On some densities in the set of permutations

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Abstract

The asymptotic density of random permutations with given properties of the kth shortest cycle length is examined. The approach is based upon the saddle point method applied for appropriate sums of independent random variables.

1 Introduction

Let $n \in \mathbf{N}$, $\mathbf{S}_n$ be the symmetric group of permutations acting on the set $\{1, 2, \ldots, n\}$, and $\mathbf{S} := \mathbf{S}_1 \cup \mathbf{S}_2 \cdots$ Set ν_n for the uniform probability measure on $\mathbf{S}_n$. By $\nu_n(A) := \nu_n(A \cap \mathbf{S}_n)$, we trivially extend it for all subsets of $\mathbf{S}$. If the limit

$$\lim_{n\to\infty} \nu_n(A) =: d(A), \quad A \subset \mathbf{S}$$

exists, $d(A)$ can be called the *asymptotic density* of A. Let $\mathcal{S} \subset \mathbf{S}$ be the class of A having an asymptotic density $d(A)$. The triple $\{\mathbf{S}, \mathcal{S}, d\}$ is far from being a probability space. However, the behavior of $d(A_m)$ for some specialized subsets $A_m \in \mathcal{S}$ as $m \to \infty$ are worth to be investigated. In this paper, we demonstrate that by taking sets connected to the ordered statistics of different cycle lengths.

Recall that each $\sigma \in \mathbf{S}_n$ can be uniquely (up to the order) written as a product of independent cycles. Let $k_j(\sigma) \geqslant 0$ be the number of cycles of length j, $1 \leqslant j \leqslant n$, in such decomposition. The *structure vector* is defined as

$$\bar{k}(\sigma):\ = \big(k_1(\sigma), \ldots, k_n(\sigma)\big).$$

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Set $\ell(\bar{k}) := 1k_1 + \cdots + nk_n$, where $\bar{k} := (k_1, \ldots, k_n) \in \mathbf{Z}_+^n$, then $\ell(\bar{k}(\sigma)) = n$. Moreover, if $\ell(\bar{k}) = n$, then the set $\{\sigma \in \mathbf{S}_n : \bar{k}(\sigma) = \bar{k}\}$ agrees with the class of conjugate permutations in $\mathbf{S}_n$. If ξ_j, $j \geqslant 1$, are independent Poisson random variables (r.vs) given on some probability space $\{\Omega, \mathcal{F}, P\}$, $\mathbf{E}\xi_j = 1/j$, and $\bar{\xi} :\ = (\xi_1, \ldots, \xi_n)$, then [2]

$$\nu_n\big(\bar{k}(\sigma) = \bar{k}\big) = \mathbf{1}\{\ell(\bar{k}) = n\} \prod_{j=1}^{n} \frac{1}{j^{k_j} k_j!} = P\big(\bar{\xi} = \bar{k} \big|\ \ell(\bar{\xi}) = n\big),$$

where $\mathbf{1}$ denotes the indicator function. Moreover,

$$\big(k_1(\sigma), \ldots k_n(\sigma), 0, \ldots,\big) \overset{\nu_n}{\Rightarrow} \big(\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots\big) \tag{1}$$

in the sense of convergence of the finite dimensional distributions. Here and in what follows we assume that $n \to \infty$. The so-called Fundamental Lemma sheds more light than (1). We state it as the following estimate of the total variation distance. If $\bar{k}(\sigma)_r := \big(k_1(\sigma), \ldots, k_r(\sigma)\big)$ and $\bar{\xi}_r := (\xi_1, \ldots, \xi_r)$, then [1]

$$\frac{1}{2} \sum_{\bar{k} \in \mathbf{Z}_+^r} \Big|\nu_n\big(\bar{k}(\sigma)_r = \bar{k}\big) - P\big(\bar{\xi}_r = \bar{k}\big)\Big| = R(n/m) \tag{2}$$

for $1 \leqslant r \leqslant n$. Here and in the sequel $R(u)$ denotes an error term which has the upper bound

$$R(u) = \mathrm{O}(\mathrm{e}^{-u \log u + \mathrm{O}(u)}) \tag{3}$$

with some absolute constants in $\mathrm{O}(\cdot)$.

In the recent decade, a lot of investigations were devoted to the limit distributions of values of additive functions with respect to ν_n. Given a real two dimensional sequence $\{h_j(k)\}$, $j, k \geqslant 1$, $h_j(0) \equiv 0$, such a function is defined as

$$h(\sigma) = \sum_{j \leqslant n} h_j(k_j(\sigma)).$$

The relevant references can be found in [2], [12], [15] and other papers. The family of additive functions

$$s(\sigma, y) = \sum_{j \leqslant y} \mathbf{1}\big\{k_j(\sigma) \geqslant 1\big\}, \quad 1 \leqslant y \leqslant n,$$

is closely related to the ordered statistics

$$j_1(\sigma) < j_2(\sigma) < \cdots < j_s(\sigma)$$

of different cycle lengths appearing in the decomposition (1). Now $s := s(\sigma, n)$ counts all such lengths. We have $s(\sigma, j_k(\sigma)) = k$ for each $1 \leqslant k \leqslant s$. The last relation and the laws of iterated logarithm for $s(\sigma, m)$ led [14] to the following result.

Denote $Lu = \log \max\{u, e\} = L_1 u, \ldots, L_r u = L(L_{r-1} u)$ for $u \in \mathbf{R}$. For $0 < \delta < 1$ and $r \geqslant 2$, set

$$\beta_{rk}(1 \pm \delta) = \left(2k\Big(L_2 k + \frac{3}{2} L_3 k + L_4 k + \cdots + (1 \pm \delta) L_r k\Big)\right)^{1/2}.$$

Theorem ([14]). *For arbitrary* $0<\delta<1$ *and* $r\geqslant 2$*, we have*

$$\lim_{n_1\to\infty}\overline{\lim_{n\to\infty}}\,\nu_n\left(\max_{n_1\leqslant k\leqslant s}\frac{|\log j_k(\sigma)-k|}{\beta_{rk}(1+\delta)}\geqslant 1\right)=0$$

and

$$\lim_{n_1\to\infty}\underline{\lim}_{n\to\infty}\,\nu_n\left(\max_{n_1\leqslant k\leqslant s}\frac{|\log j_k(\sigma)-k|}{\beta_{rk}(1-\delta)}\geqslant 1\right)=1.$$

Thus, we may say that "for almost all $\sigma\in\mathbf{S}_n$"

$$|\log j_k(\sigma)-k|\leqslant\beta_{rk}(1+\delta)$$

uniformly in k, $n_1\leqslant k\leqslant s$, where $n_1\to\infty$ arbitrarily slowly. This assertion is sharp in the sense that we can not change δ by $-\delta$. It can be compared with the following corollary of the invariance principle (see [3], [14]) where the convergence of distributions is examined.

Theorem ([14]). *Uniformly in* $x\in\mathbf{R}$*,*

$$\nu_n\big(\max_{k\leqslant s}|\log j_k(\sigma)-k|\leqslant x\sqrt{\log n}\big)=\frac{1}{\sqrt{2\pi}}\sum_{l\in\mathbf{Z}}(-1)^l\int_{-x}^{x}\mathrm{e}^{-(u-2lx)^2/2}\mathrm{d}u+\mathrm{o}(1).$$

Instead of $j_k(\sigma)$, $1\leqslant k\leqslant s$, one can deal with the sequence

$$J_1(\sigma)\leqslant J_2(\sigma)\leqslant\cdots\leqslant J_w(\sigma)$$

of all cycle lengths appearing in the decomposition of σ. The behavior of these ordered statistics is similar, however, some technical differences do arise in their analysis. Section 3.2 of the paper by D. Panario and B. Richmond [16] contains rather complicated asymptotical formulas for $\nu_n(J_k(\sigma)=m)$ as $m,n\to\infty$ if k is fixed. We are more interested into the case when k is unbounded therefore we now include V.F. Kolchin's result for the so-called middle region.

Theorem ([9]). *Let* $0<\alpha<1$ *be fixed,* $k=\alpha\log n+\mathrm{o}(\sqrt{\log n})$*, and* $n\to\infty$*. Then*

$$\nu_n\big(\log J_k(\sigma)\leqslant k+x\sqrt{k}\big)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-u^2/2}\mathrm{d}u+\mathrm{o}(1)=:\Phi(x)+\mathrm{o}(1).$$

Despite such variety of results, the frequency

$$\nu_n\big(j_k(\sigma)=m\big)=\nu_n\big(k_m(\sigma)\geqslant 1,s(\sigma,m-1)=k-1\big),\tag{4}$$

where $1\leqslant k\leqslant m\leqslant n$, can further be examined. Observe that the event under frequency is described in terms of the first m components $k_j(\sigma)$ of the structure vector. If $m=\mathrm{o}(n)$, then by (2) its frequency can be approximated by an appropriate probability for independent random variables $\xi_j, 1\leqslant j\leqslant m$.

Introduce the independent Bernoulli r. vs η_j, $j\geqslant 1$, such that

$$P(\eta_j=1)=1-\mathrm{e}^{-1/j}=1-P(\eta_j=0)$$

and set $X_y=\sum_{j\leqslant y}\eta_j$ where $1\leqslant y\leqslant n$.

Proposition 1. *For* $1 \leqslant k \leqslant m = \mathrm{o}(n)$*,*

$$\begin{aligned}
&\nu_n\big(j_k(\sigma)=m\big) = P(\eta_m=1)P(X_{m-1}=k-1)+R(n/m) \\
= \quad &\Big(1-\mathrm{e}^{-1/m}\Big)\exp\Big\{-\sum_{j<m}\frac{1}{j}\Big\}\sum_{1\leqslant j_1<\cdots<j_{k-1}<m}\prod_{r=1}^{k-1}\Big(\mathrm{e}^{1/j_r}-1\Big) \\
&+R(n/m),
\end{aligned}$$

where the remainder term is estimated in (3)*.*

This simple corollary following from (2) motivates our interest to the probabilities $P_m(k) := P(X_m = k)$ if $k \geqslant 0$ and $m \geqslant 3$. Of course, we can exclude the trivial cases $P_m(k) = 0$ if $k > m$,

$$P_m(0) = \exp\Big\{-\sum_{j\leqslant m}\frac{1}{j}\Big\} \sim \frac{\mathrm{e}^{-\gamma}}{m},$$

and

$$P_m(m) = \prod_{j\leqslant m}(1-\mathrm{e}^{-1/j}) \sim C_0\frac{\sqrt{P_m(0)}}{m!}.$$

Here $m \to \infty$, γ is the Euler constant, and

$$C_0 = \prod_{j\geqslant 1}\Big(1+\frac{1}{2!j}+\frac{1}{3!j^2}+\cdots\Big)\mathrm{e}^{-1/(2j)}.$$

Contemporary probability theory provides a lot of local theorems for sums of independent Bernoulli r. vs. Since

$$\lambda_m := \mathbf{E}X_m = \sum_{j\leqslant m}(1-\mathrm{e}^{-1/j}) = \log m + C + \mathrm{O}\Big(\frac{1}{m}\Big),$$

$$C := \gamma + \sum_{j\geqslant 1}\Big(1-\mathrm{e}^{-1/j}-\frac{1}{j}\Big),$$

and, similarly,

$$\mathbf{V}ar X_m = \log m + C_1 + \mathrm{O}\Big(\frac{1}{m}\Big),$$

where C_1 is a constant, and $m \geqslant 3$, the results on the so-called large deviations imply the approximations for $P_m(k)$ in the region $k - \log m = \mathrm{o}(\log m)$ as $m \to \infty$ (see [17], Chapter VIII or [7]). H.-K. Hwang's work [8] as well as many others can be used in this zone. However, we still have a great *terra incognita* if $(1+\varepsilon)\log m \leqslant k \leqslant m-1$ where $\varepsilon > 0$. The present paper sheds some light to it. First, we prove some new asymptotic formulas for $P_m(k)$ which are nontrivial outside the region of classical large deviations. Further, we apply them by inserting into the equalities given in Proposition 1. The very idea goes back to the number-theoretical paper by P. Erdős and G. Tenenbaum [6].

We now introduce some notation. Denote

$$F(z,m)=\sum_{0\leqslant k\leqslant m} q_k(m)z^k := \prod_{j\leqslant m}\Big(1+\Big(\mathrm{e}^{1/j}-1\Big)z\Big),\quad z\in\mathbf{C}.$$

Then $P_m(k)=q_k(m)/F(1,m)$. Let $\rho(t,m)$ satisfy the saddle point equation

$$x\,\frac{F'(x,m)}{F(x,m)}=\sum_{j\leqslant m}\frac{x}{a_j+x}=t,\quad a_j:=(\mathrm{e}^{1/j}-1)^{-1} \tag{5}$$

for $0\leqslant t\leqslant m-1$. Set

$$W(t)=\Gamma(t+1)t^{-t}\mathrm{e}^t \tag{6}$$

if $t>0$ and $W(0)=1$, where $\Gamma(t)$ denotes the Euler gamma-function.

Theorem 1. *Let $0<\varepsilon<1$ be arbitrary, $m\geqslant 3$, and $0^0:=1$. Then*

$$P_m(k)=\frac{F(\rho(k,m),m)}{F(1,m)}\,\frac{1}{\rho(k,m)^kW(k)}\Big(1+\mathrm{O}\Big(\frac{1}{\log m}\Big)\Big)$$

uniformly in $0\leqslant k\leqslant m^{1-\varepsilon}$.

Further analysis of the involved quantities leads to interesting simpler formulae. Set

$$L(t,m)=\log\frac{m}{1+t/\log m}.$$

Corollary. *If $0\leqslant k\leqslant m^{1-\varepsilon}$, then*

$$P_m(k)=\frac{1}{F(1,m)}\,\frac{L(k,m)^k}{k!}\,\exp\Big\{\mathrm{O}\Big(\frac{k}{\log m}\Big)\Big\}, \tag{7}$$

and

$$P_m(k)=\frac{F(k/L(k,m),m)}{F(1,m)}\Big(\frac{L(k,m)}{\mathrm{e}}\Big)^k\frac{1}{k!}\,\exp\Big\{\mathrm{O}\Big(\frac{k}{\log^2 m}+\frac{1}{\log m}\Big)\Big\}. \tag{8}$$

The first formula in the corollary implies an asymptotic expression only in the region $k=\mathrm{o}(\log m)$, however, it yields an effective estimate of $P_m(k)$ for $0\leqslant k\leqslant m^{1-\varepsilon}$. The classical results for $k-\log m=\mathrm{o}(\log m)$ are hidden in the second one. Instead of going into the details of that, we return to the cycle lengths and exploit Proposition 1. Set

$$d_k(m)=d\big(j_k(\sigma)=m\big)=(1-\mathrm{e}^{-1/m})P_{m-1}(k-1). \tag{9}$$

Theorem 2. *Let $m\geqslant 3$, $0<\varepsilon<1$, and $1\leqslant k\leqslant m^{1-\varepsilon}$. Then*

$$\frac{d_{k+1}(m)}{d_k(m)}=\frac{L(k,m)}{k}\Big(1+\mathrm{O}\Big(\frac{1}{\log m}\Big)\Big). \tag{10}$$

Moreover,

$$\max_{1\leqslant k\leqslant m} d_k(m)=\frac{1}{m\sqrt{2\pi\log m}}\Big(1+\mathrm{O}\Big(\frac{1}{\log m}\Big)\Big) \tag{11}$$

and the maximum is achieved at

$$k=k_m=\log m+\mathrm{O}(1).$$

Having this and some other arguments in mind, we conjecture that $d_k(m)$ is unimodal for $1 \leqslant k \leqslant m$ and all $m \geqslant 3$.

Going further, we can exploit an idea from the renewal theory. The event $\{\sigma : \ j_k(\sigma) > y\}$ occurs if and only if $\sigma \in \mathbf{S}_n$ has less than k cycles with lengths in $[1, y]$. Hence, by (2),

$$d\big(j_k(\sigma) > y\big) = d\big(s(\sigma, y) < k\big) = P(X_y < k).$$

The last probability is traditionally examined as $y \to \infty$ and $k = k(y)$ belonging to specified regions. This can be exploited. For instance, applying formula (16) from [4], we have

$$\sup_{k \geqslant 1} \left| d\big(j_k(\sigma) > y\big) - \Pi_{\lambda_y}(k) \right| = \frac{1}{2\lambda_y \sqrt{2\pi \mathrm{e}}} \sum_{j \leqslant y} (1 - \mathrm{e}^{-1/j})^2 + \mathrm{O}\big((\log y)^{-3/2}\big),$$

where $\Pi_{\lambda_y}(\cdot)$ is the Poisson distribution with the parameter λ_y defined above. The paper [4] and many other works published in the last decade provide even more exact approximations applicable for $d\big(j_k(\sigma) > y\big)$.

We now seek an asymptotical formula for it as $k \to \infty$, where $y = y(k)$ is a suitably chosen function of k. That may be ascribed to the renewal theory when the summands η_j, $j \geqslant 1$ in X_y are independent but non-identically distributed. The next our result resembles in its form the Kolchin's theorem. The very idea goes back to the number-theoretical paper [5] by J.-M. De Koninck and G. Tenenbaum.

Theorem 3. *We have*

$$d\big(\log j_k(\sigma) \leqslant k + x\sqrt{k}\big) = \Phi(x) - \frac{x^2 - 1 - 3C}{3\sqrt{2\pi k}} \, \mathrm{e}^{-x^2/2} + \mathrm{O}\Big(\frac{1}{k}\Big) \tag{12}$$

uniformly in $k \geqslant 1$ and $x \in \mathbf{R}$.

Finally, we observe that these results can be used to obtain asymptotical formulas for $\max_{m \geqslant k} d_k(m)$ as $k \to \infty$. We intend to discuss that in a forthcoming paper.

2 The saddle point method

Since $P_m(k) = q_k(m)/F(1, m)$, it suffices to analyze the Cauchy integral

$$q_k(m) = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{F(z, m)}{z^{k+1}} \, \mathrm{d}z. \tag{13}$$

Similar but more complicated integrals have been the main task in the work on the number of permutations missing long cycles (see [10] and [11]). We now exploit this experience and the ideas coming from paper [6].

Henceforth let $k \geqslant 0$, $m \geqslant 3$, and $0 < \varepsilon < 1$. For $0 \leqslant t \leqslant m^{1-\varepsilon}$, we have $L := L(t, m) \asymp \log m$. Here and in the sequel the symbol $a \asymp b$ means $a \ll b$ and $b \ll a$ while

$\ll$ is an analog of $\mathrm{O}(\cdot)$. The implicit constants in estimates depend at most on ε therefore the remainder $\mathrm{O}(1/L)$ is just a shorter form of $\mathrm{O}(1/\log m)$.

Observe that the functions $x/(a_j+x)$, $1 \leqslant j \leqslant m-2$, are strictly increasing for $x \in [0,\infty) = \mathbf{R}_+$ therefore the sum over j varies from 0 to the value $m-0$. This proves the existence of a unique $\rho(t,m) > 0$ for $0 < t \leqslant m-1$ and $m \geqslant 3$. Moreover, $\rho(0,m) = 0$. The main task of this section is to prove the following proposition.

Proposition 2. *Let $\rho(k,m)$ be the solution to equation (5) for $t = k$. Then*

$$q_k(m) = \frac{F(\rho(k,m),m)}{\rho(k,m)^k W(k)}\left(1 + \mathrm{O}\left(\frac{1}{L}\right)\right)$$

uniformly in $0 \leqslant k \leqslant m^{1-\varepsilon}$. The function $W(t)$ has been defined in (6).

Firstly, we prove a few lemmas.

Lemma 1. *For all $0 \leqslant t \leqslant m^{1-\varepsilon}$,*

$$\rho(t,m) = \frac{t}{L(t,m)}\left(1 + \mathrm{O}\left(\frac{1}{L}\right)\right). \tag{14}$$

Proof. By the definition, using the inequalities $0 < j - a_j \leqslant 2$ for all $j \geqslant 1$ and the abbreviation $\rho := \rho(t,m)$, for $t > 0$, we obtain

$$\begin{aligned}\frac{t}{\rho} &= \sum_{j \leqslant m} \frac{1}{j+\rho} + \sum_{j \leqslant m}\left(\frac{1}{a_j+\rho} - \frac{1}{j+\rho}\right) \\ &= \int_1^m \frac{du}{u+\rho} + \mathrm{O}(1) = \log\frac{m+\rho}{1+\rho} + \mathrm{O}(1).\end{aligned} \tag{15}$$

By virtue of

$$m^{1-\varepsilon} \geqslant t \geqslant \frac{m\rho}{m+\rho},$$

we have $\rho \ll m^{1-\varepsilon}$. Now (15) reduces to

$$\frac{t}{\rho} = \log\frac{m}{1+\rho} + \mathrm{O}(1). \tag{16}$$

If $\rho \ll 1$, then $t/\rho = \log m + \mathrm{O}(1)$. The last equality is equivalent to (14).

In the remaining case $1 \ll \rho \ll m^{1-\varepsilon}$, where $m \geqslant m(\varepsilon)$ is sufficiently large, by (16),

$$\log m + \mathrm{O}(1) \geqslant \frac{t}{\rho} \geqslant \varepsilon \log m + \mathrm{O}(1).$$

Hence $\rho = Bt/\log m$ with some $B = B(t,m)$, where $0 < c(\varepsilon) \leqslant B \leqslant C(\varepsilon)$ for all $m \geqslant 3$ and some constants $c(\varepsilon)$ and $C(\varepsilon)$ depending on ε. Since

$$\begin{aligned}\log(1+\rho) &= \log\left(1 + \frac{t}{\log m}\right) + \log\frac{\log m + Bt}{\log m + t} \\ &= \log\left(1 + \frac{t}{\log m}\right) + \mathrm{O}(1),\end{aligned}$$

from (16) we obtain the desired formula (14).

The lemma is proved.

We set $b_j := a_j^{-1} = \mathrm{e}^{1/j} - 1$ and examine an analytic function $\varphi(z)$ which, for $|z| < a_1$ or $\Re z > 0$, is defined by

$$\varphi(z) := \sum_{j \leqslant m} \log(1 + b_j z) = \log F(z, m).$$

Denote

$$s_r = \left.\frac{\mathrm{d}^r \varphi(\rho \mathrm{e}^w)}{\mathrm{d}w^r}\right|_{w=0}.$$

Lemma 2. *In the above notation, if $1 \leqslant k \leqslant m^{1-\varepsilon}$, then*

$$\varphi(\rho) = k + \mathrm{O}(\rho) = k\big(1 + \mathrm{O}(L^{-1})\big). \tag{17}$$

If $|z - 1| \leqslant (1 + 2\rho)/4\rho$, then

$$f(z) := \varphi(\rho z) - \varphi(\rho) = k(z - 1) + \mathrm{O}(kL^{-1}|z - 1|^2). \tag{18}$$

Moreover, $s_1 = k$ and

$$s_r = k\big(1 + \mathrm{O}(L^{-1})\big), \quad r \geqslant 2. \tag{19}$$

Proof. We will use the estimates $0 < b_j \leqslant \min\{2, \mathrm{e}/j\}$ and $0 < b_j - 1/j \leqslant 2j^{-2}$ for $j \geqslant 1$. It suffices to take sufficiently large m.

In the proof of Lemma 1, we observed that $\rho \leqslant C_1(\varepsilon) m^{1-\varepsilon}$. If $\rho \geqslant (2\mathrm{e})^{-1}$, then

$$\begin{aligned}
\varphi(\rho) &= \left(\sum_{2\mathrm{e}\rho < j \leqslant m} + \sum_{j \leqslant 2\mathrm{e}\rho}\right) \log(1 + b_j \rho) \\
&= \rho \sum_{2\mathrm{e}\rho < j \leqslant m} b_j + \mathrm{O}\left(\rho^2 \sum_{2\mathrm{e}\rho < j \leqslant m} \frac{1}{j^2}\right) + \mathrm{O}\left(\sum_{j \leqslant 2\mathrm{e}\rho} \log(3\mathrm{e}\rho/j)\right) \\
&= \rho \log \frac{m}{\rho + 1} + \mathrm{O}(\rho) \\
&= k + \mathrm{O}(\rho) = k\big(1 + \mathrm{O}(L^{-1})\big).
\end{aligned}$$

In the last step we applied formula (14) and in the step before that we applied formula (16). The derived expression for $\varphi(\rho)$ also holds in the easier case $\rho < (2\mathrm{e})^{-1}$. We omit the details.

To prove formula (18), we observe that

$$\frac{b_j \rho |z - 1|}{1 + b_j \rho} \leqslant \frac{2\rho}{1 + 2\rho} \cdot \frac{1 + 2\rho}{4\rho} = \frac{1}{2}$$

in the given region, therefore the function

$$f(z) = \sum_{j \leqslant m} \log\left(1 + \frac{b_j \rho (z - 1)}{1 + b_j \rho}\right)$$

is analytic in it. Expanding the logarithm, we obtain

$$\begin{aligned} f(z) &= (z-1)\sum_{j\leqslant m}\frac{b_j\rho}{1+b_j\rho}+\mathrm{O}\Big(\rho^2|z-1|^2\sum_{j\leqslant m}\frac{1}{j^2+\rho^2}\Big)\\ &= k(z-1)+\mathrm{O}\Big(\frac{\rho^2|z-1|^2}{\rho+1}\Big)\\ &= k(z-1)+\mathrm{O}\big(kL^{-1}|z-1|^2\big). \end{aligned}$$

To derive relations (19), it suffices to apply (18) and Cauchy's formula on the circumference $|w|=c$, where $c>0$ is chosen so that $|e^w-1|\leqslant c\mathrm{e}^c\leqslant 1/2\leqslant(1+2\rho)/(4\rho)$.

The lemma is proved.

Remark. The argument mentioned in the last step actually yields the Taylor expansion of $f(\mathrm{e}^w)$ in the region $|w|\leqslant c$.

Lemma 3. *There exists an absolute positive constant c_1 such that*

$$|F(\rho\mathrm{e}^{i\tau},m)|\ll\exp\big\{-c_1k\tau^2\big\}F(\rho,m)$$

uniformly in $1\leqslant k\leqslant m^{1-\varepsilon}$ and $|\tau|\leqslant\pi$.

Proof. We apply the identity

$$\frac{|1+x\mathrm{e}^{i\tau}|^2}{(1+x)^2}=1-\frac{2x(1-\cos\tau)}{(1+x)^2}$$

with $x=b_j\rho\leqslant\mathrm{e}\rho/j\leqslant 1/4$ for $j>4\mathrm{e}\rho$ and obtain

$$\begin{aligned} \frac{|F(\rho\mathrm{e}^{i\tau},m)|^2}{F(\rho,m)^2} &\leqslant \prod_{4\mathrm{e}\rho\leqslant j\leqslant m}\Big(1-\frac{2b_j\rho(1-\cos\tau)}{(1+b_j\rho)^2}\Big)\\ &\leqslant \exp\Big\{\sum_{4\mathrm{e}\rho<j\leqslant m}\log\Big(1-\frac{2b_j\rho(1-\cos\tau)}{(1+b_j\rho)^2}\Big)\Big\}\\ &\leqslant \exp\Big\{-\sum_{4\mathrm{e}\rho<j\leqslant m}\frac{b_j\rho(1-\cos\tau)}{(1+b_j\rho)^2}\Big)\Big\}\\ &\leqslant \exp\Big\{-\frac{4}{5}(1-\cos\tau)\sum_{4\mathrm{e}\rho<j\leqslant m}\frac{b_j\rho}{1+b_j\rho}\Big\}. \end{aligned}$$

Now, to involve k, using the definition of ρ we complete the sum in the exponent by the quantity

$$\sum_{j\leqslant 4\mathrm{e}\rho}\frac{b_j\rho}{1+b_j\rho}=\mathrm{O}(\rho)=\mathrm{O}\Big(\frac{k}{L}\Big).$$

By virtue of the inequality $1-\cos\tau\geqslant 2\tau^2/\pi^2$, we now obtain

$$\frac{|F(\rho\mathrm{e}^{i\tau},m)|^2}{F(\rho,m)^2}\leqslant\exp\Big\{-\frac{8\tau^2}{5\pi^2}k\Big(1+\mathrm{O}\Big(\frac{1}{L}\Big)\Big)\Big\}$$

for $1 \leqslant k \leqslant m^{1-\varepsilon}$. This implies the desired estimate if $m \geqslant m(\varepsilon)$ is sufficiently large. For $3 \leqslant m \leqslant m(\varepsilon)$, the claim of Lemma 3 is trivial.

The lemma is proved.

Proof of Proposition 2. For $k = 0$, its claim is evident therefore we assume that $k \geqslant 1$. Firstly, we separate a special case. If $\rho \leqslant 1/6$, then $|z| = 1$ implies $|z-1| \leqslant 2 \leqslant (2\rho+1)/4\rho$. Thus estimate (18) is at our disposal. Moreover, as we have observed in the proof of Lemma 1, $k/\rho = \log m + \mathrm{O}(1)$ and $L \asymp \log m$. So, using Lemma 2, we obtain

$$\begin{aligned} q_k(m) &= \frac{F(\rho,m)}{\rho^k 2\pi i}\int_{|w|=1}\frac{\exp\{f(w)\}}{w^{k+1}}\mathrm{d}w \\ &= \frac{F(\rho,m)}{(\mathrm{e}\rho)^k 2\pi i}\int_{|w|=1}\frac{\mathrm{e}^{kw}}{w^{k+1}}\big(1+\mathrm{O}\big(kL^{-1}|w-1|^2\big)\big)\mathrm{d}w \\ &= \frac{F(\rho,m)}{(\mathrm{e}\rho)^k}\bigg(\frac{k^k}{k!}+\mathrm{O}\bigg(\frac{k\mathrm{e}^k}{L}\int_{-\pi}^{\pi}\mathrm{e}^{-k(1-\cos\tau)}(1-\cos\tau)\mathrm{d}\tau\bigg)\bigg). \end{aligned}$$

By virtue of $1-\cos\tau \asymp \tau^2$ for $|\tau| \leqslant \pi$, the last integral is of order $k^{-3/2}$. This and Stirling's formula yield the desired asymptotic formula in the selected case.

If $1/6 \leqslant \rho \ll m^{1-\varepsilon}$, then we can start with

$$q_k(m) = \frac{F(\rho,m)}{\rho^k 2\pi}\int_{-\pi}^{\pi}\exp\{f(\mathrm{e}^{i\tau}) - ik\tau\}\mathrm{d}\tau.$$

Using the expansion of $f(\mathrm{e}^{i\tau})$ mentioned in Remark after Lemma 2, for $|\tau| \leqslant c$, we have

$$f(\mathrm{e}^{i\tau}) = ik\tau - \frac{1}{2}s_2\tau^2 - \frac{i}{6}s_3\tau^3 + \mathrm{O}(k\tau^4).$$

Hence

$$\exp\{f(\mathrm{e}^{i\tau}) - ik\tau\} = \mathrm{e}^{-s_2\tau^2/2}\Big(1 - \frac{is_3\tau^3}{6} + \mathrm{O}(k\tau^4 + k^2\tau^6)\Big)$$

for $|\tau| \leqslant ck^{-1/3}$. Exploiting the symmetry of the term with τ^3 we have

$$\begin{aligned} I_1 \; : &= \frac{1}{2\pi}\int_{|\tau|\leqslant ck^{-1/3}}\exp\{f(\mathrm{e}^{i\tau}) - ik\tau\}\mathrm{d}\tau \\ &= \frac{1}{2\pi}\bigg(\int_{\mathbf{R}} - \int_{|\tau|>ck^{-1/3}}\bigg)\mathrm{e}^{-s_2\tau^2/2}\mathrm{d}\tau + \mathrm{O}(k^{-3/2}) \\ &= \frac{1}{\sqrt{2\pi s_2}} + \mathrm{O}(k^{-3/2}) = \frac{1}{\sqrt{2\pi k}}\Big(1+\mathrm{O}(L^{-1})\Big). \end{aligned}$$

In the last step we used (19) and the inequality $k \gg L$ following from $\rho \geqslant 1/6$.

Applying Lemma 3 we obtain

$$\begin{aligned} I_2 \; : &= \int_{ck^{-1/3}\leqslant|\tau|\leqslant\pi}\frac{|F(\rho\mathrm{e}^{i\tau},m)|}{F(\rho,m)}\mathrm{d}\tau \\ &\ll \int_{|\tau|\geqslant ck^{-1/3}}\mathrm{e}^{-c_1k\tau^2}\mathrm{d}\tau \ll k^{-3/2}. \end{aligned}$$

Collecting the estimates of I_1 and I_2, we see that

$$q_k(m) = \frac{F(\rho, m)}{\rho^k}\Big(I_1 + \mathrm{O}(I_2)\Big) = \frac{F(\rho, m)}{\rho^k\sqrt{2\pi k}}\Big(1 + \mathrm{O}(L^{-1})\Big).$$

Again Stirling's formula yields the desired result.

The proposition is proved.

3 Proofs of Theorems and Corollaries

The claim of Theorem 1 follows from Proposition 2. We will discuss only the remaining statements.

Proof of Corollary. The case $k = 0$ is trivial. If $k \geqslant 1$, formula (7) follows from Proposition 2, (14), and (17).

To prove (8), let us set $\rho = \rho(k, m)$ and $L = L(k, m)$. Applying Proposition 2 we approximate $F(\rho, m)$ by $F(k/L, m)$. That is available because of the inequality $|k/(\rho L) - 1| \leqslant 1/2$ following from (14) provided that m is sufficiently large. By (18) and (14), we have

$$\begin{aligned} \varphi(\rho) &= \varphi\Big(\rho\,\frac{k}{\rho L}\Big) - k\Big(\frac{k}{\rho L} - 1\Big) + \mathrm{O}\Big(\frac{k}{L}\Big|\frac{k}{\rho L} - 1\Big|^2\Big) \\ &= \varphi\Big(\frac{k}{L}\Big) - k\log\Big(\frac{k}{\rho L}\Big) + \mathrm{O}\Big(\frac{k}{L^2}\Big). \end{aligned}$$

Inserting this into the equality in Proposition 2 we complete the proof of (8).

Proof of Theorem 2. Observe that, for $1 \leqslant k-1 \leqslant t \leqslant k \leqslant m^{1-\varepsilon}$, we have $L(t, m-1) = L(k, m-1) + \mathrm{O}(1/\log m)$ and $L(k, m-1) = L(k, m) + \mathrm{O}(1/m)$ therefore afterwards, for different arguments, we may use $L = L(k, m)$. Denote $r(t) := \rho(t, m-1)$, $r_0 = r(k-1)$, and $r_1 = r(k)$. Then, by Proposition 2,

$$\begin{aligned} \frac{d_{k+1}(m)}{d_k(m)} &= \frac{q_{m-1}(k)}{q_{m-1}(k-1)} \\ &= \frac{F(r_1, m-1)}{F(r_0, m-1)}\,\frac{r_0^{k-1}}{r_1^k}\,\frac{1}{\mathrm{e}}\Big(\frac{k}{k-1}\Big)^{k-1}\Big(1 + \mathrm{O}\Big(\frac{1}{L}\Big)\Big) \end{aligned}$$

for $k \geqslant 1$. Set

$$K(t) = \log F\big(r(t), m-1\big) - t\log r(t).$$

We have

$$\frac{d_{k+1}(m)}{d_k(m)} = \exp\Big\{\int_{k-1}^{k} K'(t)dt - 1 + (k-1)\log\Big(\frac{k}{k-1}\Big) + \mathrm{O}\Big(\frac{1}{L}\Big)\Big\}.$$

By the definitions,

$$\begin{aligned} K'(t) &= r'(t)\left(\frac{F'\big(r(t),m-1\big)}{F\big(r(t),m-1\big)}-\frac{t}{r(t)}\right)-\log r(t) \\ &= -\log r(t)=\log L-\log t+\mathrm{O}(L^{-1}). \end{aligned}$$

Inserting this expression into the previous equality and integrating we obtain the desired result (10).

To prove (11), we first restrict the region to $1\leqslant k\leqslant\sqrt{m}$ where all obtained remainder term estimates contain absolute constants. Hence, by virtue of (10), we can find absolute positive constants C_2 and C_3 such that $d_k(m)$ is increasing for $1\leqslant k\leqslant \log m-C_2$ and decreasing for $\log m+C_3\leqslant k\leqslant\sqrt{m}$. If $k=\log m+\mathrm{O}(1)$, then

$$L(k,m)=\log m+\mathrm{O}(1), \qquad \rho=\rho(k,m)=1+\mathrm{O}(\log^{-1} m),$$

and, by (18) applied with $z=1/\rho$,

$$\varphi(\rho)-k\log\rho-\varphi(1)\ll 1/\log m.$$

This and Proposition 2 imply

$$\begin{aligned} P_m(k) &= \frac{\exp\big\{\varphi(\rho)-k\log\rho-\varphi(1)\big\}}{W(k)}\left(1+\mathrm{O}\Big(\frac{1}{\log m}\Big)\right) \\ &= \frac{1}{\sqrt{2\pi\log m}}\left(1+\mathrm{O}\Big(\frac{1}{\log m}\Big)\right) \end{aligned}$$

for $k=\log m+\mathrm{O}(1)$. The same holds for $P_{m-1}(k-1)$. Recalling (9) we obtain

$$\max_{1\leqslant k\leqslant\sqrt{m}} d_k(m)=\frac{1}{m\sqrt{2\pi\log m}}\left(1+\mathrm{O}\Big(\frac{1}{\log m}\Big)\right). \tag{20}$$

It remains to estimate $d_k(m)$ for $\sqrt{m}\leqslant k\leqslant m$. Differentiating the function $F(z,m)$ we have the estimate

$$\begin{aligned} q_k(m) &\leqslant \frac{1}{k!}\Bigg(\sum_{j\leqslant m}(\mathrm{e}^{-1/j}-1)\Bigg)^k \\ &\leqslant \exp\big\{k\log(2\log m+C_4)-k\log k+\mathrm{O}(k)\big\} \\ &\ll \exp\{-(1/3)\sqrt{m}\log m\} \end{aligned}$$

which shows that the maximum of $d_k(m)=q_{k-1}(m-1)/F(1,m-1)$ over $1\leqslant k\leqslant m$ is given by (20).

The theorem is proved.

Proof of Theorem 3. Let

$$g(z)=\sum_{j\geqslant 1}\left(\log(1+b_jz)-\frac{b_jz}{1+b_j}\right)+Cz, \qquad G(z)=\mathrm{e}^{g(z)}.$$

The function $g(z)$ has an analytic continuation to the region $\mathbf{C}\backslash[-(\mathrm{e}-1)^{-1}, -\infty)$, however, we prefer to use it as a function of real variable. Observe that

$$G(u) = \mathrm{e}^{\gamma}\Big(1 + C(u-1) + \mathrm{O}\big((1-u)^2\big)\Big). \tag{21}$$

for $u \in [0, T]$, where $T > 0$ is an arbitrary fixed number and the constant in $\mathrm{O}(\cdot)$ depends on T. For such u, using elementary inequalities we can rewrite

$$\begin{aligned} F(u,y) &= \exp\Big\{\sum_{j\leqslant y}\Big(\log(1+b_j u) - \frac{b_j u}{1+b_j}\Big) + u\sum_{j\leqslant y}\Big(\frac{b_j}{1+b_j} - \frac{1}{j}\Big) + u\sum_{j\leqslant y}\frac{1}{j}\Big\} \\ &= G(u)y^u\big(1 + \mathrm{O}(1/y)\big), \end{aligned} \tag{22}$$

where the remainder depends on T only.

Let $k \geqslant 1$ be arbitrary, $0 \leqslant l \leqslant k$, and let $y \geqslant 3$ be such that $k/10 \leqslant \log y \leqslant 10k$. Then $|L(l,y) - \log y| \leqslant \log 11$ and

$$\frac{l}{L(l,y)} = \frac{l}{\log y}\Big(1 + \mathrm{O}\Big(\frac{1}{\log y}\Big)\Big),$$

where the constant in $\mathrm{O}(\cdot)$ is absolute. From formula (8) with y instead of m and (22) with $u = l/L(l,y)$, we obtain

$$\begin{aligned} q_l(y) &= F(1,y)P_y(l) \\ &= F\Big(\frac{l}{L(l,y)}, y\Big)\Big(\frac{L(l,y)}{\mathrm{e}}\Big)^l \frac{1}{l!}\Big(1 + \mathrm{O}\Big(\frac{1}{k}\Big)\Big) \\ &= G\Big(\frac{l}{L(l,y)}\Big)\exp\Big\{\frac{l\log y}{L(l,y)} + l\log L(l,y) - l\Big\}\frac{1}{l!}\Big(1 + \mathrm{O}\Big(\frac{1}{k}\Big)\Big) \\ &= G\Big(\frac{l}{\log y}\Big)\frac{(\log y)^l}{l!}\Big(1 + \mathrm{O}\Big(\frac{1}{k}\Big)\Big) \end{aligned}$$

for $0 \leqslant l \leqslant k$ with an absolute constant in the remainder term. In the exponent, we have used the second order approximation of the logarithmic function.

If $k/10 \leqslant \log y \leqslant 10k$, then recalling (21) and (22), we arrive at

$$\begin{aligned} d\big(j_k(\sigma) > y\big) &= \frac{1}{F(1,y)}\sum_{0\leqslant l<k} q_l(y) \\ &= \sum_{0\leqslant l<k}\frac{(\log y)^l}{l!y}\Big\{1 + C\Big(\frac{l}{\log y} - 1\Big) + \mathrm{O}\Big(\frac{1}{k} + \Big(\frac{l}{\log y} - 1\Big)^2\Big)\Big\} \\ &= \sum_{0\leqslant l<k}\frac{(\log y)^l \mathrm{e}^{-\log y}}{l!} - C\frac{(\log y)^{k-1}}{(k-1)!y} + \mathrm{O}\Big(\frac{1}{k}\Big). \end{aligned}$$

Consequently, if $|x| \leqslant k^{1/6}$ and $k \geqslant 3$, the last equality for $y = \mathrm{e}^{k+x\sqrt{k}}$ implies

$$\begin{aligned} d\big(\log j_k(\sigma) > k + x\sqrt{k}\big) &= S_k(\log y) - \frac{(k+x\sqrt{k})^k \mathrm{e}^{-k-x\sqrt{k}}}{k!} \\ &\quad - C\frac{(k+x\sqrt{k})^{k-1}}{(k-1)!}\mathrm{e}^{-k-x\sqrt{k}} + \mathrm{O}\Big(\frac{1}{k}\Big), \end{aligned} \tag{23}$$

where, by Lemma 2.1 from [5],

$$S_k(\log y) := \sum_{0\leqslant l\leqslant k} \frac{(k+x\sqrt{k})^l \mathrm{e}^{-k-x\sqrt{k}}}{l!} = 1-\Phi(x) + \frac{(2+x^2)\mathrm{e}^{-x^2/2}}{3\sqrt{2\pi k}} + \mathrm{O}\Big(\frac{1}{k}\Big).$$

For the other terms in (23), Stirling's formula yields

$$\frac{(k+x\sqrt{k})^k \mathrm{e}^{-k-x\sqrt{k}}}{k!} = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi k}}\bigg(1+\mathrm{O}\Big(\frac{|x|^3}{\sqrt{k}}\Big)\bigg)\bigg(1+\mathrm{O}\Big(\frac{1}{k}\Big)\bigg) = \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi k}} + \mathrm{O}\Big(\frac{1}{k}\Big)$$

provided that $|x| \leqslant k^{1/6}$. The last quantity on the right-hand side is also equal to the factor of $-C$ in (23). Inserting these estimates into (23) we obtain the desired result (12) for $|x| \leqslant k^{1/6}$ and $k \geqslant 3$.

If $x \geqslant k^{1/6}$, by monotonicity of $x \mapsto d\big(\log j_k(\sigma) > k + x\sqrt{k}\big)$ and the just proved relation,

$$d\big(\log j_k(\sigma) > k + x\sqrt{k}\big) \leqslant d\big(\log j_k(\sigma) > k + k^{2/3}\big) \ll k^{-1}.$$

Consequently, formula (12) trivially holds in this region. Similarly, if $x \leqslant -k^{1/6}$, to verify equality (12), we can use the estimate

$$d\big(\log j_k(\sigma) \geqslant k + x\sqrt{k}\big) \leqslant d\big(\log j_k(\sigma) \geqslant k - k^{2/3}\big) \ll k^{-1}.$$

The theorem is proved.

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