Rank three residually connected geometries for M_{22}, revisited

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Abstract

The rank 3 residually connected flag transitive geometries Γ for M_{22} for which the stabilizer of each object in Γ is a maximal subgroup of M_{22} are determined. As a result this deals with the infelicities in Theorem 3 of Kilic and Rowley, *On rank 2 and rank 3 residually connected geometries for* M_{22}. Note di Matematica, **22**(2003), 107–154.

1 Introduction

Here we report on calculations carried out using Magma[2] on certain rank 3 geometries for M_{22}, the Mathieu group of degree 22. Putting $G = M_{22}$, the main conclusion is as follows.

Theorem 1. *Up to conjugacy in $Aut(G)$ there are 431 rank 3 residually connected flag transitive geometries Γ satisfying the condition that $Stab_G(x)$ is a maximal subgroup of G for all $x \in \Gamma$.*

These 431 geometries are tabulated in Section 2 where they are described in terms of the action of M_{22} on a 22-element set. These geometries may also be downloaded from [6]. This list supersedes that given in Theorem 3 of [5], which not only omits some of the geometries but also contains geometries that should not be there (usually because they fail to be flag transitive). We next introduce some notation, mostly for use in describing the geometries in Section 2. Our notation for geometries is standard, as may be found in

[1]. So a geometry Γ consists of a triple $(\Gamma, I, \star)$ where Γ is a set, I the set of types and $\star$ a symmetric incidence relation on Γ for which

(i) $\Gamma = \dot{\bigcup}_{i \in I} \Gamma_i$ with each Γ_i a non-empty subset of Γ; and

(ii) if $x \in \Gamma_i$, $y \in \Gamma_j (i, j \in I)$ and $x \star y$, then $i \neq j$.

The rank of Γ, rank Γ, is the cardinality of I – if $|I| = n$ we shall take $I = \{1, ..., n\}$. If $F \subseteq \Gamma$ has the property that for all $x, y \in \Gamma$ with $x \neq y$, we have $x \star y$, then we call F a flag of Γ. The rank of F is just $|F|$, its corank is $|\{i \in I | F \cap \Gamma_i = \emptyset\}|$ and its type is $\{i \in I | F \cap \Gamma_i \neq \emptyset\}$.The geometries we consider will be assumed to possess at least one flag of rank $|I|$. Let H be a subgroup of the group of automorphisms of Γ, $Aut\Gamma$, which consists of all permutations of Γ preserving the sets Γ_i and the incidence relation. By saying that Γ is a flag transitive geometry for H we mean that if F_1 and F_2 are flags of Γ which have the same type, then ${F_1}^h = F_2$ for some $h \in H$. Assume that Γ is a flag transitive geometry for H, and let $F = \{x_1, ..., x_n\}$ be a maximal flag of Γ (that is F has rank $|I|$). For $i \in I$ set $H_i = Stab_H(x_i)$ and for $\emptyset \neq J \subseteq I$ set $H_J = \bigcap_{j \in J} H_{x_j}$ and if $J = \{i_1, .., i_j\}$ we also write H_J as $H_{i_1 \dots i_j}$.A geometry Γ is said to be residually connected if for all flags F of Γ of corank at least 2, the incidence graph of $\Gamma_F = \{x \in \Gamma | y \star x$ for all $y \in F\}$ is connected.

For the remainder of this paper G will denote M_{22}, the Mathieu group of degree 22 and Ω a 24-element set whose elements will be labelled as in Curtis [4]. So

$$\Omega = \begin{array}{|cc|cc|cc|} \hline \infty & 14 & 17 & 11 & 22 & 19 \\ 0 & 8 & 4 & 13 & 1 & 9 \\ 3 & 20 & 16 & 7 & 12 & 5 \\ 15 & 18 & 10 & 2 & 21 & 6 \\ \hline \end{array}$$

and we use the MOG [4] to give us a Steiner system $S(24, 8, 5)$ for Ω. Further we shall identify G with $Stab_{M_{24}}(\infty) \cap Stab_{M_{24}}(14)$ where M_{24} is the Mathieu group of degree 24 leaving invariant the Steiner system on Ω given by the MOG. Set $\Lambda = \Omega \setminus \{\infty, 14\}$. An 8-element block of the Steiner system is referred to as an octad of Ω and a dodecad is the symmetric difference of two octads of Ω which intersect in a set of size two.

We shall follow the notation in [5]. So
$\mathcal{H}$=$\{X \subseteq \Lambda | X \cup \{\infty, 14\}$ is an octad of $\Omega\}$ (hexads of Λ),
$\mathcal{H}_{\mathrm{p}}$=$\{X \subseteq \Lambda | X \cup \{14\}$ is an octad of $\Omega\}$ (heptads of Λ),
$\mathcal{H}_{\mathrm{p}\infty}$=$\{X \subseteq \Lambda | X \cup \{\infty\}$ is an octad of $\Omega\}$ (heptads of Λ),
$\mathcal{O}$=$\{X \subseteq \Lambda | X$ is an octad of $\Omega\}$ (octads of Λ),
$\mathcal{D}$=$\{X \subseteq \Lambda || X| = 2\}$ (duads of Λ).
$\mathcal{D}_{\mathrm{o}}$=$\{X \subseteq \Lambda | X$ is a dodecad of $\Omega\}$ (dodecads of Λ) and
$\mathcal{E}$=$\{X \subseteq \Lambda |$ one of $X \cup \{\infty\}$ and $X \cup \{14\}$ is a dodecad of $\Omega\}$ (endecads of Λ).

Set $\mathfrak{X}$= $\Lambda \cup \mathcal{H} \cup \mathcal{H}_{\mathrm{p}} \cup \mathcal{H}_{\mathrm{p}\infty} \cup \mathcal{O} \cup \mathcal{D} \cup \mathcal{D}_{\mathrm{o}} \cup \mathcal{E}$. Up to conjugacy, the maximal subgroups M_i of G are as follows (see [3]).

M_i	*Description*
$M_1 \cong L_3(4)$	$M_1 = Stab_G(a),\ a \in \Lambda$
$M_2 \cong 2^4 : A_6$	$M_2 = Stab_G(X),\ X \in \mathcal{H}$
$M_3 \cong A_7$	$M_3 = Stab_G(X),\ X \in \mathcal{H}_{\rm p}$
$M_4 \cong A_7$	$M_4 = Stab_G(X),\ X \in \mathcal{H}_{{\rm p}\infty}$
$M_5 \cong 2^3 : L_3(2)$	$M_5 = Stab_G(X),\ X \in \mathcal{O}$
$M_6 \cong 2^4 : S_5$	$M_6 = Stab_G(X),\ X \in \mathcal{D}$
$M_7 \cong M_{10}$	$M_7 = Stab_G(X),\ X \in \mathcal{D}_{\rm o}$
$M_8 \cong L_2(11)$	$M_8 = Stab_G(X),\ X \in \mathcal{E}$

For $i \in \{1, ..., 8\}$, $\mathfrak{M}_{\rm i}$ will denote the G-conjugacy class of M_i. We observe that if $X, Y \in \mathfrak{X} \setminus \mathcal{E}$, then we have $X = Y$ if and only if $Stab_G(X) = Stab_G(Y)$.

The description of the geometries listed in Section 2 follows the following scheme. By $\Gamma(G, \{G_i, G_j, G_k\}) = \mathfrak{M}_{\rm abc}(t_{ab}, t_{ac}, t_{bc})$ we mean that $G_i = Stab_G(X_i) \in \mathfrak{M}_a$, $G_j = Stab_G(X_j) \in \mathfrak{M}_b$, $G_k = Stab_G(X_k) \in \mathfrak{M}_c$ with $|X_i \cap X_j| = t_{ab}$, $|X_i \cap X_k| = t_{ac}$ and $|X_j \cap X_k| = t_{bc}$. Some care is needed with this notation when, say $X_k \in \mathcal{E}$ (as then we have two endecads to choose from, namely X_k and $\Lambda \setminus X_k$). So, concerning Theorem 2 of [5], $\mathfrak{M}_{18}(1)$ and $\mathfrak{M}_{18}(0)$ describe the same geometry (up to $Aut(G)$ conjugacy). Just as $\mathfrak{M}_{28}(5)$ and $\mathfrak{M}_{28}(1)$ are the same (up to $Aut(G)$ conjugacy). With the removal of duplicates such as those mentioned, that is $\mathfrak{M}_{18}(1)$, $\mathfrak{M}_{28}(5)$, $\mathfrak{M}_{58}(6)$, $\mathfrak{M}_{68}(2)$ and $\mathfrak{M}_{78}(8)$, the list in Theorem 2 of [5] is correct (so yielding that there are 81 such rank 2 geometries). For the meaning of subscripts (such as, for example, $\mathfrak{M}_{155}(1, 0, 4_1)$) and the number following a colon (as in $\mathfrak{M}_{378}(2, 2, 4 : 2)$) we refer the reader to [5]. Finally the column in the tables called Number gives the number of the geometry in [6]. There the geometries are given as an ordered sequence called *geo* – so for $1 \leqslant j \leqslant 431$, $\{G_1, G_2, G_3\} = \{geo[j][1], geo[j][2], geo[j][3]\}$.

2 The Rank Three Geometries

Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{111}(0,0,0)$	31	$2^4 3$		$\mathfrak{M}_{112}(0,0,0)$	34	S_4
$\mathfrak{M}_{112}(0,0,1)$	14	A_5		$\mathfrak{M}_{112}(0,1,1)$	12	$2^4 A_4$
$\mathfrak{M}_{113}(0,0,0)$	36	A_4		$\mathfrak{M}_{113}(0,0,1)$	37	S_4
$\mathfrak{M}_{113}(0,1,1)$	32	A_5		$\mathfrak{M}_{115}(0,0,1)$	38	A_4
$\mathfrak{M}_{115}(0,1,1)$	33	S_4		$\mathfrak{M}_{116}(0,1,0)$	2	$2^4 3$
$\mathfrak{M}_{117}(0,1,0)$	40	S_3		$\mathfrak{M}_{117}(0,0,0)$	35	Q_8
$\mathfrak{M}_{118}(0,1,1)$	39	S_3				
***********	****	****		***********	****	****
$\mathfrak{M}_{122}(0,0,0)$	42	$3^2 4$		$\mathfrak{M}_{122}(0,1,0)$	13	A_5
$\mathfrak{M}_{122}(0,0,2)$	49	D_8		$\mathfrak{M}_{122}(1,0,2)$	18	S_4
$\mathfrak{M}_{122}(1,1,2)$	15	$2^2 A_4$		$\mathfrak{M}_{123}(0,0,1)$	57	S_3
$\mathfrak{M}_{123}(0,1,1)$	51	D_{10}		$\mathfrak{M}_{123}(1,0,1)$	27	A_4
$\mathfrak{M}_{123}(0,0,3)$	56	S_3		$\mathfrak{M}_{123}(0,1,3)$	47	$3^2 2$
$\mathfrak{M}_{123}(1,0,3)$	19	S_4		$\mathfrak{M}_{123}(1,1,3)$	20	S_4

Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{125}(0,0,0)$	44	S_4		$\mathfrak{M}_{125}(0,1,0)$	45	S_4
$\mathfrak{M}_{125}(1,0,0)$	16	2^2D_8		$\mathfrak{M}_{125}(0,1,2)$	53	4
$\mathfrak{M}_{125}(1,0,2)$	28	S_3		$\mathfrak{M}_{125}(1,1,2)$	21	A_4
$\mathfrak{M}_{125}(0,0,4)$	48	D_8		$\mathfrak{M}_{125}(0,1,4)$	43	S_4
$\mathfrak{M}_{125}(1,1,4)$	17	S_4		$\mathfrak{M}_{126}(1,0,0)$	24	2^3
$\mathfrak{M}_{126}(0,1,0)$	5	S_4		$\mathfrak{M}_{126}(0,0,1)$	54	2^2
$\mathfrak{M}_{126}(1,0,1)$	22	A_4		$\mathfrak{M}_{126}(0,0,2)$	46	S_4
$\mathfrak{M}_{126}(1,1,2)$	1	2^4A_4		$\mathfrak{M}_{127}(0,0,2)$	52	4
$\mathfrak{M}_{127}(1,0,2)$	26	S_3		$\mathfrak{M}_{127}(0,0,4)$	59	2
$\mathfrak{M}_{127}(0,1,4)$	58	2		$\mathfrak{M}_{127}(1,1,4)$	29	2^2
$\mathfrak{M}_{127}(1,0,4)$	23	Q_8		$\mathfrak{M}_{127}(0,0,6)$	41	3^24
$\mathfrak{M}_{128}(0,1,1)$	55	S_3		$\mathfrak{M}_{128}(0,0,1)$	50	D_{10}
$\mathfrak{M}_{128}(1,0,1)$	25	A_4		$\mathfrak{M}_{128}(1,0,3)$	30	2^2
***********	****	****		***********	****	****
$\mathfrak{M}_{133}(0,0,1)$	115	4		$\mathfrak{M}_{133}(1,0,1)$	73	S_3
$\mathfrak{M}_{133}(0,3,1)$	72	S_3		$\mathfrak{M}_{133}(1,1,3)$	68	D_8
$\mathfrak{M}_{134}(0,0,0)$	108	F_{21}		$\mathfrak{M}_{134}(1,0,0)$	61	S_4
$\mathfrak{M}_{134}(0,0,2)$	118	2		$\mathfrak{M}_{134}(0,1,2)$	77	4
$\mathfrak{M}_{134}(1,1,2)$	67	D_{10}		$\mathfrak{M}_{134}(0,0,4)$	112	S_3
$\mathfrak{M}_{134}(1,1,4)$	64	3^22		$\mathfrak{M}_{134}(1,0,4)$	60	S_4
$\mathfrak{M}_{135}(0,1,0)$	83	F_{21}		$\mathfrak{M}_{135}(0,0,0)$	102	S_4
$\mathfrak{M}_{135}(1,0,0)$	61	S_4		$\mathfrak{M}_{135}(0,1,2)$	100	2
$\mathfrak{M}_{135}(1,1,2)$	74	S_3		$\mathfrak{M}_{135}(1,1,4)$	75	S_3
$\mathfrak{M}_{135}(0,1,4)$	92	S_3		$\mathfrak{M}_{135}(1,0,4)$	69	D_8
$\mathfrak{M}_{136}(0,1,0)$	9	A_4		$\mathfrak{M}_{136}(1,0,1)$	78	2^2
$\mathfrak{M}_{136}(0,0,2)$	105	D_8		$\mathfrak{M}_{136}(1,0,2)$	63	S_4
$\mathfrak{M}_{136}(1,1,2)$	3	A_5		$\mathfrak{M}_{137}(0,1,2)$	117	2
$\mathfrak{M}_{137}(0,0,2)$	114	4		$\mathfrak{M}_{137}(1,0,2)$	76	4
$\mathfrak{M}_{137}(1,1,2)$	66	D_{10}		$\mathfrak{M}_{137}(1,0,4)$	80	2
$\mathfrak{M}_{137}(0,0,6)$	113	4		$\mathfrak{M}_{137}(0,1,6)$	111	S_3
$\mathfrak{M}_{137}(1,1,6)$	71	S_3		$\mathfrak{M}_{138}(0,0,2)$	116	2
$\mathfrak{M}_{138}(1,1,2)$	70	S_3		$\mathfrak{M}_{138}(1,0,4)$	79	2
$\mathfrak{M}_{138}(0,0,6)$	110	S_3		$\mathfrak{M}_{138}(1,1,6)$	65	D_{10}
$\mathfrak{M}_{138}(0,1,6)$	109	A_4				
***********	****	****		***********	****	****
$\mathfrak{M}_{155}(0,1,0)$	81	S_4		$\mathfrak{M}_{155}(1,1,2)$	90	4
$\mathfrak{M}_{155}(1,0,4_1)$	93	3		$\mathfrak{M}_{155}(1,1,4_1)$	94	3
$\mathfrak{M}_{155}(0,1,4_2)$	85	D_8		$\mathfrak{M}_{155}(1,1,4_2)$	84	D_8
$\mathfrak{M}_{156}(0,0,0_1)$	101	2^22^2		$\mathfrak{M}_{156}(1,0,0_1)$	82	S_4
$\mathfrak{M}_{156}(1,0,0_2)$	95	2		$\mathfrak{M}_{156}(0,1,0_2)$	8	2^3
$\mathfrak{M}_{156}(1,0,2)$	86	2^3		$\mathfrak{M}_{156}(1,1,2)$	4	S_4

Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{157}(1,0,2)$	89	4		$\mathfrak{M}_{157}(0,0,2)$	103	S_3
$\mathfrak{M}_{157}(1,0,4_1)$	104	4		$\mathfrak{M}_{157}(1,0,4_1)$	87	S_3
$\mathfrak{M}_{157}(1,0,4_2)$	99	1		$\mathfrak{M}_{157}(1,1,4_2)$	98	1
$\mathfrak{M}_{157}(1,0,6)$	88	4		$\mathfrak{M}_{158}(1,0,2)$	97	2
$\mathfrak{M}_{158}(1,0,4_1)$	96	2		$\mathfrak{M}_{158}(1,1,2)$	91	S_3
***********	****	****		***********	****	****
$\mathfrak{M}_{166}(1,0,0_1)$	10	S_3		$\mathfrak{M}_{167}(0,0,0)$	107	2
$\mathfrak{M}_{167}(1,0,0)$	7	Q_8		$\mathfrak{M}_{167}(0,1,2_1)$	106	2
$\mathfrak{M}_{167}(1,1,2_1)$	6	D_{10}		$\mathfrak{M}_{168}(1,0,0)$	11	S_3
***********	****	****		***********	****	****
$\mathfrak{M}_{177}(0,1,4)$	121	2		$\mathfrak{M}_{177}(1,1,4)$	125	2^2
$\mathfrak{M}_{177}(0,0,4)$	119	Q_8		$\mathfrak{M}_{177}(1,0,8_1)$	124	1
$\mathfrak{M}_{177}(0,0,8_2)$	120	4		$\mathfrak{M}_{178}(0,0,6_1)$	123	2
$\mathfrak{M}_{178}(0,0,4)$	122	2		$\mathfrak{M}_{188}(0,0,3)$	126	2^2
$\mathfrak{M}_{188}(0,0,5_2)$	128	2		$\mathfrak{M}_{188}(1,0,7_2)$	127	3
***********	****	****		***********	****	****
$\mathfrak{M}_{222}(0,2,2)$	162	D_8		$\mathfrak{M}_{222}(0,0,2)$	159	S_4
$\mathfrak{M}_{223}(0,3,1)$	170	S_3		$\mathfrak{M}_{223}(0,1,1)$	164	D_{10}
$\mathfrak{M}_{223}(0,3,3)$	160	$3^2 2$		$\mathfrak{M}_{225}(0,2,2)$	168	2
$\mathfrak{M}_{225}(0,4,2)$	165	D_8		$\mathfrak{M}_{225}(0,0,4)$	158	S_4
$\mathfrak{M}_{225}(2,2,0)$	181	2^2		$\mathfrak{M}_{225}(2,0,0)$	173	$2^4 2$
$\mathfrak{M}_{226}(0,0,1)$	166	S_3		$\mathfrak{M}_{226}(0,0,0)$	161	D_8
$\mathfrak{M}_{226}(0,0,2)$	130	S_4		$\mathfrak{M}_{226}(2,1,0)$	202	2
$\mathfrak{M}_{226}(2,1,2)$	135	A_4		$\mathfrak{M}_{226}(2,0,2)$	133	$2 \times D_8$
$\mathfrak{M}_{227}(2,6,4)$	154	D_8		$\mathfrak{M}_{227}(0,4,2)$	171	2
$\mathfrak{M}_{227}(0,4,4)$	167	2		$\mathfrak{M}_{228}(0,3,5)$	169	S_3
$\mathfrak{M}_{228}(0,1,5)$	163	D_{10}		$\mathfrak{M}_{228}(2,1,1)$	201	S_3
$\mathfrak{M}_{228}(2,5,1)$	200	A_4				
***********	****	****		***********	****	****
$\mathfrak{M}_{233}(1,1,1)$	241	2		$\mathfrak{M}_{234}(1,1,0)$	236	A_4
$\mathfrak{M}_{234}(3,1,0)$	226	S_3		$\mathfrak{M}_{234}(3,1,4)$	225	S_3
$\mathfrak{M}_{235}(3,2,0)$	227	S_3		$\mathfrak{M}_{235}(3,0,0)$	175	S_4
$\mathfrak{M}_{235}(1,0,4)$	185	2^2		$\mathfrak{M}_{235}(1,0,2)$	186	2^2
$\mathfrak{M}_{235}(1,2,0)$	237	S_3		$\mathfrak{M}_{235}(1,4,4)$	195	S_3
$\mathfrak{M}_{235}(1,4,0)$	190	A_4		$\mathfrak{M}_{236}(1,2,0)$	145	S_3
$\mathfrak{M}_{236}(3,1,0)$	213	2		$\mathfrak{M}_{236}(3,1,2)$	206	S_3
$\mathfrak{M}_{236}(3,2,1)$	141	D_8		$\mathfrak{M}_{236}(3,0,2)$	216	D_{12}
$\mathfrak{M}_{236}(3,2,2)$	134	S_4		$\mathfrak{M}_{236}(1,0,2)$	218	2^2
$\mathfrak{M}_{237}(3,2,6)$	224	S_3		$\mathfrak{M}_{237}(3,6,6)$	151	$3^2 2$
$\mathfrak{M}_{237}(1,4,6)$	240	2		$\mathfrak{M}_{237}(1,2,2)$	239	2

Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{237}(1,6,2)$	155	D_{10}		$\mathfrak{M}_{238}(3,5,6)$	222	S_3
$\mathfrak{M}_{238}(1,3,6)$	238	2		$\mathfrak{M}_{238}(1,1,2)$	228	2^2
$\mathfrak{M}_{238}(3,1,5)$	223	S_3				
***********	****	****		***********	****	****
$\mathfrak{M}_{255}(0,2,4_1)$	187	2		$\mathfrak{M}_{255}(4,4,4_1)$	196	3
$\mathfrak{M}_{255}(2,2,0)$	243	2^2		$\mathfrak{M}_{255}(2,4,4_2)$	193	2^2
$\mathfrak{M}_{255}(4,0,2)$	179	2^2		$\mathfrak{M}_{255}(0,0,4_2)$	174	2^4
$\mathfrak{M}_{255}(4,0,0)$	172	$2^4 2$		$\mathfrak{M}_{256}(0,0,1)$	182	3
$\mathfrak{M}_{256}(0,1,0_2)$	180	2^2		$\mathfrak{M}_{256}(0,0,0_2)$	176	2^3
$\mathfrak{M}_{256}(0,2,0_1)$	129	$2^4 2^2$		$\mathfrak{M}_{256}(2,1,2)$	208	2
$\mathfrak{M}_{256}(2,2,1)$	146	3		$\mathfrak{M}_{256}(2,2,0_2)$	142	2^2
$\mathfrak{M}_{256}(2,1,0_1)$	203	S_3		$\mathfrak{M}_{256}(2,0,0_1)$	215	D_8
$\mathfrak{M}_{256}(4,0,1)$	197	2		$\mathfrak{M}_{256}(4,1,0_1)$	194	2^2
$\mathfrak{M}_{256}(4,1,2)$	191	S_3		$\mathfrak{M}_{256}(4,2,2)$	131	$2^2 2^2$
$\mathfrak{M}_{256}(4,0,0_1)$	188	$2^2 2^2$		$\mathfrak{M}_{257}(0,2,4_2)$	184	2
$\mathfrak{M}_{257}(0,4,6)$	177	2^2		$\mathfrak{M}_{257}(0,4,2)$	178	2^2
$\mathfrak{M}_{257}(2,6,4_2)$	157	2		$\mathfrak{M}_{257}(4,4,2)$	189	D_8
$\mathfrak{M}_{257}(4,4,4_1)$	192	2^2		$\mathfrak{M}_{258}(0,5,2)$	183	2^2
$\mathfrak{M}_{258}(4,3,4_2)$	199	1		$\mathfrak{M}_{258}(4,1,4_1)$	198	2
$\mathfrak{M}_{258}(2,5,6)$	234	2				
***********	****	****		***********	****	****
$\mathfrak{M}_{266}(1,1,0_1)$	214	1		$\mathfrak{M}_{266}(1,1,0_2)$	204	2^2
$\mathfrak{M}_{266}(2,0,0_1)$	143	2^2		$\mathfrak{M}_{266}(1,0,0_2)$	205	2^2
$\mathfrak{M}_{266}(2,0,0_2)$	132	$2 \times D_8$		$\mathfrak{M}_{266}(2,1,0_1)$	137	S_3
$\mathfrak{M}_{267}(0,4,2_1)$	221	1		$\mathfrak{M}_{267}(0,6,0)$	153	D_8
$\mathfrak{M}_{267}(1,2,0)$	211	1		$\mathfrak{M}_{267}(1,4,2_2)$	212	1
$\mathfrak{M}_{267}(1,4,0)$	210	1		$\mathfrak{M}_{267}(1,2,2_1)$	207	2
$\mathfrak{M}_{267}(1,6,2_1)$	150	D_{10}		$\mathfrak{M}_{267}(2,4,1)$	149	2
$\mathfrak{M}_{267}(2,2,0)$	139	2^2		$\mathfrak{M}_{267}(2,4,2_1)$	140	4
$\mathfrak{M}_{267}(2,4,2_2)$	136	D_8		$\mathfrak{M}_{268}(0,1,1_1)$	219	2
$\mathfrak{M}_{268}(0,1,1_2)$	220	2		$\mathfrak{M}_{268}(0,1,0)$	217	2^2
$\mathfrak{M}_{268}(1,1,0)$	209	2		$\mathfrak{M}_{268}(2,3,1_1)$	147	2
$\mathfrak{M}_{268}(2,3,0)$	148	2^2		$\mathfrak{M}_{268}(2,3,1_2)$	138	2^2
$\mathfrak{M}_{268}(2,1,0)$	144	S_3				
***********	****	****		***********	****	****
$\mathfrak{M}_{277}(4,6,8_1)$	156	2		$\mathfrak{M}_{277}(2,2,4)$	242	2^2
$\mathfrak{M}_{277}(2,6,4)$	152	D_8		$\mathfrak{M}_{278}(4,5,4)$	233	2
$\mathfrak{M}_{278}(4,5,3)$	232	2				
***********	****	****		***********	****	****
$\mathfrak{M}_{288}(5,1,5_1)$	235	1		$\mathfrak{M}_{288}(1,3,7_2)$	230	2
$\mathfrak{M}_{288}(3,5,7_2)$	231	2		$\mathfrak{M}_{288}(1,1,7_2)$	229	3
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Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{333}(1,1,1)$	422	2		$\mathfrak{M}_{334}(3,2,0)$	405	2
$\mathfrak{M}_{334}(1,2,0)$	402	4		$\mathfrak{M}_{334}(1,4,0)$	400	S_3
$\mathfrak{M}_{334}(3,0,0)$	399	A_4		$\mathfrak{M}_{334}(1,4,4)$	406	$3^2 2$
$\mathfrak{M}_{335}(3,0,2)$	371	2		$\mathfrak{M}_{335}(1,4,4)$	394	2
$\mathfrak{M}_{335}(1,2,0)$	361	S_3		$\mathfrak{M}_{335}(1,0,4)$	364	4
$\mathfrak{M}_{336}(3,0,1:2)$	346	2		$\mathfrak{M}_{336}(3,2,1)$	294	2
$\mathfrak{M}_{336}(3,2,0)$	285	2^2		$\mathfrak{M}_{336}(3,2,2)$	278	D_8
$\mathfrak{M}_{337}(3,6,6)$	420	2		$\mathfrak{M}_{337}(3,6,2)$	416	4
$\mathfrak{M}_{337}(1,2,2)$	421	4		$\mathfrak{M}_{338}(1,6,4)$	412	2
$\mathfrak{M}_{338}(3,6,2)$	409	2^2		$\mathfrak{M}_{338}(3,6,6)$	408	S_3
***********	****	****		***********	****	****
$\mathfrak{M}_{345}(2,2,0)$	372	2		$\mathfrak{M}_{345}(0,4,2)$	393	2
$\mathfrak{M}_{345}(2,4,0)$	365	4		$\mathfrak{M}_{345}(4,0,2)$	362	S_3
$\mathfrak{M}_{345}(0,0,4)$	358	A_4		$\mathfrak{M}_{345}(4,0,0)$	357	S_4
$\mathfrak{M}_{346}(4,1,2)$	282	S_3		$\mathfrak{M}_{346}(4,2,2)$	276	D_{12}
$\mathfrak{M}_{346}(0,0,0)$	347	S_3		$\mathfrak{M}_{346}(0,2,0)$	279	D_8
$\mathfrak{M}_{346}(0,0,1)$	344	3		$\mathfrak{M}_{346}(2,2,1)$	295	2
$\mathfrak{M}_{346}(2,2,0)$	296	2		$\mathfrak{M}_{347}(0,2,4)$	404	2
$\mathfrak{M}_{347}(0,5,1)$	401	4		$\mathfrak{M}_{347}(2,6,6)$	417	4
$\mathfrak{M}_{348}(2,1,6)$	407	D_{10}		$\mathfrak{M}_{348}(0,2,3)$	403	2
$\mathfrak{M}_{348}(0,6,1)$	398	A_4		$\mathfrak{M}_{348}(2,2,1)$	413	2
***********	****	****		***********	****	****
$\mathfrak{M}_{355}(0,2,2)$	374	2		$\mathfrak{M}_{355}(0,2,4_1)$	375	2
$\mathfrak{M}_{355}(0,4,2)$	373	2		$\mathfrak{M}_{355}(0,2,4_2)$	366	2^2
$\mathfrak{M}_{355}(4,2,0)$	355	2^2		$\mathfrak{M}_{356}(4,2,1)$	297	2
$\mathfrak{M}_{356}(4,0,2)$	306	2^2		$\mathfrak{M}_{356}(4,2,2)$	286	2^2
$\mathfrak{M}_{356}(4,0,0_1)$	250	S_3		$\mathfrak{M}_{356}(2,1,0_1)$	257	2^2
$\mathfrak{M}_{356}(0,1,0_2)$	341	2^2		$\mathfrak{M}_{356}(2,0,0_1)$	258	2^2
$\mathfrak{M}_{356}(0,0,1)$	348	3		$\mathfrak{M}_{356}(0,0,2)$	304	S_3
$\mathfrak{M}_{356}(0,2,0_2)$	280	D_8		$\mathfrak{M}_{357}(2,2,2)$	383	2
$\mathfrak{M}_{357}(4,2,4_1)$	387	2		$\mathfrak{M}_{357}(4,2,6)$	392	2
$\mathfrak{M}_{357}(0,2,6)$	370	2		$\mathfrak{M}_{357}(0,6,2)$	360	S_3
$\mathfrak{M}_{358}(2,6,4_2)$	390	2		$\mathfrak{M}_{358}(4,2,6)$	391	2
$\mathfrak{M}_{358}(4,6,4_2)$	388	3		$\mathfrak{M}_{358}(0,2,4_1)$	369	2
$\mathfrak{M}_{358}(0,2,6)$	368	2		$\mathfrak{M}_{358}(0,4,2)$	367	2
$\mathfrak{M}_{358}(0,2,4_2)$	363	2^2		$\mathfrak{M}_{358}(0,6,2)$	359	S_3
***********	****	****		***********	****	****
$\mathfrak{M}_{366}(0,0,0_1:2)$	271	2		$\mathfrak{M}_{366}(1,2,0_1)$	298	2
$\mathfrak{M}_{366}(2,1,0_2)$	266	D_8		$\mathfrak{M}_{366}(2,2,1)$	277	A_4
$\mathfrak{M}_{367}(1,6,2_1)$	331	2		$\mathfrak{M}_{367}(1,6,2_2)$	319	2
$\mathfrak{M}_{367}(1,2,2_1)$	330	2		$\mathfrak{M}_{367}(1,2,2_2)$	320	2
$\mathfrak{M}_{367}(0,2,0)$	338	2		$\mathfrak{M}_{367}(0,6,2_1)$	322	4

Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{367}(2,4,0)$	291	2		$\mathfrak{M}_{367}(2,4,2_1)$	293	2
$\mathfrak{M}_{367}(2,2,1)$	290	2		$\mathfrak{M}_{367}(2,2,0)$	292	2
$\mathfrak{M}_{367}(2,6,2_1)$	284	4		$\mathfrak{M}_{367}(2,6,2_2)$	281	S_3
$\mathfrak{M}_{368}(0,6,1_1)$	349	2		$\mathfrak{M}_{368}(1,6,2)$	345	2
$\mathfrak{M}_{368}(2,4,1_2)$	299	1		$\mathfrak{M}_{368}(2,4,0)$	288	2
$\mathfrak{M}_{368}(2,2,1_1)$	289	2		$\mathfrak{M}_{368}(2,2,1_2)$	287	3
$\mathfrak{M}_{368}(2,6,2)$	283	2^2				
***********	****	****		***********	****	****
$\mathfrak{M}_{377}(2,2,8_2)$	424	2		$\mathfrak{M}_{377}(4,6,4)$	419	2
$\mathfrak{M}_{377}(6,6,8_2)$	415	S_3		$\mathfrak{M}_{378}(6,2,6_1)$	418	2
$\mathfrak{M}_{378}(2,2,4:2)$	423	2		$\mathfrak{M}_{378}(2,6,6_2)$	414	1
***********	****	****		***********	****	****
$\mathfrak{M}_{388}(2,6,5_2)$	410	2		$\mathfrak{M}_{388}(6,4,7_1)$	411	2
***********	****	****		***********	****	****
$\mathfrak{M}_{555}(0,4_1,2)$	356	2		$\mathfrak{M}_{555}(4_2,4_1,4_2)$	378	2
$\mathfrak{M}_{556}(4_1,2,0_2)$	307	2		$\mathfrak{M}_{556}(4_1,2,2)$	308	2
$\mathfrak{M}_{556}(4_2,1,2)$	309	2		$\mathfrak{M}_{556}(2,0_1,1)$	259	2
$\mathfrak{M}_{556}(4_1,0_1,1)$	251	3		$\mathfrak{M}_{556}(0,2,0_2)$	300	2^3
$\mathfrak{M}_{556}(0,0_1,0_1)$	244	$2^4 2^2$		$\mathfrak{M}_{557}(0,4_2,6)$	354	2
$\mathfrak{M}_{557}(4_1,4_2,2)$	379	1		$\mathfrak{M}_{558}(0,4_1,4_1)$	353	1
$\mathfrak{M}_{558}(4_2,4_2,4_2)$	377	1		$\mathfrak{M}_{558}(4_1,6,2)$	395	2
$\mathfrak{M}_{558}(0,4_2,4_2)$	352	3		$\mathfrak{M}_{558}(4_2,2,6)$	376	2^2
***********	****	****		***********	****	****
$\mathfrak{M}_{566}(1,0_2,0_1)$	275	1		$\mathfrak{M}_{566}(0_1,1,0_1)$	260	2
$\mathfrak{M}_{566}(2,0_2,0_1)$	267	2		$\mathfrak{M}_{566}(2,2,1)$	302	2^2
$\mathfrak{M}_{566}(0_1,0_2,0_1)$	249	2^2		$\mathfrak{M}_{566}(2,2,0_1)$	303	2^2
$\mathfrak{M}_{566}(0_1,0_1,0_2)$	245	$2^3 2^2$		$\mathfrak{M}_{566}(0_1,2,0_2)$	246	$2^3 2$
$\mathfrak{M}_{567}(0_1,6,0)$	248	2^2		$\mathfrak{M}_{567}(0_1,4_1,0)$	247	D_8
$\mathfrak{M}_{567}(0_2,4_1,0)$	334	2		$\mathfrak{M}_{567}(0_2,4_1,2_1)$	256	2
$\mathfrak{M}_{567}(1,2,0)$	340	1		$\mathfrak{M}_{567}(1,4_1,0)$	339	1
$\mathfrak{M}_{567}(1,2,2_1)$	324	2		$\mathfrak{M}_{567}(0_1,4_2,1)$	262	1
$\mathfrak{M}_{567}(0_2,4_1,2_1)$	323	2		$\mathfrak{M}_{567}(2,4_2,2_1)$	314	1
$\mathfrak{M}_{567}(2,4_2,2_2)$	305	2		$\mathfrak{M}_{567}(2,4_1,0)$	301	2^2
$\mathfrak{M}_{568}(0_1,4_1,1_1)$	252	2		$\mathfrak{M}_{568}(0_1,4_2,1_1)$	255	2
$\mathfrak{M}_{568}(0_1,4_1,2)$	261	2		$\mathfrak{M}_{568}(0_1,6,1_2)$	253	2^2
$\mathfrak{M}_{568}(0_1,6,0)$	254	2^2		$\mathfrak{M}_{568}(2,4_2,1_1)$	312	1
$\mathfrak{M}_{568}(2,4_2,2)$	311	2		$\mathfrak{M}_{568}(2,6,1_1)$	313	2
$\mathfrak{M}_{568}(2,6,1_2)$	310	2				
***********	****	****		***********	****	****
$\mathfrak{M}_{577}(4_1,4_2,4)$	386	1		$\mathfrak{M}_{577}(4_1,4_1,8_1)$	384	2
$\mathfrak{M}_{577}(6,2,8_1)$	382	2		$\mathfrak{M}_{577}(4_1,6,4)$	385	2
$\mathfrak{M}_{577}(2,4_2,4)$	380	2		$\mathfrak{M}_{577}(2,4_1,8_1)$	381	2

Γ	Number	G_{123}		Γ	Number	G_{123}
$\mathfrak{M}_{588}(4_2, 6, 8)$	389	2		$\mathfrak{M}_{588}(2, 2, 7_1)$	397	2
$\mathfrak{M}_{588}(2, 2, 5_2)$	396	2				
***********	****	****		***********	****	****
$\mathfrak{M}_{666}(0_1, 0_2, 0_2)$	268	2^2		$\mathfrak{M}_{666}(0_1, 0_2, 1)$	272	2
$\mathfrak{M}_{667}(0_2, 1, 0)$	263	2^2		$\mathfrak{M}_{667}(0_2, 2_1, 2_1)$	264	4
$\mathfrak{M}_{667}(0_2, 0, 1)$	274	1		$\mathfrak{M}_{667}(1, 2_1, 2_1)$	315	2
$\mathfrak{M}_{667}(1, 0, 0)$	316	1		$\mathfrak{M}_{667}(1, 2_1, 2_2)$	317	2
$\mathfrak{M}_{667}(0_1, 2_1, 2_1)$	332	1		$\mathfrak{M}_{668}(0_2, 1_2, 1_2)$	265	2
$\mathfrak{M}_{668}(0_2, 1_1, 1_2)$	269	2		$\mathfrak{M}_{668}(0_2, 2, 2)$	270	2
$\mathfrak{M}_{668}(0_2, 2, 1)$	273	2				
***********	****	****		***********	****	****
$\mathfrak{M}_{677}(2_2, 1, 4)$	318	1		$\mathfrak{M}_{677}(2_1, 2_1, 8_2)$	321	2
$\mathfrak{M}_{677}(2_1, 0, 4)$	328	1		$\mathfrak{M}_{677}(2_1, 0, 8_1)$	329	1
$\mathfrak{M}_{677}(0, 1, 4)$	337	1		$\mathfrak{M}_{678}(2_1, 1_2, 6_1)$	325	1
$\mathfrak{M}_{678}(2_1, 0, 6_1)$	326	2		$\mathfrak{M}_{678}(2_1, 0, 8)$	327	2
$\mathfrak{M}_{678}(2_1, 1_2, 8)$	333	1		$\mathfrak{M}_{678}(0, 1_1, 6_1)$	335	1
$\mathfrak{M}_{678}(0, 0, 4)$	336	2				
***********	****	****		***********	****	****
$\mathfrak{M}_{688}(1_2, 0, 3)$	342	1		$\mathfrak{M}_{688}(1_2, 2, 5_2)$	343	1
$\mathfrak{M}_{688}(1_1, 2, 7_2)$	350	1		$\mathfrak{M}_{688}(2, 2, 7_2)$	351	2
***********	****	****		***********	****	****
$\mathfrak{M}_{777}(8_2, 4, 4)$	425	2^2		$\mathfrak{M}_{778}(4, 6_1, 6_1)$	426	1
***********	****	****		***********	****	****
$\mathfrak{M}_{888}(7_2, 3, 7_2)$	427	3		$\mathfrak{M}_{888}(7_2, 5, 5)$	428	1
$\mathfrak{M}_{888}(7_2, 3, 3)$	429	1		$\mathfrak{M}_{888}(3, 5_2, 5_2)$	430	1
$\mathfrak{M}_{888}(3, 7_1, 7_1)$	431	1				

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