) = \Delta(x_1, x_2, \ldots, x_n)^{-1} \begin{vmatrix} f(x_1) & f(x_2) & \cdots & f(x_n) \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} \tag{2.2}$$

$$= \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j\neq i}(x_i - x_j)}, \tag{2.3}$$

where $\partial_a f(x) := \frac{f(x)-f(a)}{x-a}$.

The following lemma is vital to our approach.

Lemma 2.2 (Main Lemma). *For $n \geqslant 2$, we have*

$$\frac{V_1}{x_1} + \frac{V_2}{x_2} + \cdots + \frac{V_n}{x_n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}, \tag{2.4}$$

where $V_m := \prod_{\substack{i=1 \\ i\neq m}}^{n} \left(1 - \frac{x_m}{x_i}\right)^{-1}$ *for* $m = 1, 2, \ldots, n$.

Proof. Let $f(x_i) = \frac{1}{x_i^2}$ for $i = 1, 2, \ldots, n$. First we claim that

$$\partial_{x_2}\partial_{x_3}\cdots\partial_{x_n} f(x_1) = \frac{(-1)^{n-1}}{x_1x_2\cdots x_n}\Big(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\Big). \tag{2.5}$$

We prove (2.5) by induction on n. Clearly, (2.5) holds when $n = 2$. Assume that (2.5) holds with n replaced by $n - 1$. Then we have

$$\begin{aligned}
\partial_{x_2}\partial_{x_3}\cdots\partial_{x_n} f(x_1) &= \partial_{x_2}\left[\frac{(-1)^{n-2}}{x_1x_3\cdots x_n}\Big(\frac{1}{x_1} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}\Big)\right] \qquad \text{by induction hypothesis} \\
&= \frac{\left[\frac{(-1)^{n-2}}{x_1x_3\cdots x_n}\Big(\frac{1}{x_1} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}\Big)\right] - \left[\frac{(-1)^{n-2}}{x_2x_3\cdots x_n}\Big(\frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}\Big)\right]}{x_1 - x_2} \\
&= \frac{(-1)^{n-2}}{x_3\cdots x_n}\left[\Big(\frac{1}{x_1^2} - \frac{1}{x_2^2}\Big) + \Big(\frac{1}{x_1x_3} - \frac{1}{x_2x_3}\Big) + \cdots + \Big(\frac{1}{x_1x_n} - \frac{1}{x_2x_n}\Big)\right]\frac{1}{x_1 - x_2} \\
&= \frac{(-1)^{n-2}}{x_3\cdots x_n}\left[-\frac{x_1 + x_2}{x_1^2x_2^2} - \frac{1}{x_1x_2x_3} - \cdots - \frac{1}{x_1x_2x_n}\right] \\
&= \frac{(-1)^{n-1}}{x_1x_2\cdots x_n}\Big(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\Big).
\end{aligned}$$

Furthermore, it follows by (2.3) that

$$\sum_{i=1}^{n} \frac{1/x_i^2}{\prod_{j\neq i}(x_i - x_j)} = \frac{(-1)^{n-1}}{x_1x_2\cdots x_n}\Big(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\Big)$$

$$\Leftrightarrow x_1x_2\cdots x_n \sum_{i=1}^{n} \frac{1/x_i^2}{\prod_{j\neq i}(x_j - x_i)} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

$$\Leftrightarrow \sum_{i=1}^{n} \frac{1}{x_i}\cdot\frac{1}{\prod_{j\neq i}(1 - x_i/x_j)} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

$$\Leftrightarrow \frac{V_1}{x_1} + \frac{V_2}{x_2} + \cdots + \frac{V_n}{x_n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}.$$

This completes the proof. □

Now we are ready to prove Theorem 1.2. Without loss of generality, we may assume $r = 1$ and $s = 2$ in Theorem 1.2.

A new approach to Theorem 1.2. By (2.4) we have

$$\frac{V_1 - 1}{x_1} + \frac{V_2 - 1}{x_2} + \cdots + \frac{V_n - 1}{x_n} = 0.$$

Multiplying both sides by $\frac{x_2}{V_4-1}$ yields

$$\frac{x_2}{x_4} = \frac{(1 - V_1)x_2}{(V_4 - 1)x_1} + \frac{1 - V_2}{V_4 - 1} + \frac{(1 - V_3)x_2}{(V_4 - 1)x_3} + \frac{(1 - V_5)x_2}{(V_4 - 1)x_5} + \cdots + \frac{(1 - V_n)x_2}{(V_4 - 1)x_n}. \tag{2.6}$$

Note that $D_n(\mathbf{x}, \mathbf{a}) = V_1^{-a_1}V_2^{-a_2}\cdots V_n^{-a_n}$, (2.6) implies that

$$\begin{aligned}\frac{x_2^2}{x_1x_4}D_n(\mathbf{x}, \mathbf{a}) &= \frac{x_2^2}{x_1x_4}\prod_{j=1}^{n} V_j^{-a_j}\\ &= \frac{x_2}{x_1}\Big[\frac{(1 - V_1)x_2}{(V_4 - 1)x_1} + \frac{1 - V_2}{V_4 - 1} + \frac{(1 - V_3)x_2}{(V_4 - 1)x_3} + \frac{(1 - V_5)x_2}{(V_4 - 1)x_5} + \cdots + \frac{(1 - V_n)x_2}{(V_4 - 1)x_n}\Big]\prod_{j=1}^{n} V_j^{-a_j}.\end{aligned} \tag{2.7}$$

Multiplying both sides by $V_4 - 1$ and taking the constant term in the x's, (2.7) can be rewritten as follows

$$F(a_1) - F(a_1 - 1) = \mathop{\mathrm{CT}}_{\mathbf{x}}\Big[\frac{x_2}{x_1}(V_2 - 1) + \frac{x_2^2}{x_1x_3}(V_3 - 1) + \cdots + \frac{x_2^2}{x_1x_n}(V_n - 1)\Big]\prod_{j=1}^{n} V_j^{-a_j}, \tag{2.8}$$

where $F(a_1) := \mathrm{CT}_{\mathbf{x}}\,\frac{x_2^2}{x_1^2}\prod_{j=1}^{n} V_j^{-a_j}$.

For $j = 3, 4, \ldots, n$, observe that

$$\begin{aligned}&\mathop{\mathrm{CT}}_{\mathbf{x}}\frac{x_2^2}{x_1x_j}(V_j - 1)\prod_{j=1}^{n} V_j^{-a_j} = \mathop{\mathrm{CT}}_{\mathbf{x}}\frac{x_2^2}{x_1x_j}D_n\big(\mathbf{x}, (a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_n)\big) - \mathop{\mathrm{CT}}_{\mathbf{x}}\frac{x_2^2}{x_1x_j}D_n(\mathbf{x}, \mathbf{a})\\ &= \Big[\frac{a_1 + a_j - 1}{1 + a - a_1 - a_j} - \frac{a_1}{a - a_1} - \frac{a_j - 1}{1 + a - a_j}\Big]\frac{a_j}{a}C_n(\mathbf{a})\\ &\qquad - \Big[\frac{a_1 + a_j}{1 + a - a_1 - a_j} - \frac{a_1}{1 + a - a_1} - \frac{a_j}{1 + a - a_j}\Big]C_n(\mathbf{a}) \qquad \text{by [9, Theorem 1.4]}\\ &= -\Big[\frac{a_1a_j}{(1 + a - a_1)(1 + a - a_1 - a_j)} + \frac{a_1a_j}{a(a - a_1)}\Big]C_n(\mathbf{a})\end{aligned} \tag{2.9}$$

and

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_2}{x_1}(V_2-1)\prod_{j=1}^{n} V_j^{-a_j} = \operatorname*{CT}_{\mathbf{x}} \frac{x_2}{x_1} D_n\big(\mathbf{x},(a_1,a_2-1,a_3,\dots,a_n)\big) - \operatorname*{CT}_{\mathbf{x}} \frac{x_2}{x_1} D_n(\mathbf{x},\mathbf{a})$$

$$= \left[-\frac{a_1}{a-a_1}\cdot\frac{a_2}{a}+\frac{a_1}{1+a-a_1}\right] C_n(\mathbf{a}) \qquad \text{by [9, Theorem 1.1]}$$

$$= \left[\frac{a_1}{1+a-a_1}-\frac{a_1a_2}{a(a-a_1)}\right] C_n(\mathbf{a}). \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we obtain the following recurrence

$$\begin{aligned}
&F(a_1)-F(a_1-1)\\
=&\left[\frac{a_1}{1+a-a_1}-\frac{a_1a_2}{a(a-a_1)}-\sum_{j=3}^{n}\left(\frac{a_1a_j}{(1+a-a_1)(1+a-a_1-a_j)}+\frac{a_1a_j}{a(a-a_1)}\right)\right]C_n(\mathbf{a})\\
=&\left[\frac{a_1}{1+a-a_1}-\frac{a_1a_2}{a(a-a_1)}-\frac{a_1(a-a_1-a_2)}{a(a-a_1)}-\sum_{j=3}^{n}\frac{a_1a_j}{(1+a-a_1)(1+a-a_1-a_j)}\right]C_n(\mathbf{a})\\
=&\left[\frac{a_1(a_1-1)}{a(1+a-a_1)}-\sum_{j=3}^{n}\frac{a_1a_j}{(1+a-a_1)(1+a-a_1-a_j)}\right]C_n(\mathbf{a}).
\end{aligned} \tag{2.11}$$

Further noting that $F(0)=0$, which can be easily verified, (2.11) finally gives

$$\begin{aligned}
F(a_1)=&\left[\sum_{k=1}^{a_1}\frac{k(k-1)(a-a_1+k)!}{(1+a-a_1)(a-a_1+k)k!}-\sum_{k=1}^{a_1}\sum_{j=3}^{n}\frac{ka_j(a-a_1+k)!}{(1+a-a_1)(1+a-a_1-a_j)k!}\right]\frac{1}{a_2!\cdots a_n!}\\
=&\left[\sum_{k=2}^{a_1}\frac{(a-a_1+k-1)!}{(1+a-a_1)(k-2)!}-\sum_{k=1}^{a_1}\sum_{j=3}^{n}\frac{ka_j(a-a_1+k)!}{(1+a-a_1)(1+a-a_1-a_j)k!}\right]\frac{1}{a_2!\cdots a_n!}\\
=&\Bigg[\frac{a_1(a_1-1)}{(1+a-a_1)(2+a-a_1)}\cdot\frac{a!}{a_1!} \qquad \text{by (2.1) for the case } n=a_1 \text{ and } m=a-a_1.\\
&-\sum_{j=3}^{n}\sum_{k=1}^{a_1}\frac{ka_j(a-a_1+k)!}{(1+a-a_1)(1+a-a_1-a_j)k!}\Bigg]\frac{1}{a_2!\cdots a_n!}\\
=&\Bigg[\frac{a_1(a_1-1)}{(1+a-a_1)(2+a-a_1)}\cdot\frac{a!}{a_1!}\\
&-\sum_{j=3}^{n}\frac{a_1a_j}{(1+a-a_1)(2+a-a_1)(1+a-a_1-a_j)}\cdot\frac{(1+a)!}{a_1!}\Bigg]\frac{1}{a_2!\cdots a_n!} \qquad \text{by (2.1)}\\
=&\frac{a_1}{(1+a^{(1)})(2+a^{(1)})}\left[(a_1-1)-\sum_{i=3}^{n}\frac{a_i(1+a)}{(1+a^{(1)}-a_i)}\right]C_n(\mathbf{a}).
\end{aligned}$$

This completes the proof. □

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