Asymptotics of Some Convolutional Recurrences

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Abstract

We study the asymptotic behavior of the terms in sequences satisfying recurrences of the form $a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n,k) a_k a_{n-k}$ where, very roughly speaking, $f(n,k)$ behaves like a product of reciprocals of binomial coefficients. Some examples of such sequences from map enumerations, Airy constants, and Painlevé I equations are discussed in detail.

1 Main results

There are many examples in the literature of sequences defined recursively using a convolution. It often seems difficult to determine the asymptotic behavior of such sequences. In this note we study the asymptotics of a general class of such sequences. We prove

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subexponential growth by using an iterative method that may be useful for other recurrences. By subexponential growth we mean that, for every constant $D > 1$, $a_n = o(D^n)$ as $n \to \infty$. Thus our motivation for this note is both the method and the applications we give.

Let $d > 0$ be a fixed integer and let $f(n,k) \geqslant 0$ be a function that behaves like a product of some powers of reciprocals of binomial coefficients, in a general sense to be specified in Theorem 1. We deal with the sequence a_n for $n \geqslant d$ where $a_d, a_{d+1}, \cdots, a_{2d-1} \geqslant 0$ are arbitrary and, when $n \geqslant 2d$,

$$a_n \;=\; a_{n-1} + \sum_{k=d}^{n-d} f(n,k) a_k a_{n-k}. \tag{1}$$

Without loss of generality,

$$\text{we assume that } f(n,k) = f(n,n-k)$$

since we can replace $f(n,k)$ and $f(n,n-k)$ in (1) with $\frac{1}{2}(f(n,k)+f(n,n-k))$.

Theorem 1 proves subexponential growth. Theorem 2 provide more accurate estimates under additional assumptions. In Section 2, we apply the corollary to some examples.

Theorem 1 (Subexponential growth) *Let a_n be defined by recursion (1) with $a_d > 0$. Suppose there is a function $R(x)$ defined on $(0, 1/2]$, an $\alpha > 0$ and an r such that*

(a) $0 < R(x) < r < 1$,

(b) $\lim_{x\to 0+} R(x) = 0$,

(c) $0 \leqslant f(n,k) = O\left(n^{-\alpha} R^{k-d}(k/n)\right)$ *uniformly for* $d \leqslant k \leqslant n/2$.

Then a_n grows subexponentially; in fact,

$$a_n \;=\; (1 + O(n^{-\alpha}))\, a_{n-1}. \tag{2}$$

Proof: We first note that the a_n are non-decreasing when $n \geqslant 2d-1$.

Our proof is in three steps. We first prove that $a_n = O(C^n)$ for some constant $C > 2$. We then prove that C can be chosen very close to 1. Finally we deduce (2) and subexponential growth.

First Step: Since the bound in (c) is bounded by some constant times the geometric series $n^{-\alpha} r^{k-d}$ with ratio less than 1, $\sum_{k=d}^{n-d} f(n,k) = O(n^{-\alpha})$. Hence we can choose M so large that $\sum_{k=d}^{n-d} f(n,k) < 1/4$ when $n > M$. Next choose $C \geqslant 2$ so large ($C = \max\{a_d, a_{d+1}, ..., a_{2d-1}, a_M, 2\}$ will do) that $a_n < 2C^n$ for $n \leqslant M$. By induction, using the recursion (1), we have for $n > M$

$$a_n \;<\; 2C^{n-1} + (1/4)4C^n \;\leqslant\; C^n + C^n \;=\; 2C^n.$$

Second Step: By (b) there is a λ in $(0,1/2)$ such that $R(x) < \frac{1}{2C}$ for $0 < x < \lambda$. Fix any $D \leqslant C$ such that $a_n = O(D^n)$, which is true for $D = C$ by the First Step.

Split the sum in (1) into $\lambda n \leqslant k \leqslant (1-\lambda)n$ and the rest, calling the first range of k the "center" and the rest the "tail". Noting $r < 1$, the center sum is bounded by

$$2\sum_{k=\lambda n+1}^{n/2} f(n,k)a_k a_{n-k} \;=\; O\left(D^n \sum_{k=\lambda n+1}^{n/2} r^{k-d}\right) \;=\; O\left((r^\lambda D)^n\right). \tag{3}$$

Since a_j are increasing, the tail sum is bounded by

$$\begin{aligned} 2\sum_{k=d}^{\lambda n} f(n,k)a_k a_{n-k} &= O(n^{-\alpha})a_{n-1}\sum_{k=d}^{\lambda n} R(x)^{k-d}D^k \\ &= O(n^{-\alpha})a_{n-1}\sum_{k=d}^{\lambda n}(DR(x))^{k-d} \;=\; O\left(n^{-\alpha}a_{n-1}\right), \end{aligned} \tag{4}$$

where the last equality follows from the fact that $DR(x) < 1/2$. Combining (3) and (4),

$$a_n \;=\; \left(1 + O(n^{-\alpha})\right)a_{n-1} + O((r^\lambda D)^n). \tag{5}$$

When $r^\lambda D > 1$, induction on n easily leads to $a_n = O((r^\lambda D')^n)$ for any $D' > D$, an exponential growth rate no larger than $r^\lambda D'$.

Since r^λ has a fixed value less than one, we can iterate this process, replacing D by $r^\lambda D'$ at the start of the Second Step. We finally obtain a growth rate $D > 1$ with $r^\lambda D < 1$. This completes the second step.

Third Step: With the value of D just obtained, the last term in (5) is exponentially small and hence is $O(n^{-\alpha}a_{n-1})$. Thus we obtain (2) which immediately implies subexponential growth of a_n, since $1 + O(n^{-\alpha}) < D$ for any $D > 1$ and sufficiently large n. ∎

To say more than (2), we need additional information about the behavior of the $f(n,k)$. When $f(n,k)/f(n,d)$ is small for each k in the range $d+1 \leqslant k \leqslant n-d-1$, the first and last terms dominate the sum. The following theorem is based on this observation.

Theorem 2 (Asymptotic behavior) *Assume (a)–(c) of Theorem 1 hold. Suppose further that there is a $\beta > 0$ such that*

$$\frac{f(n,k)}{f(n,d)} \;=\; O(n^{-\beta}r^{k-d-1}) \quad \textit{uniformly for} \quad d+1 \leqslant k \leqslant n/2. \tag{6}$$

Then

$$\log a_n \;=\; 2a_d\sum_{k=2d+1}^{n} f(k,d) + O\left(\sum_{k=2d}^{n} f(k,d)\left(k^{-\alpha} + k^{-\beta}\right)\right). \tag{7}$$

Proof: We assume $n > 2d$. Remove the $k = d$ and $k = n - d$ terms from the sum in (1). We first deal with the remaining sum. Theorem 1 gives $a_k = O(D^k)$ for all $D > 1$, so we can assume $D < 1/r$. Using (6)

$$\begin{aligned}\sum_{k=d+1}^{n-d-1} f(n,k)a_k a_{n-k} &= O\Big(f(n,d)n^{-\beta}a_{n-1}\Big)\sum_{k=d+1}^{n/2} r^{k-d-1}D^k \\ &= O\Big(f(n,d)n^{-\beta}a_{n-1}\Big).\end{aligned}$$

Combining this with (1), we obtain

$$\begin{aligned}a_n &= a_{n-1} + 2a_d f(n,d)a_{n-d} + f(n,d)O(n^{-\beta})a_{n-1} \\ &= a_{n-1}\Big(1 + 2a_d f(n,d) + \{O(n^{-\alpha}) + O(n^{-\beta})\}f(n,d)\Big),\end{aligned}$$

Taking logarithms and noting for expansion purposes that $f(n,d) = O(n^{-\alpha})$, we obtain

$$\log a_n - \log a_{n-1} = 2a_d f(n,d) + O\left(\left(n^{-\alpha} + n^{-\beta}\right) f(n,d)\right).$$

Sum over n starting with $n = 2d + 1$. The theorem follows immediately when we note that the constant terms can be incorporated into the $O(\ )$ in (7) since the sum therein is bounded below by a nonzero constant. ∎

Corollary 1 *Assume the conditions of Theorem 2 hold and* $f(n,d) = \Theta(n^{-\alpha})$.

- *If* $\alpha < 1$, *then* $a_n = \exp\left(\Theta\left(n^{1-\alpha}\right)\right)$.
- *If* $\alpha > 1$, *then* $a_n = K + O(n^{1-\alpha})$ *for some constant* K.
- *If* $f(n,d) - A/n$ *are the terms of a convergent series, then* $a_n \sim Cn^{2Aa_d}$ *for some positive constant* C.

Proof: Since $\alpha > 0$ and $\beta > 0$, (7) gives $\log a_n = \Theta(\sum_{k=2d+1}^{n} k^{-\alpha})$. The case $\alpha < 1$ follows immediately; for $\alpha > 1$, we see that a_n is bounded and nondecreasing and therefore has a limit K. For $m > n$, (2) gives $\log(a_m/a_n) = O(n^{1-\alpha})$ uniformly in m. Letting $m \to \infty$, we obtain the claim regarding $\alpha > 1$.

For $\alpha = 1$, the first sum in (7) is $A\log n + B + o(1)$ for some constant B, and the last sum in (7) converges. ∎

2 Examples

We apply Theorem 2 and Corollary 1 to some recursions which arise from combinatorial applications. In our examples, $f(n,k)$ behaves like a product of the reciprocal of binomial coefficients, which satisfies the conditions of Theorems 1 and 2. A more general case of interest is when $f(n,k)$ takes the form of the product of functions like

$$g(n,k) = \frac{[a]_k\ [a]_{n-k}}{[a]_n}$$

for some constant $a > 0$, where $[x]_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(k)}$, the rising factorial. We note that when $a = 1$, $g(n,k) = \binom{n}{k}^{-1}$.

We begin with some useful bounds. When $a > 0$ and $1 \leqslant k \leqslant n/2$,

$$\begin{aligned} g(n,k) &= \prod_{j=0}^{k-1} \frac{a+j}{a+n-k+j} \;<\; \left(\frac{a+k}{a+n}\right)^k \qquad (8) \\ &\leqslant (k/n)^k \left(\frac{1+a/k}{1+a/n}\right)^k \;=\; O\left((k/n)^k\right) \;=\; O\left(n^{-1}(3k/2n)^{k-1}\right) \end{aligned}$$

since $k(2/3)^{k-1}$ is bounded. So g satisfies the condition on f in Theorem 1(c), with $\alpha = 1$. Similarly, when $a > 0$ and $d \leqslant k \leqslant n/2$,

$$\frac{g(n,k)}{g(n,d)} \;=\; \prod_{j=0}^{k-d-1} \frac{a+d+j}{a+n-k+d+j} \;=\; O\left(n^{-1}(3k/2n)^{k-d-1}\right). \qquad (9)$$

This is in accordance with (6) with $\beta = 1$.

Example 1 (Map enumeration constants) There are numbers t_n appearing in the asymptotic enumeration of maps in an orientable surface of genus n, whose value does not concern us here. Define u_n by

$$t_n \;=\; 8\frac{[1/5]_n\,[4/5]_{n-1}}{\Gamma\left(\frac{5n-1}{2}\right)}\left(\frac{25}{96}\right)^n u_n.$$

Then $u_1 = 1/10$ and u_n satisfies the following recursion [3]

$$u_n \;=\; u_{n-1} + \sum_{k=1}^{n-1} f(n,k)u_k u_{n-k} \quad \text{for} \quad n \geqslant 2, \qquad (10)$$

where

$$f(n,k) \;=\; \frac{[1/5]_k\,[1/5]_{n-k}}{[1/5]_n}\,\frac{[4/5]_{k-1}\,[4/5]_{n-k-1}}{[4/5]_{n-1}}.$$

From the observations above, the conditions of Theorem 2 are satisfied with $d = 1$, $R(\lambda) = (3\lambda/2)^2$ and $\alpha = \beta = 2$. Hence, $u_n \sim K$ for some constant K. Unlike the proof in [3], this does not depend on the value of u_1. ∎

Example 2 (Airy constants) The *Airy constants* Ω_n are determined by $\Omega_1 = 1/2$ and the recurrence [7]

$$\Omega_n \;=\; (3n-4)n\Omega_{n-1} + \sum_{k=1}^{n-1}\binom{n}{k}\Omega_k\Omega_{n-k} \quad \text{for} \quad n \geqslant 2.$$

Let $\Omega_n = n!\,[2/3]_{n-1}\,3^n a_n$. Then a_n satisfies (1) with $d = 1$ and

$$f(n,k) = \frac{[2/3]_{k-1}\,[2/3]_{n-k-1}}{[2/3]_{n-1}}.$$

Theorem 2 applies with $d = 1$, $R(\lambda) = 3\lambda/2$ and $\alpha = \beta = 1$. Since

$$f(n,1) \;=\; \frac{1}{n-4/3} \;=\; \frac{1}{n} + \frac{4/3}{n(n-4/3)}$$

and $a_1 = 1/6$, we have $a_n \sim Cn^{1/3}$ for some constant C.

We note that it is possible to apply the result of Olde Daalhuis [13] to obtain a full asymptotic expansion for Ω_n. Let

$$A_n = \frac{\Omega_n}{3^n n!}.$$

Then the recursion for Ω_n becomes

$$A_n = (n-4/3)\,A_{n-1} + \sum_{k=1}^{n-1} A_k A_{n-k},\ n \geqslant 2.$$

It follows that the formal series

$$F(z) = \sum_{n\geqslant 1} \frac{A_n}{z^n}$$

satisfies the Riccati equation

$$F^{'}(z) + \left(1 + \frac{1}{3z}\right) F(z) - F^2(z) - \frac{1}{6z} = 0.$$

It then follows from the result of Olde Daalhuis [13] that

$$A_n \sim \frac{1}{2\pi} \sum_{k=0}^{\infty} b_k \Gamma(n-k),\ \text{as } n \to \infty,$$

where $b_0 = 1$ and b_k can be computed using the recursion

$$b_k = \frac{-2}{k} \sum_{j=2}^{k+1} b_{k+1-j} A_j, \quad k \geqslant 1.$$

In particular, we have

$$\Omega_n \sim \frac{1}{2\pi}\Gamma(n) 3^n n! = \frac{1}{2\pi n}(n!)^2 3^n, \quad \text{as } n \to \infty.$$

It is well known that solutions to the Riccati equation have infinitely many singularities, hence $F(z)$ (via its Borel transform [2]) cannot satisfy a linear ODE with polynomial coefficients. This implies that the sequence A_n (and hence Ω_n) is not holonomic. ∎

Example 3 The following recursion, with $\ell > 0$ and $\ell \neq 1/2$, appeared in [6]. The Airy constants are the special case $\ell = 1$. The case $\ell = 2$ corresponds to the recursion studied in [9, 10], which arises in the study of the Wiener index of Catalan trees. We have $C_1 = \frac{\Gamma(\ell-1/2)}{\sqrt{\pi}}$ and, for $n \geqslant 2$,

$$C_n \;=\; n\frac{\Gamma(n\ell+(n/2)-1)}{\Gamma((n-1)\ell+(n/2)-1)}C_{n-1}+\frac{1}{4}\sum_{k=1}^{n-1}\binom{n}{k}C_kC_{n-k}. \tag{11}$$

Define a_n by $C_n = n!\,g(n)a_n$ where $g(1) = 1$ and

$$g(m) \;=\; \prod_{k=2}^{m}\frac{\Gamma(k\ell+(k/2)-1)}{\Gamma((k-1)\ell+(k/2)-1)}.$$

Then (11) becomes

$$a_n \;=\; a_{n-1}+\sum_{k=1}^{n-1}\frac{g(k)g(n-k)}{4g(n)}a_ka_{n-k},$$

so $f(n,k) = g(k)g(n-k)/4g(n)$.

With a fixed and $x \to \infty$ and using 6.1.47 on p.257 of [1] (or using Stirling's formula), we have

$$\begin{aligned}\frac{\Gamma(x+a)}{\Gamma(x)} \;&=\; x^a\left(1+\frac{a(a-1)}{2x}+O(1/x^2)\right)\\ &=\; x^a\left(1+\frac{a-1}{2x}\right)^a\Big(1+O(1/x^2)\Big) \qquad (12)\\ &=\; \left(x+\frac{a-1}{2}\right)^a\Big(1+O(1/x^2)\Big). \qquad (13)\end{aligned}$$

When $m > 1$, (13) gives us

$$\begin{aligned}g(m) \;&=\; \prod_{k=2}^{m}\left(\frac{(2\ell+1)k-\ell-3}{2}\right)^{\ell}\Big(1+O(1/k^2)\Big)\\ &=\; \Theta(1)\left((\ell+1/2)^m\prod_{k=2}^{m}\left(k-\frac{\ell+3}{2\ell+1}\right)\right)^{\ell}\\ &=\; \Theta(1)\left((\ell+1/2)^m\,[a]_{m-1}\right)^{\ell}, \quad \text{where } a \;=\; \frac{3\ell-1}{2\ell+1}.\end{aligned}$$

Hence

$$f(n,k) \;=\; \Theta(1)\left|\frac{[a]_{k-1}\,[a]_{n-k-1}}{[a]_{n-1}}\right|^{\ell}.$$

where the absolute values have been introduced to allow for $a < 0$. A slight adjustment of the argument leading to (8) and (9) leads to

$$f(n,k) = O(n^{-\ell}(3k/2n)^{\ell(k-1)}) \quad \text{and} \quad \frac{f(n,k)}{f(n,1)} = O(n^{-\ell}(3k/2n)^{\ell(k-d-1)})$$

for $1 \leqslant k \leqslant n/2$. Hence Theorem 2 applies with $\alpha = \beta = \ell$, and a_n converges to a constant when $\ell > 1$ by Corollary 1. ∎

It is interesting to note that there is a simple relation between the sequence u_n in Example 1 and the sequence a_n in Example 3 with $\ell = 2$. It is not difficult to check that the $f(n,k)$ defined in Example 3 is exactly five times the $f(n,k)$ in Example 1: since $a_1 = 5u_1$, we have $a_n = 5u_n$ for all $n \geqslant 1$. This simple relation suggests a relationship between the number of maps on an orientable surface of genus g and the gth moment of a particular toll function on a certain type of trees. Using a bijective approach, Chapuy [4] recently found an expression for t_g as the gth moment of the labels in a random well-labelled tree.

3 A convolutional recursion arising from Painlevé I

The following is recursion (44) in [11].

$$\alpha_n = (n-1)^2\alpha_{n-1} + \sum_{k=2}^{n-2} \alpha_k\alpha_{n-k},\ n \geqslant 1,\ n \geqslant 3. \tag{14}$$

It follows from Proposition 14 of [11] that, for $0 < \alpha_1 < 1$ and $\alpha_2 = \alpha_1 - \alpha_1^2$,

$$\alpha_n = c(\alpha_1)((n-1)!)^2 \left(1 - \frac{2\alpha_2(n-3)}{3(n-1)^2(n-2)^2} + \delta_n\right), \tag{15}$$

where $c(\alpha_1)$ depends only on α_1, and

$$\delta_n = O(1/n^4).$$

We note that α_n for $n \geqslant 3$ depends only on α_2. The proof of (15) relies on the fact that $0 < \alpha_2 < 1/4$ for $0 < \alpha_1 < 1$. It is conjectured in [11] that the asymptotic expression (15) actually holds for a wider range of values of α_1.

For $n \geqslant 1$, let

$$p_n = \frac{\alpha_n}{((n-1)!)^2}.$$

Then, as shown in [11], p_n satisfies recursion (1) with $d = 2$ and

$$f(n,k) = \left(\frac{(n-k-1)!(k-1)!}{(n-1)!}\right)^2.$$

We note here $f(n,2) = O(n^{-4})$. It follows from Theorem 2 that

$$p_n = p(1+\epsilon_n) \text{ for any } \alpha_2 > 0,$$

where $p = p(\alpha_2)$ is a positive constant and $\epsilon_n = O(1/n^3)$.

It is also interesting to note that, with $\alpha_1 = 1/50$, $\alpha_2 = 49/2500$, the sequence α_n is related to the sequence u_n in Example 1 by

$$\alpha_n = [1/5]_n\,[4/5]_{n-1}\,u_n.$$

As mentioned in [11], the formal series $v(t) = \sum_{n\geqslant 1} \alpha_n t^{-n}$ satisfies

$$t^2 v'' + tv' - (t + 2\alpha_1)v + tv^2 + \alpha_1 = 0, \tag{16}$$

and hence, with

$$t = \frac{8\sqrt{6}}{25} x^{5/2},$$

$y(x) = (x/6)^{1/2}(1 - 2v(t))$ satisfies the following Painlevé I:

$$y'' = 6y^2 - x.$$

This connection with Painlevé I is used in [8] to show that the sequence α_n is not holonomic (It follows that u_n and t_n in Example 1 are also not holonomic). The proof uses the fact that solutions to Painlevé I have infinitely many singularities and hence cannot satisfy a linear ODE with polynomial coefficients.

In the following we apply the techniques of [14] to prove that (15) holds for any complex constant α_1. It is convenient to introduce the formal series

$$u_0(z) = v(z^2) = \sum_{n=2}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} \alpha_n z^{-2n}.$$

It follows from (16) that $u = u_0(z)$ is a formal solution to the differential equation

$$\frac{1}{4}u'' + \frac{1}{4z}u' - \left(1 + \frac{2\alpha_1}{z^2}\right)u + u^2 + \frac{\alpha_1}{z^2} = 0.$$

The Stokes lines for this differential equation are the positive and the negative real axes. When the negative real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_1(z) = Ke^{2z} z^{-1/2} \sum_{n=0}^{\infty} c_n z^{-n},$$

in which the Stokes multiplier K is a constant (depending on the constant α_1). To determine the coefficients c_n we observe that u_1 is a solution of the linear differential equation

$$\frac{1}{4}u_1'' + \frac{1}{4z}u_1' - \left(1 + \frac{2\alpha_1}{z^2} - 2u_0\right)u_1 = 0.$$

Hence, for the coefficients c_n we can take $c_0 = 1$ and for the others we have

$$nc_n = \frac{1}{4}\left(n - \frac{1}{2}\right)^2 c_{n-1} + 2\sum_{k=4}^{n+1} b_k c_{n+1-k}, \qquad n \geqslant 1.$$

The first five coefficients are

$$c_0 = 1, \quad c_1 = \frac{1}{16}, \quad c_2 = \frac{9}{512}, \quad c_3 = \frac{75}{8192} + \frac{2}{3}\alpha_2, \quad c_4 = \frac{3675}{524288} + \frac{13}{24}\alpha_2.$$

In a similar manner it can be shown that when the positive real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_2(z) = iKe^{-2z}z^{-1/2}\sum_{n=0}^{\infty}(-1)^n c_n z^{-n}.$$

This is all the information that is needed to conclude that

$$\alpha_n = b_{2n} \sim \frac{K}{\pi}\sum_{k=0}^{\infty}(-1)^k c_k \frac{\Gamma(2n-k-\frac{1}{2})}{2^{2n-k-(1/2)}}, \qquad \text{as} \quad n \to \infty.$$

By taking the first 4 terms in this expansion we can verify that (15) holds for any complex constant α_1.

For more details see [12], [13] and [14]. (It's best to get the version of the first reference on the website `http://www.maths.ed.ac.uk/ adri/public.htm`.)

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