Upper and lower bounds for $F_v(4,4;5)$

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Abstract

In this note we give a computer assisted proof showing that the unique $(5,3)$-Ramsey graph is the unique K_5-free graph of order 13 giving $F_v(3,4;5) \leqslant 13$, then we prove that $17 \leqslant F_v(2,2,2,4;5) \leqslant F_v(4,4;5) \leqslant 23$. This improves the previous best bounds $16 \leqslant F_v(4,4;5) \leqslant 25$ provided by Nenov and Kolev.

1 Introduction

In this note, we shall only consider graphs without multiple edges or loops. If G is a graph, then the set of vertices of G is denoted by $V(G)$, the set of edges of G by $E(G)$, the cardinality of $V(G)$ by $|V(G)|$, and the complementary graph of G by $\overline{G}$. The subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$. A cycle of order n is denoted by C_n. Given a positive integer n, $Z_n = \{0,1,2,\cdots,n-1\}$, and $S \subseteq \{1,2,\cdots,\lfloor n/2 \rfloor\}$, let G be a graph with the vertex set $V(G) = Z_n$ and the edge set $E(G) = \{(x,y) : \min\{|x-y|, n-|x-y|\} \in S\}$, then G is called a cyclic graph of order n, denoted by $G_n(S)$. G is an (s,t)-graph if G contains neither clique of order s nor independent set of order t. We denote by $\mathcal{R}(s,t)$ the set of all (s,t)-graphs. An (s,t)-graph of order n is called an $(s,t;n)$-graph. We denote by $\mathcal{R}(s,t;n)$ the set of all $(s,t;n)$-graphs. The Ramsey number $R(s,t)$ is defined to be the minimum number n for which $\mathcal{R}(s,t;n)$ is not empty. In [3], it was proved that $R(4,3) = 9$ and $R(5,3) = 14$ which are useful in the following.

For a graph G and positive integers $a_1, a_2, \cdots, a_r$, we write $G \to (a_1, a_2, \cdots, v_r)^v$ if every r-coloring of the vertices must result in a monochromatic a_i-clique of color i for some $i \in \{1, 2, \cdots, r\}$. Let

$$\mathcal{F}_v(a_1, a_2, \cdots, a_r; k) = \{G : G \to (a_1, a_2, \cdots, a_r)^v \text{ and } K_k \nsubseteq G\}.$$

The graphs in $\mathcal{F}_v(a_1, a_2, \cdots, a_r; k)$ are called $(a_1, a_2, \cdots, a_r; k)^v$ graphs. An $(a_1, a_2, \cdots, a_r; k)^v$ graph of order n is called an $(a_1, a_2, \cdots, a_r; k; n)^v$ graph.

The vertex Folkman number is defined as

$$F_v(a_1, a_2, \cdots, a_r; k) = \min\{|V(G)| : G \in \mathcal{F}_v(a_1, a_2, \cdots, a_r; k)\}.$$

In 1970, Folkman [2] proved that for positive integers k and $a_1, a_2, \cdots, a_r$, $F_v(a_1, a_2, \cdots, a_r; k)$ exists if and only if $k > \max\{a_1, \cdots, a_r\}$. Recently Dudek and Rödl gave a new proof with a relatively small upper bound (see [1]). Until now, even with the help of computer, very little is known about the exact values of vertex Folkman numbers. It is easy to see that $F_v(2, 2; 3) = 5$. In 1981, Nenov [10] obtained the upper bound for the number $F_v(3, 3; 4) = 14$, while the lower bound for this number was obtained using a computer in the paper [15]; in 2001, Nenov [13] proved that $F_v(3, 4; 5) = 13$. It might be not easy to determine the exact value of $F_v(4, 4; 5)$. In 2006, Kolev and Nenov [7] proved that $F_v(4, 4; 5) \leqslant 26$. Later in 2007, Kolev [5] pushed down this bound to 25. In [12], Nenov proved that $F_v(4, 4; 5) \geqslant 16$.

In this note, we will improve the upper and lower bounds for $F_v(4, 4; 5)$. With the help of computer, we obtain that there is exactly one graph in the set of $(2, 2, 4; 5; 13)^v$ graphs. Then we prove that $F_v(4, 4; 5) \geqslant F_v(2, 3, 4; 5) \geqslant F_v(2, 2, 2, 4; 5) \geqslant 17$. In addition, we find a $(4, 4; 5; 23)^v$ graph to show that $F(4, 4; 5) \leqslant 23$.

2 The lower bound

For a graph G, a complete graph K and vertex set $S \subseteq V(G)$, we say that S is $(G, +v, K)$ maximal if and only if $K \nsubseteq G[S]$ and $K \subseteq G[S \cup \{v\}]$, for every vertex $v \in V(G) - S$; we say that G is $(+e, K)$ maximal if and only if $K \nsubseteq G$ and $K \subseteq G + e$, for every edge $e \in E(\overline{G})$.

Let us define two special graphs. The first one is the cyclic graph $G_{13}(S)$ with $S = \{1, 4, 5, 6\}$, which is denoted by F_1 and was constructed by Greenwood and Gleason in [3] for proving $R(3, 5) \geqslant 14$. It was proved that every 13-vertex (5, 3)-graph is isomorphic to the graph F_1 (see [4]).

The second one is denoted by F_2, which is defined as follows. $V(F_2) = \{1, 2, \cdots, 10\}$, for $1 \leqslant x, y \leqslant 9$, if $\min\{|x - y|, 9 - |x - y|\} = 1$, then $(x, y) \notin E(F_2)$, otherwise $(x, y) \in E(F_2)$; the edges $(3, 10), (6, 10), (9, 10) \in E(F_2)$. We can see that $F_2[\{1, 2, \cdots, 9\}] \cong \overline{C_9}$.

Graphs F_1 and F_2 are shown in Figures 1 and 2 .

Our computational approach is based on the following lemmas and observations.

Lemma 1. *For $r \geqslant 2$ and positive integers $a_1, a_2, \cdots, a_r$, if $G \to (a_1, a_2, \cdots, a_r)^v$, u is a vertex of G and $d_G(u) < \sum_{i=1}^{r} a_i - r$, then $G - \{u\} \to (a_1, a_2, \cdots, a_r)^v$.*

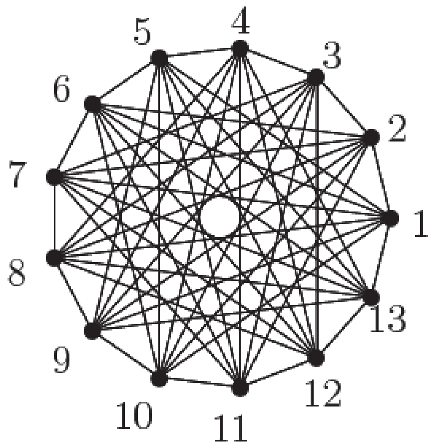

Figure 1: F_1

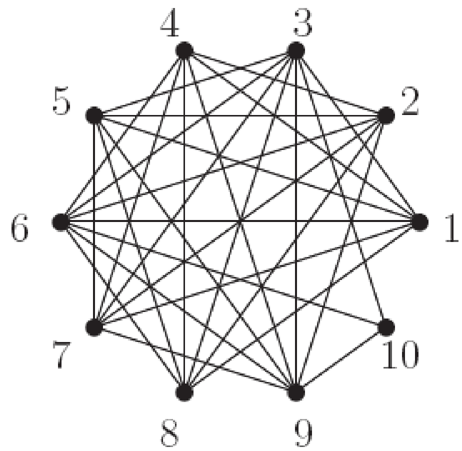

Figure 2: F_2

Proof. Suppose to the contrary that $G - \{u\} \nrightarrow (a_1, a_2, \cdots, a_r)^v$, then there exists a r-coloring of the vertices of $G - \{u\}$ such that $G - \{u\}$ contains no K_{a_i} for each i with color i. Since $d_G(u) < \sum_{i=1}^{r} a_i - r$, we have there exists some j such that there are x vertices with color j in the neighborhood of u and $x < a_j - 1$. Then we color the vertex u with color j and we have G contains no K_{a_j}. Thus, $G \nrightarrow (a_1, a_2, \cdots, a_r)^v$, a contradiction. □

Observation 1. *If $G \in \mathcal{F}_v(2, 2, 4; 5)$ and $G \notin \mathcal{R}(5, 3)$, then G contains an independent set of order* 3.

Proof. Since $G \in \mathcal{F}_v(2, 2, 4; 5)$, then we have G contains no K_5. Since $G \notin \mathcal{R}(5, 3)$, then we have G contains an independent set of order 3. □

Observation 2. *If $G \in \mathcal{F}_v(2, 2, 4; 5)$ and $G \notin \mathcal{R}(5, 3)$, G is $(+e, K_5)$ maximal, and H is obtained from G by removing an independent set of order* 3*, then*

(1) *H contains no K_5,*

(2) *$H \rightarrow (2, 4)^v$,*

(3) *$3 \leqslant \delta(H) \leqslant \Delta(H) \leqslant 7$, and $\delta(H) = 3$ if and only if $H \cong F_2$.*

Proof. (1) Since H is a subgraph of G and G contains no K_5, so H contains no K_5.

(2) Let I be an independent set of order 3 in G, suppose to the contrary that $H \nrightarrow (2, 4)^v$, then there exists a 2-coloring, say color 2 and color 3, of the vertices of H such that H neither contain K_2 with color 2 nor K_4 with color 3. We color the independent set I with color 1. Thus, $G \nrightarrow (2, 2, 4)^v$, a contradiction.

(3) It is not difficult to see $\delta(G) \geqslant 5$. In fact, let v be any vertex in $V(G)$, from $F_v(2, 2, 4; 5) = 13$ (see [11]) we know the subgraph of G induced by $V(G) - \{v\}$, denoted

by J, can not satisfy $J \to (2,2,4)^v$. But $G \to (2,2,4)^v$, so there must be two 1-cliques and a 3-clique without common vertex in the neighborhood of v in G. Therefore the degree of v in G is at least 5, so $\delta(G) \geqslant 5$. Therefore $\delta(H) \geqslant 2$.

Now, let us give $\delta(H)$ a lower bound.

If $2 \leqslant \delta(H) \leqslant 3$, and the degree of u in H is $\delta(H)$, since $H \to (2,4)^v$ and by Lemma 1 we have $H - \{u\} \to (2,4)^v$ and $H - \{u\}$ is K_5-free graph of order 9. In [8], it was used that $\overline{C_9}$ is the unique $(2,4;5;9)^v$ graph (the result that is used in the text is a special case of a more general theorem). So $H - \{u\}$ is isomorphic to $\overline{C_9}$. We suppose the vertex set of $H - \{u\}$ is Z_9, where i and j are not adjacent if and only if $\min\{|i-j|, 9-|i-j|\} = 1$. Before continue to work, we have the following claims.

Claim 1. *$\delta(H) \neq 2$.*

Proof. If $\delta(H) = 2$, let v_1 and v_2 be the neighbors of u in H, v_3 be a non-neighbor of v_1 in $H - \{u\}$ which is different from v_2. Then we can add the edge (u, v_3) to graph G to get a new K_5-free graph, which contradicts with that G is $(+e, K_5)$ maximal. □

Claim 2. *$\delta(H) = 3$ if and only if $H \cong F_2$.*

Proof. If $\delta(H) = 3$, since G is $(+e, K_5)$ maximal and $H - \{u\} \cong \overline{C_9}$, it is not difficult to see $H \cong F_2$ with some simple computation. □

Now, we continue to show that $\Delta(H) \leqslant 7$. In fact, let v be any vertex in $V(H)$, since H is K_5-free we know the subgraph of H induced by the neighbors of v in H is K_4-free. So since $H \to (2,4)^v$, we know there must be 2-clique in the subgraph of H induced by the non-neighbors of v in H. So the degree of v in H is at most 7. Therefore we have $\Delta(H) \leqslant 7$.
From above, we have part (3) holds. □

Observation 1 and Observation 2 guarantee that the following algorithm generates all $(+e, K_5)$ maximal graphs in the set of $(2,2,4;5;13)^v$ graphs which contain independent set of order 3. In Algorithm 1, Step 4 is used to speed up the processing, which reduces the graphs from $\mathcal{C}$ with the cardinality 368 to $\mathcal{D}$ with the cardinality 114. So the number of graphs in Step 5 need to be processed is reduced.

Algorithm 1

Step 1. Generate the set $\mathcal{A}$ of all nonisomorphic graphs of order 10 such that for each graph $H \in \mathcal{A}$ the degree of each vertex of H ranges from 4 to 7, then set $\mathcal{A} = \mathcal{A} \cup \{F_2\}$.

Step 2. Obtain the set $\mathcal{B}$ from $\mathcal{A}$ by removing the graphs containing K_5.

Step 3. Obtain the set $\mathcal{C}$ such that $\mathcal{C} = \{H \in \mathcal{B} : H \to (2,4)^v\}$.

Step 4. Initial a new set $\mathcal{D} = \emptyset$. Then for each graph $H \in \mathcal{C}$, find the family $\mathcal{M} = \{S \subseteq V(H) : S \text{ is } (H, +v, K_4) \text{ maximal }\}$. Let $m = |\mathcal{M}|$ and $\mathcal{M} = \{S_1, S_2, \cdots, S_m\}$, construct a graph F by adding m vertices $v_1, v_2, \cdots, v_m$ such that $N_F(v_i) = S_i$ for $1 \leqslant i \leqslant m$, if $F \to (2, 2, 4; 5)^v$, add H to the set $\mathcal{D}$.

Step 5. Initial a new set $\mathcal{E} = \emptyset$. Then for every graph $H \in \mathcal{D}$, find the family $\mathcal{M} = \{S \subseteq V(H) : S \text{ is } (H, +v, K_4) \text{ maximal }\}$, for every triple S_1, S_2, $S_3 \in \mathcal{M}$, construct a graph F by adding three vertices v_1, v_2, v_3 to H such that $N_F(v_i) = S_i$ for $i = 1, 2, 3$, if $F \to (2, 2, 4; 5)^v$, add F to the set $\mathcal{E}$.

By Algorithm 1, we generate 754465, 640548, 368, 114 elements in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ respectively, and do not produce any graph in $\mathcal{E}$. For any $(2, 2, 4; 5; 13)^v$ graph G, where G is $(+e, K_5)$ maximal, G must be isomorphic to F_1 as the proper subgraphs of F_1 are not maximal. With the help of computer, we can have Lemma 2.

Lemma 2. *There are only two nonisomorphic subgraphs of F_1 obtained by deleting one edge from F_1. None of them is a $(2, 2; 4)^v$ graph.*

We know any $(3, 4, 5; 13)^v$ graph is also a $(2, 2, 4, 5; 13)^v$ graph. So by Lemma 2, we have

Theorem 1. *F_1 is both the unique $(2, 2, 4; 5; 13)^v$ graph and the unique $(3, 4; 5; 13)^v$ graph.*

Theorem 2. *$F_v(2, 2, 2, 4; 5) \geqslant 17$.*

Proof. Suppose $F_v(2, 2, 2, 4; 5) \leqslant 16$. Let G_0 be a K_5-free graph of order 16 such that $G_0 \to (2, 2, 2, 4)^v$. We can see there must be 3-independent set in G_0. Let $V_0 = \{v_1, v_2, v_3\}$ be a 3-independent set in G_0. Then the subgraph of G_0 induced by $V(G_0) - V_0$, say G', must satisfy $G' \to (2, 2, 4)^v$, and from Theorem 1 above we know G' must be isomorphic to F_1. Since $R(5, 3) = 14$, the subgraph of G_0 induced by $V(G_0) - \{v_2, v_3\}$ is of order 14 and must contain a 3-independent set since it is K_5-free. Let such a 3-independent set be V_1. We know the subgraph of G_0 induced by $V(G_0) - V_0$ is isomorphic to F_1, so v_1 must be in V_1. Suppose $V_1 = \{v_1, v_4, v_5\}$, we know the subgraph of G_0 induced by $V(G_0) - V_1$, say G'', must satisfy $G'' \to (2, 2, 4)^v$, and then from Theorem 1 we know G'' must be isomorphic to F_1. For any $v_i \in V_0$, the subgraph of G_0 induced by the neighbors of v_i, say G_i, is a subgraph of G'. Since G' is a $(5, 3)$-graph and G_i is induced by the neighbors of v_i in G', we have G_i must be a $(4, 3)$-graph. So the degree of both v_2 and v_3 in G'' is no more than 8 because $R(3, 4) = 9$. But both v_2 and v_3 is in G'' which is isomorphic to F_1, so the degree of v_2 and v_3 in G_0 can not be less than 8. So $d_{G_0}(v_2) = d_{G_0}(v_3) = 8$, similarly we can get $d_{G_0}(v_4) = d_{G_0}(v_5) = 8$. Since G' is isomorphic to F_1, we have G' is 8-regular. So we have $d_{G'}(v_4) = d_{G'}(v_5) = 8$. Therefore $\{v_2, v_3, v_4, v_5\}$ is an independent set in G_0. The subgraph of G_0 induced by $V(G_0) - \{v_2, v_3, v_4, v_5\}$ is of order 12, which is denoted by H. From $F_v(2, 2, 4; 5) = 13$ we know $H \nrightarrow (2, 2, 4)^v$, which contradicts with $\{v_2, v_3, v_4, v_5\}$ is an independent set in G_0 and $G_0 \to (2, 2, 2, 4)^v$. Therefore we have $F_v(2, 2, 2, 4; 5) \geqslant 17$. □

1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
2	1	0	1	1	1	1	0	1	1	0	0	1	0	1	0	1	1	1	0	1	0	1	0
3	1	1	0	1	1	0	1	1	1	0	0	0	1	0	1	1	1	0	1	0	1	0	1
4	1	1	1	0	0	1	1	0	0	1	1	0	1	0	1	0	1	1	1	0	1	1	0
5	1	1	1	0	0	1	1	0	0	1	1	1	0	1	0	1	0	1	1	1	0	0	1
6	1	1	0	1	1	0	0	1	1	1	1	0	0	0	1	1	1	0	1	1	1	0	0
7	1	0	1	1	1	0	0	1	1	1	1	0	0	1	0	1	1	1	0	1	1	0	0
8	1	1	1	0	0	1	1	0	0	1	0	0	1	0	1	1	0	1	1	1	0	1	1
9	1	1	1	0	0	1	1	0	0	0	1	1	0	1	0	0	1	1	1	0	1	1	1
10	1	0	0	1	1	1	1	1	0	0	0	1	1	0	0	1	1	0	1	0	1	1	1
11	1	0	0	1	1	1	1	0	1	0	0	1	1	0	0	1	1	1	0	1	0	1	1
12	1	1	0	0	1	0	0	0	1	1	1	0	1	0	1	0	1	1	1	1	1	1	0
13	1	0	1	1	0	0	0	1	0	1	1	1	0	1	0	1	0	1	1	1	1	0	1
14	1	1	0	0	1	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1	0	1	1
15	1	0	1	1	0	1	0	1	0	0	0	1	0	1	0	1	1	1	1	0	1	1	1
16	0	1	1	0	1	1	1	1	0	1	1	0	1	1	1	0	1	0	0	0	1	1	0
17	0	1	1	1	0	1	1	0	1	1	1	1	0	1	1	1	0	0	0	1	0	0	1
18	0	1	0	1	1	0	1	1	1	0	1	1	1	1	1	0	0	0	1	0	1	0	1
19	0	0	1	1	1	1	0	1	1	1	0	1	1	1	1	0	0	1	0	1	0	1	0
20	0	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	0	1	0	1	1	1
21	0	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	0	1	0	1	1
22	0	1	0	1	0	0	0	1	1	1	1	1	0	1	1	1	0	0	1	1	1	0	1
23	0	0	1	0	1	0	0	1	1	1	1	0	1	1	1	0	1	1	0	1	1	1	0

Figure 3: Adjacency matrix of a $(4, 4; 5; 23)^v$ graph

In [14], it was proved that

Lemma 3. *[14] Let* $G \rightarrow (a_1, a_2, \cdots, a_r)^v$ *and let for some* i, $a_i \geqslant 2$. *Then* $G \rightarrow (a_1, \cdots, a_{i-1}, 2, a_i - 1, a_{i+1}, \cdots, a_r)^v$

By Lemma 3, $F_v(4, 4; 5) \geqslant F_v(2, 3, 4; 5) \geqslant F_v(2, 2, 2, 4; 5)$, by Theorem 2, we have

Theorem 3. $F_v(4, 4; 5) \geqslant 17$*.*

3 The upper bound

We investigate some vertex transitive graphs, which can be found on the website [16]. With the help of computer, we find a $(4, 4; 5; 23)^v$ graph, which is the 154th graph in the file "trans23.g6.gz" and is shown in Figure 3. Thus, we have $F_v(4, 4; 5) \leqslant 23$.

Some subgraphs obtained from this graph by deleting some edges are in $\mathcal{F}_v(4, 4; 5; 23)$ too, but the graphs obtained by deleting one vertex are not in $\mathcal{F}_v(4, 4; 5; 23)$.

4 Remark

The powerful programs `shortg` and `geng`, which were developed by Mckay [9], are used as an important tool in this work. We use `shortg` for fast isomorph rejection and `geng` for generating all nonisomorphic graphs of order 10 with minimum degree 4 and maximum degree 7 as follows: `geng -d4D7 10 file10d4D7.g6`

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