On zero-sum free subsets of length 7

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Abstract

Let G be a finite additively written abelian group, and let X be a subset of 7 elements in G. We show that if X contains no nonempty subset with sum zero, then the number of the elements which can be expressed as the sum over a nonempty subsequence of X is at least 24.

1 Introduction

Let G be an additive abelian group and $X \subseteq G$ a subset of G. We denote by $f(G, X) = f(X)$ the number of nonzero group elements which can be expressed as a sum of a nonempty subset of X. For a positive integer $k \in \mathbb{N}$, let $f(k)$ denote the minimum of all $f(G, X)$, where the minimum is taken over all finite abelian groups G and all zero-sum free subsets $X \subset G$ with $|X| = k$. The invariant $f(k)$ was first studied by R. B. Eggleton and P. Erdős in 1972 [1]. For every $k \in \mathbb{N}$ they obtained a subset X in a cyclic group G with $|X| = k$ such that

$$f(k) \leqslant f(G, X) = \left\lfloor \frac{1}{2}k^2 \right\rfloor + 1. \tag{1}$$

And J. E. Olson [2] proved that

$$f(k) \geqslant \frac{1}{9}k^2.$$

Moreover, Eggleton and Erdős determined $f(k)$ for all $k \leqslant 5$, and they stated the following conjecture (which holds true for $k \leqslant 5$):

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Conjecture 1.1. *For every $k \in \mathbb{N}$ there is a cyclic group G and a zero-sum free subset $X \subset G$ with $|X| = k$ such that $f(k) = f(G, X)$.*

Recently, Weidong Gao et al. [3] proved that $f(6) = 19$ and G.Bhowmik et al. [5] showed that $f(G, X) \geqslant 24$ (the lower bound is sharp), where G is a cyclic group, $|X| = 7$. Together with the conjecture above, we have that $f(7) = 24$. The main aim of the present paper is to show the following theorem.

Theorem 1.1. $f(7) = 24$.

In Section 2, we fix the notation. Sections 3 and 4 are devoted to the tools and lemmas needed in the proof of Theorem 1.1. In Section 5, we prove Theorem 1.1 with the help of a C++ program.

Throughout this paper, let G denote an additive finite abelian group.

2 Notation

We follow the conventions of [6] and [3] for notation concerning sequences over an abelian group.

We denote by $\mathbb{N}$ the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers a, $b \in \mathbb{R}$ we set $[a, b] = \{x \in \mathbb{Z} | a \leqslant x \leqslant b\}$.

Let $\mathcal{F}(G)$ denote the multiplicative, free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. An element $X \in \mathcal{F}(G)$ will be written in the form

$$X = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{v_g(X)}$$

where $v_g(X) \in \mathbb{N}_0$ is the *multiplicity* of g in X. For a sequence X above we have:

$$|X| = l = \sum_{g \in G} v_g(X) \in \mathbb{N}_0 \quad \textit{the length of } X,$$

$$\sigma(X) = \sum_{i=1}^{l} g_i = \sum_{g \in G} v_g(X) g \in G \quad \textit{the sum of } X,$$

$$\sum(X) = \{\sum_{i \in I} g_i | \emptyset \neq I \subset [1, l]\} \quad \textit{the set of subsums of } X.$$

We say that X is

- *zero-sum free if* $0 \notin \sum(X)$,
- *a zero-sum sequence if* $\sigma(X) = 0$,
- *squarefree if* $v_g(X) \leqslant 1$ *for all* $g \in G$*, moreover, a squarefree sequence can be considered as a subset of* G.

For a zero-sum free sequence X over G, we have:

$$f(G,X) = f(X) = |\sum(X)|,$$

$$f(G,k) = \min\{f(X)|X \in \mathcal{F}(G) \text{ zero-sum free, squarefree and } |X| = k\}$$

and set $f(G,k) = \infty$ when there are no sequences in G of the above form.

$$f(k) = \min\{f(G,k)|G \text{ run over all finite abelian groups}\}$$

- Let $D(G)$ denote the Davenport's constant of G and $r(G)$ the rank of G.
- Let $ol(G)$ denote the maximal length of a sequence X over G which is zero-sum free and squarefree. The invariant $ol(G)$ is called the *Olson constant* of G.

3 Preliminaries

Lemma 3.1. *1. If $k \in \mathbb{N}$ and $X = X_1 \cdot \ldots \cdot X_k \in \mathcal{F}(G)$ is a zero-sum free sequence, then*

$$f(X) \geqslant f(X_1) + \cdots + f(X_k).$$

2. If $X \subset G$ is zero-sum free, $|X| = k$ and $k \in \mathbb{N}$, then

$$f(X) \begin{cases} = 1, & if \;\; k = 1 \\ = 3, & if \;\; k = 2 \\ \geqslant 5, & if \;\; k = 3 \\ \geqslant 6, & if \;\; k = 3 \text{ and } 2g \neq 0 \text{ for all } g \in X \\ \geqslant 2k, & if \;\; k \geqslant 4. \end{cases}$$

Proof. 1. See [6] Theorem 5.3.1. □
2. See [6] Corollary 5.3.4.

Lemma 3.2. ([3]) *$f(5) = 13$, $f(6) = 19$.*

Lemma 3.3. ([5]) *$f(G,7) \geqslant 24$, where G is a cyclic group. Furthermore, let $G = C_{25}$ and $X = \{5, 10, 1, 6, 11, 16, 21\}$, then $f(X) = 24$.*

Lemma 3.4. *Let $X \subset G$ be a zero-sum free subset of G and $|X| = 7$. If X contains an element of order 2, then $f(X) \geqslant 25$.*

Proof. See [3] Theorem 3.2. □

4 Some bounds on subset S

The lemmas in this section follows mainly from A. Pixton [7].

Lemma 4.1. ([7] Lemma 4.3) *Let G be a finite abelian group and let $X \subseteq G\backslash\{0\}$ be a generating set for G. Suppose S is a nonempty proper subset of G, then*

$$\sum_{x\in X} |(S+x)\backslash S| \geqslant |X|.$$

Lemma 4.2. ([7] Lemma 4.4) *Let G be a finite abelian group and let $X \subseteq G\backslash\{0\}$ be a generating set for G. Suppose $f : G \to \mathbb{Z}$ is a function on G. Then*

$$\sum_{\substack{x\in X\\ g\in G}} max\{f(g+x) - f(g), 0\} \geqslant (max(f) - min(f))\,|X|.$$

The proofs of the following two Lemmas are essential from A. Pixton ([7] Theorem 4.5). For the convenience of the reader, we present the proof here.

Lemma 4.3. *Let G be a finite abelian group and let $X \subseteq G\backslash\{0\}$ be a generating set for G. Suppose $S \subseteq G$ satisfies $|(S+x)\backslash S| \leqslant m$, for $m \in \mathbb{N}$ and all $x \in X$, and for $Y \subset X$ and $H =< Y >\not\subseteq G$, define a function $f : G/H \to \mathbb{Z}$ by $f(a) = |(a+H)\cap S|$ for $a \in G/H$. Then*

$$max(f) - min(f) \leqslant m.$$

Proof. First, without loss of generality, we may replace Y by a minimal subset of Y that still generates H, and we still denote it by Y. Then we may replace X by a minimal subset X_0 of X that satisfies $Y \subset X_0 \subset X$ and $\langle X_0\rangle = G$. For the convenience, we still label it by X. Also, if $|H| \leqslant m$, the result is trivial. Since

$$\begin{aligned}
|(S+x)\backslash S| &= |(S-x)\backslash S| \\
&= \sum_{a\in G/H} |((S-x)\backslash S)\cap(a+H)| \\
&= \sum_{a\in G/H} |(S-x)\cap(a+H)| - |(S-x)\cap S\cap(a+H)| \\
&= \sum_{a\in G/H} |S\cap(a+x+H)| - |(S-x)\cap S\cap(a+H)| \\
&\geqslant \sum_{a\in G/H} \max\{f(a+x)-f(a),\,0\}
\end{aligned}$$

It follows that

$$\begin{aligned}
m|X\backslash Y| &\geqslant \sum_{x\in X\backslash Y} |(S+x)\backslash S| \\
&\geqslant \sum_{x\in X\backslash Y}\sum_{a\in G/H} \max\{f(a+x)-f(a),\,0\} \\
&\geqslant (\max(f)-\min(f))|X\backslash Y| \qquad (2)
\end{aligned}$$

by Lemma 4.2. Since $H \not\subseteq G$, $|X\backslash Y| > 0$, then the result follows immediately from (2). $\square$

Lemma 4.4. *Let G be a finite abelian group, $X \subseteq G\backslash\{0\}$ a generating set for G, $Y \subset X$ and $H =< Y >\not\subseteq G$. Suppose $|H| > m$ and $|G/H| > m$, where $m \in \mathbb{N}$, and suppose $S \subseteq G$ satisfies $|(S+x)\backslash S| \leqslant m$ for all $x \in X$. Then*

$$min\{|S|, |G\backslash S|\} \leqslant m^2.$$

Proof. Define a function $f: G/H \to \mathbb{Z}$ by $f(a) = |(a+H) \cap S|$ for $a \in G/H$.We have

$$\max(f) - \min(f) \leqslant m$$

by Lemma 4.3. Then by replacing S by G/S if necessary, we can assume that $f(a) \neq |H|$ for any $a \in G/H$. The reason is that

$$|(G\backslash S + x)\backslash(G\backslash S)| = |(S+x)\backslash S|$$

Thus we can apply Lemma 4.1 to obtain that

$$\begin{aligned} m|Y| &\geqslant \sum_{x\in Y} |(S+x)\backslash S| \\ &= \sum_{a\in G/H}\sum_{x\in Y} |(S\cap(a+H)+x)\backslash(S\cap(a+H))| \\ &\geqslant |\text{supp}(f)||Y| \end{aligned}$$

where $\text{supp}(f) = \{a \in G/H\,|f(a) \neq 0\}$ is the support of f. Since $|G/H| > m$, this implies that $f(a) = 0$ for some a, and thus $f(a) \leqslant m$ for all $a \in G/H$. Then $|S| = \sum_{a\in G/H} f(a) \leqslant \max(f)|\text{supp}(f)| \leqslant m^2$, as desired. $\square$

Lemma 4.5. ([7] Theorem 5.3) *Let G be a finite abelian group of rank greater than 2 and let $X \subset G\backslash\{0\}$ be a generating set for G consisting only of elements of order greater than 2. Suppose $S \subset G$ satisfies $|(S+x)\backslash S| \leqslant 3$ for all $x \in X$. Then $min\{|S|, |G\backslash S|\} \leqslant 5$.*

Lemma 4.6. *Let G be a finite abelian group of rank greater than 2, and $X \subseteq G$ a zero-sum free generating set consisting of only elements of order greater than 2, and let $S = \sum(X)$. Suppose that $|X| \geqslant 5$ and $|G| \geqslant 29$, then $|S| \geqslant 24$, or $|S| \geqslant 4|X|-3$, or there is some $x \in X$ satisfies $\langle X\backslash\{x\}\rangle = G$ and $|S| - |\sum(X\backslash\{x\})| \geqslant 4$.*

Proof. If there is an element $x \in X$ such that $\langle X\backslash\{x\}\rangle \neq G$, then

$$S = \sum(X\backslash\{x\}) \uplus \{x\} \uplus \sum(X\backslash\{x\}) + x$$

is a disjoint union. It follows that $|S| = 2|\sum(X\backslash\{x\})| + 1 \geqslant 2\times 2(|X|-1)+1 = 4|X|-3$ by Lemma 3.1. Hence we may assume that $\langle X\backslash\{x\}\rangle = G$ for all $x \in X$.

Now if $|S| - |\sum(X\backslash\{x\})| \leqslant 3$ for all $x \in X$, since $\sum(X\backslash\{x\}) \subset (S-x) \cap S$, then we have

$$\begin{aligned} |(S-x)\backslash S| &= |S-x| - |(S-x) \cap S| \\ &\leqslant |S| - |\sum(X\backslash\{x\})| \\ &\leqslant 3. \end{aligned}$$

It follows from Lemma 4.5 that $\min\{|S|, |G\backslash S|\} \leqslant 5$. Notice that $|S| \geqslant 2|X| \geqslant 10$, then $|G\backslash S| \leqslant 5$ and $|S| \geqslant |G| - 5 \geqslant 24$, as desired. □

5 Other lemmas before the proof

In this section, we present some Lemmas that will be used in the proof of the main result.

Lemma 5.1. *Let $X \subseteq G$ be a zero-sum free generating set of G, $|X| = 4$, and X has no element of order 2. If $r(G) \geqslant 3$ and $G \ncong C_2 \oplus C_2 \oplus C_4$, then $f(X) \geqslant 12$.*

Proof. Let $X = x_1 \cdot x_2 \cdot x_3 \cdot x_4$. If there are distinct indices $i, j, k \subset [1, 4]$ such that $x_i = x_j + x_k$, without loss of generality, we may assume that $x_1 = x_2 + x_3$. Since $r(G) \geqslant 3$, then $x_4 \notin \langle x_1, x_2, x_3 \rangle$, and so

$$\sum(X) = \sum(x_1x_2x_3) \biguplus \{x_4\} \biguplus (x_4 + \sum(x_1x_2x_3))$$

is a disjoint union. It follows from Lemma 3.1 that $f(X) \geqslant 2f(x_1x_2x_3) + 1 \geqslant 13$.

Now we consider the case that $x_i \neq x_j + x_k$ for all distinct indices $i, j, k \in [1, 4]$. If there is no index $i \in [1, 4]$ such that $x_i = \sum_{j \neq i} x_j$, then $x_1, x_2, x_3, x_4, x_1 + x_2, x_1 + x_3, x_1 + x_4, x_1 + x_2 + x_3, x_1 + x_2 + x_4, x_1 + x_3 + x_4, x_2 + x_3 + x_4, x_1 + x_2 + x_3 + x_4$ are pairwise distinct, so $f(X) \geqslant 12$.

Otherwise, we can assume $x_1 = x_2 + x_3 + x_4$.

If $x_{\tau(1)} + x_{\tau(2)} = x_{\tau(3)} + x_{\tau(4)}$, where τ is an element of the symmetric group on $[1, 4]$, then the two equations imply that there is some x_i of order 2, a contradiction.

If there is an index $i \in [1, 4]$ such that $x_i \neq \sum_{j \neq i} x_j$, say, $x_4 \neq x_1 + x_2 + x_3$, then $x_1, x_2, x_3, x_4, x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4, x_1 + x_2 + x_3 + x_4$ are pairwise distinct, so $f(X) \geqslant 12$.

If $x_i = \sum_{j \neq i} x_j$ for all indices $i \in [1, 4]$, then the 4 equations imply $4x_4 = 0$, $x_1 = x_4 + g_1$, $x_2 = x_4 + g_2$ and $x_3 = -x_4 + g_1 + g_2$, where g_1, g_2 is of order 2. It follows that $G = \langle X \rangle \cong C_2 \oplus C_2 \oplus C_4$, again a contradiction. We are done. □

Lemma 5.2. *Let C_n be a cyclic group of order n, $S \subset C_n$ a subset of G. Suppose that d is a generator of C_n and $x \in C_n$ is an element of order greater than 2. Then we have:*

1. $|(S + x)\backslash S| = |(S - x)\backslash S|$.

2. *If S is an arithmetic progression of difference d, then*

$$|(S + x)\backslash S| = min\{|S|, n - |S|, k, n - k\},$$

where $k \in [1, n-1]$ is the integer with $x = kd$.

3. *If $S = S_1 \uplus S_2$ is a disjoint union, where S_1, S_2 are arithmetic progressions of difference d and S is not an arithmetic progressions of difference d. Suppose that $2 \leqslant |S| \leqslant n-2$, then $|(S+x)\backslash S| \geqslant 1$.*

4. *Let S be as in 3, and moreover $5 \leqslant |S| \leqslant n-5$ and $n = 2r, r$ is a positive integer, then $|(S+x)\backslash S| \geqslant 2$. Furthermore the equality holds only when x is one of the following cases:*

(a): *$x = \pm d$.*

(b): *$x = \pm 2d$. In the case, $S = \{g, g+d, \ldots, g+(t-1)d, g+td\} \uplus \{g+(t+2)d\}$ or $S = \{g\} \uplus \{g+2d, g+3d, \ldots, g+(t-1)d, g+td\}$, $g \in G$ and $t \in [3, n-5]$.*

(c): *$x = \pm(r-1)d$. In the case, $||S_1| - |S_2|| \leqslant 2$.*

5. *If $S = S_1 \cup S_2 \cup S_3$ is a disjoint union of 3 arithmetic progressions of difference d, $x = \pm 3d$, and $8 \leqslant |S| \leqslant n-7$, then $|(S+x)\backslash S| \geqslant 2$.*

Proof. **1.** It is obvious.

2. Obviously.

3. For a counterexample, we may assume that $S_1 = \{g_1, g_1+d, \ldots, g_1+t_1d\}$, $S_2 = \{g_2, g_2+d, \ldots, g_2+t_2d\}$ and $|(S+x)\backslash S| = 0$. The proof is divided into the following two cases:

Case 3.1: If $g_1 + x \in S_1$, then there is an integer $k \in [0, t_1-1]$ such that $g_1+kd+x = g_1+t_1d$, however $g_1+(k+1)d \in S_1$ and $g_1+(k+1)d+x \notin S$ yield a contradiction. The proof of the case $g_2+x \in S_2$ is similar.

Case 3.2: If $g_1+x \in S_2$ and $g_2+x \in S_1$, then $S_1+x \subseteq S_2$ and $S_2+x \subseteq S_1$. Hence $|S_1| = |S_1+x| \leqslant |S_2|$ and similarly $|S_2| \leqslant |S_1|$. It follows that $|S_1| = |S_2|$, $g_1+x = g_2$ and $g_2+x = g_1$, and hence $g_1 = g_2+x = g_1+x+x$ and $2x = 0$, again a contradiction. We are done.

4. Without loss of generality, we may assume $|S_1| \geqslant |S_2|$. Let $r = \frac{n}{2}$, $S_1 = \{g_1, g_1+d, \ldots, g_1+t_1d\}$, $S_2 = \{g_2, g_2+d, \ldots, g_2+t_2d\}$, $U_1 = \{g_1+(t_1+1)d, g_1+(t_1+2)d, \ldots, g_2-d\}$ and $U_2 = \{g_2+(t_2+1)d, g_2+(t_2+2)d, \ldots, g_1-d\}$. If $x = \pm d$ then $|(S+x)\backslash S| = 2$. Since $|(S+x)\backslash S| = |(S-x)\backslash S|$, $n = 2r$ and x is an element of order greater than 2, then without loss of generality, we may assume that $x = kd$, $k \in [2, r-1]$.

Case 4.1: $k \in [3, r-2]$.

Subcase 4.1.1: If $|S_1| \geqslant k$, then $|U_1| + |U_2| = n - |S| \geqslant 5$ implies that $|U_1| \geqslant 3$ or $|U_2| \geqslant 3$. If $|U_2| \geqslant 3$, then $U_0 = \{g_1-3d, g_1-2d, g_1-d\} \subset U_2$ and $U_0+x \subset S_1$, so $|(S+x)\backslash S| = |(S-x)\backslash S| = |(S-x) \cap U_1| + |(S-x) \cap U_2| \geqslant |U_0| = 3$. The proof of the case $|U_1| \geqslant 3$ is similar.

Subcase 4.1.2: If $|S_1| < k$, let $H_1 = \{g_1, g_1+kd, g_1+2kd\}$, $H_2 = H_1+d$ and $H_3 = H_1+2d$. Obviously, each of the 3 disjoint subsets of C_n has 3 elements. Now we first prove that $H_i \not\subset S,\ i = 1,2,3$. If $g_1+(i-1+k)d \in S$, then $g_1+(i-1+k)d \in S_2$ since $|S_1| < k$, and so $g_1+(i-1+2k)d \notin S_2$. Notice that $g_1+(i-1+2k)d \notin S_1$, otherwise, we would have $g_1+(i-1+2k)d == g_1+md$ for some integer m, $m \in [0, |S_1|-1]$, and so $2k+2 \geqslant n = 2r$, which is impossible. It follows that $g_1+(i-1+2k)d \notin S$. Since $g_1+(i-1)d \in S_1, i = 1,2,3$, so each of $H_i \not\subset S,\ i = 1,2,3$ contributes at least one element to $(S+x)\backslash S$. Hence $|(S+x)\backslash S| \geqslant 3$.

Case 4.2: $k = 2$. Since $|S_1| \geqslant |S_2|$ and $|S| \geqslant 5$, we have$|S_1| \geqslant 3$. Note that $|U_1| + |U_2| = n - |S| \geqslant 5$, so $|U_1| \geqslant 3$ or $|U_2| \geqslant 3$.

Subcase 4.2.1: If $|U_2| \geqslant 3$, then $(S_1-x)\backslash S = \{g_1-2d, g_1-d\}$, and so $|(S_1-x)\backslash S| = 2$. If $S_2 - x \not\subset S$ then $|(S+x)\backslash S| = |(S_1-x)\backslash S| + |(S_2-x)\backslash S| \geqslant 3$. So we may assume $S_2 - x \subset S$. If $|S_2| \geqslant 2$, then $g_2 + d \in S_2$ and $g_2 + d - 2d \notin S$, a contradiction. Hence $S_2 = \{g_2\}$, and $g_2 - 2d \notin S_2$ implies that $g_2 - 2d \in S_1$. Since S is not an arithmetic progression of difference d, S must have the form of case (b).

Subcase 4.2.2: If $|U_1| \geqslant 3$, it is similar to the Subcase 4.2.1.

Case 4.3: $k = r - 1$.

Subcase 4.3.1: $|S_1| \geqslant k$. By a similar argument as in Subcase 4.1.1 we have that $|(S+x)\backslash S| \geqslant 3$

Subcase 4.3.2: $|S_1| < k$. Obviously, both $H_1 = \{g_1, g_1+x, g_1+2x\}$ and $H_2 = H_1+d$ have 3 elements. By the same argument as in Subcase 4.1.2, we derive $H_i \not\subset S,\ i = 1, 2$ and $|(S+x)\backslash S| \geqslant 2$. To get the equality, we must have $\{g_1+2d, g_1+3d, \ldots, g_1+t_1d\}+x \subset S_2$, then $|S_1| \geqslant |S_2| \geqslant |S_1| - 2$. It is just the case (c), which completes the proof of this case.

5. Let $S_1 = \{g_1, g_1 + d, \ldots, g_1 + t_1d\}$, $S_2 = \{g_2, g_2 + d, \ldots, g_2 + t_2d\}$, $S_3 = \{g_3, g_3 + d, \ldots, g_3 + t_3d\}$ and let $U_1 = \{g_1 + t_1d + d, g_1 + t_1d + 2d, \ldots, g_2 - d\}$, $U_2 = \{g_2 + t_2d + d, g_2 + t_2d + 2d, \ldots, g_3 - d\}$, $U_3 = \{g_3 + t_3d + d, g_3 + t_3d + 2d, \ldots, g_1 - d\}$. Without loss of generality, we may assume that S_1 has the maximal length. Since $8 \leqslant |S| \leqslant n - 7$, then $|S_1| \geqslant 3$. If $|U_1| \geqslant 2$ or $|U_3| \geqslant 2$, then it is easy to verify that $|(S_1 + 3d)\backslash S| \geqslant 2$ or $|(S_1 - 3d)\backslash S| \geqslant 2$, so both of them imply the result. Now we assume that $|U_1| = |U_3| = 1$, then $|U_2| \geqslant 5$, and so $|(S_2 + 3d)\backslash S| \geqslant 1$ and $|(S_1 + 3d)\backslash S| \geqslant 1$, the result follows. This completes the proof of Lemma 5.2. ∎

Now, we give some remarks about Lemma 5.2:

1. The equality of part 1 holds for all abelian groups G and any element $x \in G$.

2. In part 2 of the Lemma, if $2 \leqslant |S| \leqslant n - 2$, then $|(S + x)\backslash S| = 1$ if and only if $x = \pm d$.

3. In part 4 of the Lemma, case (b) and case (c) do not hold simultaneously.

6 Proof of the Theorem 1.1

Proof. Let $X \subset G$ be a zero-sum free subset with $|X| = 7$, and let $S = \sum(X)$. Without loss of generality, we may assume $G = \langle X \rangle$ and $|S| \leqslant 23$ for the contrary. By Lemmas 3.3 and 3.4, we may assume $r(G) \geqslant 2$ and all elements of X have order greater than 2. By Lemmas 3.1 and 3.2, $f(X) \geqslant f(X\backslash\{x\}) + f(x) \geqslant 19 + 1 = 20$ where $x \in X$, then we have $|G| \geqslant f(X) + 1 \geqslant 21$. If there is an element $x \in X$ such that $|(S - x)\backslash S| \geqslant 5$, since $\sum(X\backslash\{x\}) \subset (S-x)\cap S$, we have that $|S| \geqslant f(X\backslash\{x\}) + |(S-x)\backslash S| \geqslant f(X\backslash\{x\}) + 5 \geqslant 24$ by Lemma 3.2. Hence we may assume that $|(S - x)\backslash S| \leqslant 4$ for all $x \in X$. So, to sum up, we may assume that $20 \leqslant |S| \leqslant 23$, $|G| \geqslant 21, < X >= G$, $r(G) \geqslant 2$ and $ord(x) > 2$, $|(S - x)\backslash S| \leqslant 4$ for all $x \in X$. The proof is divided to the following six cases.

Case 1: $r(G) \geqslant 3$ and $|G| \geqslant 29$.

Since $|S| \leqslant 23$, by Lemma 4.6, there is an element $x_1 \in X$ such that $\langle X\backslash\{x_1\}\rangle = G$ and $|\sum(X\backslash\{x_1\})| \leqslant |S| - 4 \leqslant 19$. Now we apply Lemma 4.6 repeatedly, we will obtain $x_2 \in X\backslash\{x_1\}$, $x_3 \in X\backslash\{x_1, x_2\}$ such that $\langle X\backslash\{x_1, x_2\}\rangle = G$, $\langle X\backslash\{x_1, x_2, x_3\}\rangle = G$ and $f(X\backslash\{x_1, x_2\}) \leqslant f(X\backslash\{x_1\}) - 4 \leqslant 15$, $f(X\backslash\{x_1, x_2, x_3\}) \leqslant f(X\backslash\{x_1, x_2\}) - 4 \leqslant 11$. But we have that $|\sum(X\backslash\{x_1, x_2, x_3\})| \geqslant 12$ by Lemma 5.1, a contradiction.

Case 2: $G \cong C_n \oplus C_{nr}$, $n \geqslant 5$ and $|G| \geqslant 40$

Subcase 2.1: There is an element $x_0 \in X$ such that $\text{ord}(x_0) \geqslant 5$. Let $H = \langle x_0\rangle$, then $|H| \geqslant 5$ and $|G/H| \geqslant 5$. Since $|(S - x)\backslash S| \leqslant 4$ for all $x \in X$, it follows from Lemma 4.4 that $\min\{|S|, |G\backslash S|\} \leqslant 4^2$. Notice that $|S| \geqslant 20$, then $|G\backslash S| \leqslant 16$ and $|S| \geqslant |G| - 16 \geqslant 24$, a contradiction.

Subcase 2.2: If $ord(x) < 5$ for all $x \in X$, then $\text{ord}(x) \in \{3, 4\}$ for all $x \in X$.

We can choose 2 elements x_0, $x_1 \in X$ such that $\text{ord}(x_0) = 3$ and $\text{ord}(x_1) = 4$. The choice is possible since otherwise we would have $ord(x) = 3$ for all $x \in X$ or $ord(x) = 4$ for all $x \in X$. Note that $r(G) = 2$, so $G \cong C_3 \oplus C_3$, $G \cong C_4 \oplus C_4$ or $G \cong C_2 \oplus C_4$, which contradicts $|G| \geqslant 40$. Let $H = \langle x_0, x_1\rangle$, then a similar discussion as in Subcase 2.1 will lead to a contradiction again.

Case 3: $G \cong C_4 \oplus C_{4r}$ and $|G| \geqslant 40$.

Subcase 3.1: There is an element $x_0 \in X$ such that $5 \leqslant \text{ord}(x_0) < 4r$. Let $H = \langle x_0\rangle$, then the remaining discussion is similar to Subcase 2.1.

Subcase 3.2: $\text{ord}(x) \in \{3, 4, 4r\}$ for all $x \in X$.

We first prove the following 2 claims.

Claim 1: There is an element $x_0 \in X$ such that $\text{ord}(x_0) = 4r$.

Proof of Claim 1: Since $f(6) \geqslant 19 > |C_4 \oplus C_4|$, then there is at most 5 elements of order 4 in X. Notice that there is at most 1 element of order 3 in X and $|X| = 7$, then Claim 1 follows.

Claim 2: Let $H = \langle x_0\rangle$, $x_0 \in X$ and $\text{ord}(x_0) = 4r$, then $H \cap X = \{x_0\}$.

Proof of Claim 2: Let $a_i + H$, $a_i \in G/H$, $i = 0, 1, 2, 3$ denote the 4 cosets of H in G. Let $S_i = (H + a_i) \cap S$ and define a function $f : G/H \to N$ by $f(a_i) = |S_i|$, then $\max(f) - \min(f) \leqslant 4$ by Lemma 4.3.

Notice that $20 \leqslant |S| \leqslant 23$, so $2 \leqslant f(a_i) \leqslant |H| - 2$ for all $i \in [0, 3]$, and hence $|(S_i - x_0)\backslash S_i| \geqslant 1$ for all $i \in [0, 3]$. Since $|(S - x_0)\backslash S| \leqslant 4$, it follows that each S_i is an arithmetic progression of difference x_0. If there is another $x_1 \in X \cap H$, say, $x_1 = kx_0, 2 \leqslant k \leqslant 4r - 2$, since $S_i, i \in [0, 3]$ are arithmetic progressions of difference x_0 and $2 \leqslant |S_i| \leqslant |H| - 2$, then $|(S_i - x_1)\backslash S_i| \geqslant 1$ and $|(S - x_1)\backslash S| \leqslant 4$ imply $|(S_i - x_1)\backslash S_i| = 1$ and each $S_i, i \in [0, 3]$ is an arithmetic progression of difference x_1. Hence $x_1 = \pm x_0$ by Lemma 5.2, a contradiction. So Claim 2 holds.

Since each element of order 3 is contained in a cyclic subgroup of order $4r$, by Claim 2, we have $\text{ord}(x) \neq 3$ for any $x \in X$. Let $X = Y \cup Z$, where Y consists of elements of order 4 and Z consists of elements of order $4r$, then $|Y| \leqslant 5$ by the proof of Claim 1. Let $b \in X$ be an element with $ord(b) = 4r$, choose $a \in G$ such that $G = \langle a\rangle \oplus \langle b\rangle$ and $\text{ord}(a) = 4$. Let $G_0 = \langle a, rb\rangle$. Obviously, $Y \subset G_0$ and $Z \cap G_0 = \emptyset$.

Subcase 3.2.1: If $2|r$, then there are only 4 cyclic subgroups of order $4r$: $\langle b\rangle$,

$\langle a+b\rangle$, $\langle -a+b\rangle$ and $\langle 2a+b\rangle$. By Claim 2, a subgroup of order $4r$ contributes at most 1 element of order $4r$ to X, so $|Z| \leqslant 4$. It is easy to see that every element of order $4r$ is of the form $ka+tb, \gcd(t,r)=1$.

If $|Y|=5$, then $S \supset \sum(Y) \uplus \{b\} \uplus (b+\sum(Y))$ is a disjoint union and $|S| \geqslant 2|\sum(Y)|+1 \geqslant 2\times 13+1=27$ by Lemma 3.2.

If $|Y|=4$, we let $Z=\{k_1a+t_1b, k_2a+t_2b, k_3a+t_3b\}, \gcd(t_1t_2t_3,r)=1$. If $|\sum(t_1,t_2,t_3) \pmod{r} \backslash\{0\}| \geqslant 2$, say, $l_1,l_2 \in \sum(t_1,t_2,t_3) \pmod{2r}$, $0 \not\equiv l_1,l_2 \pmod{r}, l_1 \not\equiv l_2 \pmod{r}$, then $S \supseteq \sum(Y) \uplus (m_1a+l_1b+\sum(Y)) \uplus (m_2a+l_2b+\sum(Y))$ and hence $|S| \geqslant 3|\sum(Y)| \geqslant 3\times 8=24$ by Lemma 3.1. If $|\sum(t_1,t_2,t_3) \pmod{r}\backslash\{0\}|<2$, then $t_1\equiv t_2\equiv t_3 \pmod{r}$ and $r=2$ since $\gcd(t_1t_2t_3,r)=1$ and $2|r$, which contradicts $|G|\geqslant 40$.

If $|Y|=3$, we let $Z=\{k_1a+t_1b,\ldots,k_4a+t_4b\}, \gcd(t_1t_2t_3t_4,r)=1$. If $|\sum(t_1,t_2,t_3,t_4) \pmod{r}\backslash\{0\}|\geqslant 3$, then similarly, we have $|S|\geqslant 4|\sum(Y)|\geqslant 4\times 6=24$ by Lemma 3.1. If $|\sum(t_1,t_2,t_3,t_4) \pmod{r}\backslash\{0\}|\leqslant 2$, then a similar discussion as in the case $|Y|=4$ shows that $r=2$ since $\gcd(t_1t_2t_3t_4,r)=1$ and $2|r$, which also contradicts $|G|\geqslant 40$.

Subcase 3.2.2: If $2 \not| r$, then there are precisely 6 cyclic subgroups of order $4r$: $\langle b\rangle$, $\langle a+b\rangle$, $\langle a+2b\rangle$, $\langle -a+b\rangle$, $\langle a+4b\rangle$ and $\langle 2a+b\rangle$. Notice that any element of order 4 is contained in one of the 6 subgroups. By the Pigeonhole Principle, there is some subgroup H which contributes at least 2 elements to X. By Claim 2, this subgroup H contributes only elements of order 4. However, 2 elements of order 4 in H leads to a contradiction since H has precisely two elements of order four: $rx, -rx$, where x is a generator of H.

Case 4: $G\cong C_3\oplus C_{3r}$ and $|G|\geqslant 40$.

Subcase 4.1: There is an element $x_0\in X$ such that $5\leqslant \mathrm{ord}(x_0)<3r$, or two elements $x_1,x_2\in X$ with $ord(x_i)\in\{3,4\}, i=1,2$. Let $H=\langle x_0\rangle$ or $<x_1,x_2>$, $G_1=G/H$. Then both H and G/H have at least 5 elements. The remaining discussion is similar to Subcase 2.1.

Claim 3: Let $H=\langle x_0\rangle$, $x_0\in X$ and $\mathrm{ord}(x_0)=3r$, then $H\cap X=\{x_0\}$.

Proof of Claim 3: Let $H+a_i$, $a_i\in G/H$, $i=0,1,2$ denote the 3 cosets of H in G. Let $S_i=(H+a_i)\cap S$ and define a function $f: G/H\to N$ by $f(a_i)=|S_i|$, then $\max(f)-\min(f)\leqslant 4$ by Lemma 4.3.

Notice that $20\leqslant |S|\leqslant 23$, so $4\leqslant f(a_i)\leqslant |H|-3$ for any $i\in[0,2]$. Since $|(S_i-x_0)\backslash S_i|\geqslant 1$ and $|(S-x_0)\backslash S|\leqslant 4$, without loss of generality, we may assume that S_0 and S_1 are arithmetic progressions of difference x_0. If there is another $x_1\in X\cap H$, then by Lemma 5.2 $|(S_0+x_1)\backslash S_0|\geqslant 2$, $|(S_1+x_1)\backslash S_1|\geqslant 2$ and $|(S_2+x_1)\backslash S_2|\geqslant 1$ imply that $|(S+x_1)\backslash S|\geqslant 5$, a contradiction, so the claim holds.

Since all the elements of order 4 are included in the cyclic subgroup of order $3r$, by Claim 3 above, we have $\mathrm{ord}(x)\neq 4$ for any $x\in X$. Let $X=Y\cup Z$, where Y consists of elements of order 3 and Z consists of elements of order $3r$, then $|Y|\leqslant 1$ by Subcase 4.1. Choose $a,b\in G$ such that $G=\langle a\rangle\oplus\langle b\rangle$, and $\mathrm{ord}(b)=3r$.

If $3|r$, then there are only 3 cyclic subgroups of order $3r$: $\langle b\rangle$, $\langle a+b\rangle$ and $\langle -a+b\rangle$. If $3 \not| r$, then there are precisely 5 cyclic subgroups of order $3r$: $\langle b\rangle$, $\langle a+b\rangle$, $\langle -a+b\rangle$, $\langle a-3b\rangle$ and $\langle a+3b\rangle$. By Claim 4, every subgroup of order $3r$ will contribute at most 1 element of order $3r$ to X, so $|Z|\leqslant 5$. It follows that $|Y|+|Z|\leqslant 6$, a contradiction.

Case 5: $G \cong C_2 \oplus C_{2r}$ and $|G| \geqslant 40$.

Subcase 5.1: There is an element $x_0 \in X$ such that $5 \leqslant \text{ord}(x_0) < r$. Let $H = \langle x_0 \rangle$, then the discussion is similar to Subcase 2.1.

Subcase 5.2: For any $x \in X$, $\text{ord}(x) \in \{3, 4, r, 2r\}$.

Choose $a, b \in G$ such that $G = \langle a \rangle \oplus \langle b \rangle$, $ord(a) = 2$ and $\text{ord}(b) = 2r$. Let $X = X_3 \cup X_4 \cup X_r \cup X_{2r}$, where X_i, $i \in \{3, 4, r, 2r\}$ consists of elements of order i. Let $G_0 =< a, 2b >\not\subseteq G$, it is easy to verify that $x \in G_0$ for any $x \in G$ with $\text{ord}(x) \in \{3, r\}$. There are at most 2 cyclic subgroups of order 4 in G: $\left\langle \frac{r}{2}b \right\rangle$ and $\left\langle a + \frac{r}{2}b \right\rangle$, each contributes at most 1 element of order 4 to X, so $|X_4| \leqslant 2$. If $|X_4| = 2$, let $H =< x_4 >$, then a similar discussion as in Subcase 2.1 leads to a contradiction. Now again we need 3 claims.

Claim 4: $X_{2r} \neq \emptyset$

Proof of Claim 4: Suppose that $X_{2r} = \emptyset$. If there is an element $x_4 \in X_4$ such that $x_4 \notin G_0$, then $S \supset \sum(X_3 \cup X_r) \uplus \{x_4\} \uplus (x_4 + \sum(X_3 \cup X_r))$ is a disjoint union and hence $|S| \geqslant 2|\sum(X_3 \cup X_r)| + 1 \geqslant 2 \times 13 + 1 = 27$, a contradiction. It follows that either $X_4 = \emptyset$ or $X_4 \subset G_0$, which implies $G =< X >\subset G_0 \not\subseteq G$, a contradiction again. So the Claim 4 holds.

Claim 5: Let $x_{2r} \in X_{2r}$ and $H =< x_{2r} >$, if there is another $y \in H \cap X$, then $y = \pm 2x_{2r}$ or $y = \pm(r-1)x_{2r}$. Furthermore, $|H \cap X| \leqslant 2$.

Proof of Claim 5: Let $H + a_i$, $a_i \in G/H$, $i = 0, 1$ denote the 2 cosets of H in G. Let $S_i = (H + a_i) \cap S$ and define a function $f : G/H \to N$ by $f(a_i) = |S_i|$, then $\max(f) - \min(f) \leqslant 4$ by Lemma 4.3

Notice that $20 \leqslant |S| \leqslant 23$, so $8 \leqslant f(a_i) \leqslant |H| - 7$ for any $i \in [0, 1]$. Since $|(S_i - x_{2r}) \backslash S_i| \geqslant 1$ and $|(S - x_{2r}) \backslash S| \leqslant 4$, then we have that S_i, $i = 0, 1$ is the union of at most 3 arithmetic progressions of difference x_{2r}.

If there is some S_i which is an arithmetic progression of difference x_{2r}, without loss of generality, we may assume that is S_0. If $y \neq \pm 2x_{2r}$, since $y \neq \pm x_{2r}$, we have $y = \pm 3x_{2r}$ or $y \in< x_{2r} > \backslash \{\pm x_{2r}, \pm 2x_{2r}, \pm 3x_{3r}\}$. If $y = \pm 3x_{2r}$, then $|(S_0 + y) \backslash S| = 3$ and $|(S_1 + y) \backslash S| \geqslant 2$ by Lemma 5.2.5. . If $y = \pm 4x_{2r}$, then $|(S_0 + y) \backslash S| = 4$ and $|(S_1 + y) \backslash S| \geqslant 1$. Otherwise, $|(S_0 + y) \backslash S| \geqslant 5$. It follows that $|(S + y) \backslash S| \geqslant 5$, a contradiction.

If there is no S_i which is an arithmetic progression, then we have that S_i, $i = 0, 1$ both are the unions of 2 arithmetic progressions of difference x_{2r}. Since $y \neq \pm x_{2r}$, we have $|(S_i + y) \backslash S_i| \geqslant 3$, or $y = \pm 2x_{2r}$, or $y = \pm(r-1)x_{2r}$ by Lemma 5.2.4, and the claim holds.

By the hypothesis, we have that $\pm 2x_{2r}$ has order r and $\pm(r-1)x_{2r}$ has order $2r$ if $2|r$ and order r if $2 \not| r$.

Claim 6: Let $x_r \in X_r$ and $K =< x_r >$, then $X_r \cap K = \{x_r\}$.

Proof of Claim 6: By a similar argument as in the proof of Claim 2, we obtain Claim 6.

Since all elements of order 3 are contained in cyclic subgroups of order $2r$, by Claim 5 and $r \neq 3$, we have $X_3 = \emptyset$.

Subcase 5.2.1: If $2 \not| r$, then there are precisely 3 cyclic subgroups of order $2r$: $< b >$, $< a + b >$ and $< a + 2b >$. In this subcase, $X_4 = \emptyset$. Claim 5 and the discussion after imply that each subgroup of order $2r$ contributes at most 1 element of order $2r$, so $|X_{2r}| \leqslant 3$ and $|X_r| = |X| - |X_{2r}| \geqslant 4$. Notice that $X_r \subset G_0$ and $< a + 2b >\subset G_0$, then

$|X_{2r}\backslash G_0| \leqslant 2$. If $|X_{2r}\backslash G_0| = 0$, then $X \subset G_0$, a contradiction. Therefore we can choose $x_{2r} \in X_{2r}\backslash G_0$. Now $S \supset \sum(G_0 \cap X) \oplus \{x_{2r}\} \oplus (x_{2r} + \sum(G_0 \cap X))$ and $|G_0 \cap X| \geqslant 5$ imply that $|S| \geqslant 2 \times 13 + 1 = 27$, a contradiction.

Subcase 5.2.2: If $2|r$, then there are only 2 cyclic subgroups of order $2r$: $H_1 =< b >$ and $H_2 = \langle a + b\rangle$. Let H be a cyclic subgroup of order $2r$, if $X_{2r} \cap H \neq \emptyset$, then $|X \cap H| \leqslant 2$ by Claim 5; if $X_{2r} \cap H = \emptyset$, then $|X_r \cap H| \leqslant 1$ by Claim 6, and so $|X \cap H| \leqslant 2$ since $|X_4 \cap H| \leqslant 1$. It follows that H contributes at most 2 elements to X.

If $4 \not| r$, then all the elements of order 4 are contained in the subset $H_1 \cup H_2$. Note that G has precisely 3 cyclic subgroups of order r: $< 2b > \subset < b >$, $< a + 2b >$ and $< a + 4b >$. A cyclic subgroups of order r contributes at most 1 element to X by Claim 6. It follows that $|X| \leqslant 2 \times 2 + 1 + 1 = 6 < 7$, a contradiction.

If $4|r$, let $r_0 = r/2$. Then G has precisely 2 cyclic subgroups of order 4: $< r_0 b > \subset < b >$ and $< a + r_0 b >$, and 2 cyclic subgroups of order r: $< 2b > \subset < b >$ and $< a + 2b >$. A cyclic subgroups of order 4 or r contributes at most 1 element to X by Claim 6. It follows that $|X| \leqslant 2 \times 2 + 1 + 1 = 6 < 7$, a contradiction again.

Case 6: G with small order.

Since $|G| \geqslant 21$ and $r(G) \geqslant 2$, the left cases of G are of the following forms: $C_3 \oplus C_3 \oplus C_3$, $C_2 \oplus C_2 \oplus C_6$, $C_6 \oplus C_6$, $C_5 \oplus C_5$, $C_4 \oplus C_8$, $C_3 \oplus C_9$, $C_3 \oplus C_{12}$, $C_2 \oplus C_{12}$, $C_2 \oplus C_{14}$, $C_2 \oplus C_{16}$ and $C_2 \oplus C_{18}$.

To begin with, since $D(C_3 \oplus C_3 \oplus C_3) = 7$, we have $f(C_3 \oplus C_3 \oplus C_3, 7) = 26$.

The remaining cases are computed with a C++ program. With the help of a computer, we obtain the following values:

Result	Running Time(sec)
$ol(C_2 \oplus C_2 \oplus C_6) = 6$	7.6
$f(C_6 \oplus C_6, 7) = 29$	98
$ol(C_5 \oplus C_5) = 6$	1.0
$f(C_4 \oplus C_8, 7) = 27$	26
$ol(C_3 \oplus C_9) = 6$	3.9
$f(C_3 \oplus C_{12}, 7) = 27$	190
$ol(C_2 \oplus C_{12}) = 6$	1.2
$f(C_2 \oplus C_{14}, 7) = 25$	8.7
$f(C_2 \oplus C_{16}, 7) = 25$	51
$f(C_2 \oplus C_{18}, 7) = 25$	285

This completes the proof. □

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