Standard character condition for table algebras

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Abstract

It is well known that the complex adjacency algebra A of an association scheme has a specific module, namely the *standard module*, that contains the regular module of A as a submodule. The character afforded by the standard module is called the *standard character*. In this paper we first define the concept of standard character for C-algebras and we say that a C-algebra has the *standard character condition* if it admits the standard character. Among other results we acquire a necessary and sufficient condition for a table algebra to originate from an association scheme. Finally, we prove that given a C-algebra admits the standard character and its all degrees are integers if and only if so its dual.

1 Introduction

A table or, equivalently, C-algebra with nonnegative structure constants was introduced by [2]. It is easy to see that the complex adjacency algebra of an association scheme (or homogeneous coherent configuration) is an integral table algebra. On the other hand, the adjacency algebra of an association scheme has a special module, namely the standard module, that contains the regular module as a submodule. The character afforded by the standard module is called the *standard character*, see [8]. This leads us to generalize the concept of standard character from adjacency algebras to table algebras. As an application of this generalization, we provide a necessary and sufficient condition for a table algebra to originate from an association scheme, see Theorem 4.7.

The paper is organized as follows. In Section 2 we recall the concept of C-algebras and table algebras and some related properties which we will use in this paper.

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In Section 3 we first define the *standard feasible trace* for C-algebras which is a generalization of the standard character in the theory of association schemes. Thereafter, we show that the standard feasible multiplicities of the characters of a table algebra and its quotient are the same. Furthermore, we prove that the set of standard feasible multiplicities preserve under C- algebras isomorphism.

In Section 4 we give an example of C-algebra for which the standard feasible trace is a character, such character is called the *standard character*. By using the standard character we obtain a necessary and sufficient condition for which a table algebra to originate from an association scheme. Finally, we prove that a C-algebra (A, B) admits the standard character and is integer degree, i.e., all degrees $|b|, b \in B$ are integers, if and only if so is its dual $(\widehat{A}, \widehat{B})$, see Corollary 4.11.

2 Preliminaries

Although in algebraic combinatorics the concept of C-algebra is used for commutative algebras, in this paper we will also consider non-commutative algebras. Hence we deal with C-algebras in the sense of [7] as the following:

Let A be a finite dimensional associative algebra over the complex field $\mathbb{C}$ with the identity element 1_A and a base B in the linear space sense. Then the pair (A, B) is called a C-*algebra* if the following conditions (I)-(IV) hold:

(I) $1_A \in B$ and the structure constants of B are real numbers, i.e., for $a, b \in B$:

$$ab = \sum_{c \in B} \lambda_{abc} c, \qquad \lambda_{abc} \in \mathbb{R}.$$

(II) There is a semilinear involutory anti-automorphism (denoted by *) of A such that $B^* = B$.

(III) For $a, b \in B$ the equality $\lambda_{ab1_A} = \delta_{ab^*}|a|$ holds where $|a| > 0$ and δ is the Kronecker symbol.

(IV) The mapping $b \to |b|, b \in B$ is a one dimensional $*$-linear representation of the algebra A, which is called the *degree map*.

Remark 2.1. *In the definition above if the algebra A is commutative, then (A, B) becomes a C-algebra in the sense of [4].*

If the structure constants of a given C-algebra (resp. commutative) are nonnegative real numbers, then it is called a *table algebra* (resp. commutative) in the sense of [2] (resp. [1]).

A C-algebra (table algebra) is called *integral* if all its structure constants λ_{abc} are integers. The value $|b|$ is called the *degree* of the basis element b. From condition (IV)

we see that $|b| = |b^*|$ for all $b \in B$, and from condition (II) for $a = \sum_{b\in B} x_b b$ we have $a^* = \sum_{b\in B} \overline{x_b} b^*$, where $\overline{x_b}$ means the complex conjugate to x_b. This implies that the Jacobson radical $J(A)$ of the algebra A is equal to $\{0\}$ which means A is semisimple.

Let (A, B) and (A', B') be two C-algebras. An $*$-algebra homomorphism $f : A \to A'$ such that $f(B) = B'$ is called a C-*algebra homomorphism* from (A, B) to (A', B'). Such C-algebra homomorphism is called C-*algebra epimorphism* (resp. *monomorphism*) if f is onto (resp. into). A C-algebra epimorphism f is called a C-*algebra isomorphism* if f is monomorphism too. Two C-algebras (A, B) and (A', B') are called *isomorphic*, if there exists a C-algebra isomorphism between them.

A nonempty subset $C \subseteq B$ is called a *closed subset*, if $C^*C \subseteq C$. We denote by $\mathcal{C}(B)$ the set of all closed subsets of B.

Let (A, B) be a table algebra with the basis B and let $C \in \mathcal{C}(B)$. From [3, Proposition 4.7], it follows that $\{CbC \mid \ b \in B\}$ is a partition of B. A subset CbC is called a C-*double coset* or *double coset* with respect to the closed subset C. Let

$$b/\!\!/C := |C^+|^{-1}(CbC)^+ = |C^+|^{-1} \sum_{x\in CbC} x$$

where $C^+ = \sum_{c\in C} c$ and $|C^+| = \sum_{c\in C} |c|$. Define $B/\!\!/C = \{b/\!\!/C \mid b \in B\}$ and let $A/\!\!/C$ be the vector space spanned by the elements $b/\!\!/C$, for $b \in B$. Then [3, Theorem 4.9] follows that the pair $(A/\!\!/C, B/\!\!/C)$ is a table algebra. The table algebra $(A/\!\!/C, B/\!\!/C)$ is called the *quotient table algebra* of (A, B) modulo C.

We refer the reader to [12] for the background of association schemes.

3 The standard feasible trace for C-algebras

In this section we first define the *standard feasible trace* for C-algebras and then we show that the standard feasible multiplicities of the characters of a table algebra and its quotient are the same. Furthermore, we prove that the set of standard feasible multiplicities preserve under C-algebras isomorphism.

Let (A, B) be a C-algebra and let $\mathrm{Irr}(A)$ be the set of irreducible characters of A. We define a linear function $\zeta \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{A}, \mathbb{C})$ by $\zeta(b) = \delta_{1_A b}|B^+|$, for $b \in B$, where $|B^+| = \sum_{b\in B} |b|$. It is easily seen that $\zeta(bc) = \zeta(cb)$, for all $b, c \in B$. This shows that ζ is a *feasible trace* in the sense of [9]. In addition, since $\mathrm{rad}\zeta = \{0\}$, where $\mathrm{rad}\zeta = \{x \in A : \zeta(xy) = 0, \ \forall y \in A\}$, it is a non-degenerate feasible trace on A. Therefore, from [9] it follows that

$$\zeta = \sum_{\chi\in \mathrm{Irr(A)}} \zeta_\chi \chi, \tag{1}$$

where $\zeta_\chi \in \mathbb{C}$ and all ζ_χ are nonzero. We call ζ the *standard feasible trace*, ζ_χ the *standard feasible multiplicity* of χ and $\{\zeta_\chi| \ \chi \in \mathrm{Irr}(A)\}$ the *set of standard feasible multiplicities* of

the C-algebra (A, B).

Let (A, B) be a C-algebra with the standard feasible trace ζ. Since A is a semisimple algebra, we have

$$A = \bigoplus_{\chi \in \mathrm{Irr}(A)} A\varepsilon_\chi$$

where ε_χ's are the central primitive idempotents.

Lemma 3.1. (i) *Let $\chi \in \mathrm{Irr}(A)$. Then*

$$\varepsilon_\chi = \frac{1}{|B^+|} \sum_{b \in B} \frac{\zeta_\chi \chi(b^*)}{|b^*|} b. \tag{2}$$

(ii) (Orthogonality Relation) *For every $\phi, \psi \in \mathrm{Irr(A)}$*

$$\frac{1}{|B^+|} \sum_{b \in B} \frac{1}{|b^*|} \phi(b^*)\psi(b) = \delta_{\phi\psi} \frac{\phi(1)}{\zeta_\phi}. \tag{3}$$

(iii) *If (A, B) is commutative then for every $b, c \in B$*

$$\sum_{\chi \in \mathrm{Irr}(A)} \zeta_\chi \chi(b)\chi(c^*) = \delta_{bc}|b||B^+|.$$

Proof. Let $B := \{b_1, b_2, \ldots, b_m\}$ and let $\widehat{b_1}, \widehat{b_2}, \ldots, \widehat{b_m}$ be the dual basis defined by $\zeta(b_i\widehat{b_j}) = \delta_{ij}$, in the sense of [9, 4.1]. On the other hand, $\zeta(b_i b_j^*) = \delta_{ij}|b_i||B^+|$, for $b_i, b_j \in B$. This follows that $\widehat{b_i} = \frac{b_i^*}{|b_i||B^+|}$, for each $b_i, 1 \leqslant i \leqslant m$. Now parts (i) and (iii) follow from [9, 5.7] and [9, 5.5′], respectively. Part (ii) follows from the equality $\varepsilon_\phi \varepsilon_\psi = \delta_{\phi\psi}\varepsilon_\phi$ by replacing b^* by 1_A. □

Remark 3.2. *From (2) one can see that if A is a commutative table algebra, ζ_χ is the coefficient of 1_A in the linear combination of $|B^+|\varepsilon_\chi$ in terms of the basis elements of B.*

Let (A, B) be a table algebra and $C \in \mathcal{C}(B)$. Set $e = |C^+|^{-1}C^+$. Then e is an idempotent of the table algebra A and the vector space eAe spanned by the elements $ebe, b \in B$ is a table algebra which is equal to the quotient table algebra $(A/\!\!/C, B/\!\!/C)$ modulo C, see [3]. Let ζ be the standard feasible trace of the table algebra (A, B). We claim that the restriction $\zeta_{A/\!\!/C}$ of the standard feasible trace ζ to the subalgebra eAe is the standard feasible trace for $(A/\!\!/C, B/\!\!/C)$. To do so, assume that $T \subseteq B$ be a complete set of representatives of C-double cosets of A. Then $B = \bigcup_{b \in T} CbC$ and $|C^+|^{-1}|B^+| = \sum_{b \in T} |b/\!\!/C|$. Since

$$\zeta_{A/\!\!/C}(b/\!\!/C) = \begin{cases} |C^+|^{-1}|B^+|, & \text{if } b = 1_A \\ 0, & \text{if } b \neq 1_A \end{cases}$$

it follows that $\zeta_{A/\!\!/C}$ is the standard feasible trace for $(A/\!\!/C, B/\!\!/C)$. Thus we proved the following lemma:

Lemma 3.3. *Let (A, B) be a table algebra with the standard feasible trace ζ and let $C \in \mathcal{C}(B)$. Then $\zeta_{A/\!/C}$ is the standard feasible trace of $(A/\!/C, B/\!/C)$.* □

The following theorem gives the relationship between the standard feasible multiplicity of a character of a table algebra (A, B) and the quotient table algebra $(A/\!/C, B/\!/C)$.

Theorem 3.4. *Let (A, B) be a table algebra with the standard feasible trace ζ and let $\chi \in \mathrm{Irr}(A)$. Then the standard feasible multiplicity of $\chi_{A/\!/C}$ is equal to that of χ if $\chi_{A/\!/C} \neq 0$, for $C \in \mathcal{C}(B)$.*

Proof. From [10, Theorem 3.2] there is a bijection between the set of $\mathrm{Irr}(A/\!/C)$ and the set $\{\chi \in \mathrm{Irr}(A) \mid \chi_{A/\!/C} \neq 0\}$. It follows that $\{e\varepsilon_\chi |\ \chi \in \mathrm{Irr}(A)\} \setminus \{0\}$ is the set of central primitive idempotents of the quotient table algebra $(A/\!/C, B/\!/C)$ where $e = |C^+|^{-1}C^+$ and $\{\varepsilon_\chi \mid \chi \in \mathrm{Irr(A)}\}$ is the set of central primitive idempotents of A. This shows that for $\chi \in \mathrm{Irr}(A/\!/C)$ we have

$$\zeta(e\varepsilon_\chi) = \zeta_{A/\!/C}(e\varepsilon_\chi). \tag{4}$$

On the other hand, by (1) we conclude that

$$\zeta(e\varepsilon_\chi) = \zeta_\chi \chi(e\varepsilon_\chi) \tag{5}$$

But from Lemma 3.3 it follows that $\zeta_{A/\!/C}(e\varepsilon_\chi) = \zeta_{\chi_{A/\!/C}}\chi_{A/\!/C}(e\varepsilon_\chi)$, where $\zeta_{\chi_{A/\!/C}}$ is the standard feasible multiplicity of $\chi_{A/\!/C}$. The latter equality along with (4) and (5) imply that $\zeta_\chi \chi(e\varepsilon_\chi) = \zeta_{\chi_{A/\!/C}}\chi_{A/\!/C}(e\varepsilon_\chi)$. Thus $\zeta_\chi = \zeta_{\chi_{A/\!/C}}$, as claimed. □

Suppose that (A, B) is a C- algebra and $\rho \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{A}, \mathbb{C})$ such that $\rho(b) = |b|$. Then ρ is an irreducible character of A, which is called the *principle character* of (A, B). From (3) by replacing ϕ and ψ by ρ we conclude that $\zeta_\rho = 1$. Moreover, if (A, B) is a commutative table algebra, then [4, Corollary 5.6] shows that $\zeta_\chi > 0$. In the following lemma we give a lower bound for the standard feasible multiplicities of the characters of a table algebra.

Lemma 3.5. *Let (A, B) be a table algebra. Then $|\zeta_\chi| \geqslant \chi(1_A)^{-1}$, for every $\chi \in \mathrm{Irr}(A)$. In particular, if (A, B) is commutative table algebra then $\zeta_\chi \geqslant 1$.*

Proof. From [10, Proposition 4.1] we have $|\chi(b)| \leqslant |b|\chi(1)$, where $b \in B$ and χ is a character of A. Now by applying the degree map $|\cdot|$ on the both sides of the equation (3) the first statement of the lemma follows.

The second statement is an immediate consequence of the first one, since $\chi(1_A) = 1$ when (A, B) is commutative. □

Lemma 3.6. *The set of standard feasible multiplicities of two isomorphic C-algebras are the same.*

Proof. Let (A, B) and (A', B') be two C-algebras and $f : (A, B) \to (A', B')$ be an isomorphism. Let ζ and ζ' be the standard feasible traces of (A, B) and (A', B'), respectively. Let $P = \{\varepsilon_\chi \mid \chi \in \mathrm{Irr}(A)\}$ be the set of central primitive idempotents

of A. Then it is easily seen that the set $P' = \{\varepsilon_{\chi^f} \mid \chi \in \mathrm{Irr}(A)\}$ is the set of central primitive idempotents of A', where $\chi^f(a') = \chi(f^{-1}(a'))$ and $a' \in A'$. It follows that for any $\chi \in \mathrm{Irr}(A)$ there exists $\psi \in \mathrm{Irr}(A)$ such that $(\varepsilon_\psi)^f = \varepsilon_{\chi^f}$, and so $\psi(1) = \chi(1)$. Therefore, by comparing the coefficient of $1_{A'}$ in the both sides of the former equality we get

$$\frac{\psi(1)}{|B^+|}\zeta_\psi = \frac{\chi(1)}{|B'^+|}\zeta'_{\chi^f}$$

where ζ_ψ and ζ'_{χ^f} are the standard feasible multiplicities of ψ and χ^f with respect to standard feasible traces ζ and ζ', respectively. This implies that $\zeta_\psi = \zeta'_{\chi^f}$. Therefore the set of standard feasible multiplicities of the C-algebras (A, B) and (A', B') are the same, as desired. □

4 The standard character

Let X be a set with n elements. According to [9] a linear subspace $\mathcal{W}$ of the algebra $\mathrm{Mat}_X(\mathbb{C})$ of all $n \times n$-complex matrices whose rows and columns are indexed by the elements of X, is called a *coherent algebra* on X if $I_n, J_n \in \mathcal{W}$; $\mathcal{W}$ is closed under the matrix and the Hadamard (componentwise) multiplications and $\mathcal{W}$ is closed under the conjugate transpose, where I_n is the identity matrix and J_n is the matrix all of whose entries are ones. Denote by $\mathcal{M}$ the set of primitive idempotents of $\mathcal{W}$ with respect to the Hadamard multiplication. Then $\mathcal{M}$ is a linear basis of $\mathcal{W}$ consisting of $\{0,1\}$-matrices such that

$$\sum_{A\in\mathcal{M}} A = J_n, \quad \text{and} \quad A \in \mathcal{M} \Leftrightarrow A^t \in \mathcal{M}.$$

Let $\mathcal{W}$ be a coherent algebra with the basis $A_0 = I_n, A_1, \ldots, A_d$ consisting of $\{0,1\}$-matrices. Define binary relations g_i, for $i = 0, 1, \ldots, d$, on X as follows:

$$\forall x, y \in X: \quad (x, y) \in g_i \Leftrightarrow (A_i)_{x,y} = 1$$

where $(A_i)_{x,y}$ is the (x, y)-entry of the matrix A_i. Now from the definition of coherent algebra it follows that $(X, \{g_i\}_{i=0}^d)$ is an association scheme whose complex adjacency algebra is $\mathcal{W}$. Conversely, any complex adjacency algebra of a given association scheme is a coherent algebra.

Let (X, G) be an association scheme and let $\mathbb{C}G = \bigoplus_{g\in G} \mathbb{C}\sigma_g$ be the complex adjacency algebra of G. Let $\mathbb{C}X$ be the $\mathbb{C}$-vector space with the basis X. Clearly $\mathbb{C}X$ is a $\mathbb{C}G$-module which is called the *standard module* of $\mathbb{C}G$. The character of $\mathbb{C}G$ afforded by the standard module is called the *standard character* of $\mathbb{C}G$, see [12]. We shall denote the standard character of $\mathbb{C}G$ by $\chi_{\mathbb{C}X}$. Moreover, $\chi_{\mathbb{C}X}(\sigma_{1_X}) = |X|$ and $\chi_{\mathbb{C}X}(\sigma_g) = 0$ for $1_X \neq g \in G$ and

$$\chi_{\mathbb{C}X} = \sum_{\chi\in\mathrm{Irr}(G)} m_\chi \chi. \tag{6}$$

In this case by setting $A = \mathbb{C}G$ and $B = \{\sigma_g : g \in G\}$, the pair (A, B) is a table algebra with the standard feasible trace $\zeta = \chi_{\mathbb{C}X}$ given in (6). Therefore, the standard feasible multiplicities $\zeta_\chi = m_\chi$ for $\chi \in \mathrm{Irr}(G)$ are nonnegative integers.

Let (A, B) be a C-algebra with the standard feasible trace ζ. In general, the standard feasible multiplicities ζ_χ are not nonnegative integers, see Example 4.3. Moreover, there exists a table algebra which does not originate from association schemes but its standard feasible trace is a character, see Example 4.2. In the case that the standard feasible trace ζ of a C-algebra (A, B) is a character, we shall call ζ the *standard character* of (A, B).

Definition 4.1. *We say that a C-algebra has standard character condition, if it possesses the standard character. We denote by $\mathcal{S}$ the class of all such C-algebras.*

Clearly association schemes belong to the class $\mathcal{S}$ and Example 4.2 below shows that the class $\mathcal{S}$ is larger than the class of association schemes. Even this class does not contain the class of integral table algebras. In fact, Example 4.3 gives an integral table algebra does not belong to $\mathcal{S}$.

For a given strongly regular graph (X, E) with parameters (n, k, λ, μ) one can find an association scheme $\mathcal{C} = (X, G)$ where $G = \{1_X, g, h\}$ with structure constants $\lambda_{gg1_X} = k, \lambda_{ggg} = \lambda, \lambda_{ggh} = \mu$. In [6] some of the necessary conditions for the existence of a strongly regular graph with parameters (n, k, λ, μ) are given. One of them is *integrality condition*. If we consider the adjacency algebra of the association scheme $\mathcal{C}$, which is an integral table algebra (A, B) of dimension 3, then one can see that the standard character condition for (A, B) is equivalent to integrality condition for the existence strongly regular graphs with parameters (n, k, λ, μ).

Example 4.2. Let A be a $\mathbb{C}$-linear space with the basis $B = \{1_A, x, y\}$ such that

$$\begin{aligned} x^2 &= 9\ 1_A + 4y \\ y^2 &= 18\ 1_A + 10x + 12y \\ xy &= yx = 8x + 5y \end{aligned}$$

Then one can see that the pair (A, B) is a table algebra. By using the orthogonality relation given in Lemma 3.1 part (ii) the character table of (A, B) is as the following:

	1_A	x	y	ζ_{χ_i}
χ_1	1	9	18	1
χ_2	1	1	-2	21
χ_3	1	-5	4	6

Table (1)

From Table (1), one can see that $(A, B) \in \mathcal{S}$. On the other hand, the fact that any association scheme of rank 2 gives a strongly regular graph along with the argument in [8, Section 12] imply that the table algebra (A, B) can not originate from an association scheme.

Example 4.3. Let A be a $\mathbb{C}$-linear space with the basis $B = \{1_A, b, c\}$ such that

$$\begin{aligned} b^2 &= 2\ 1_A + b \\ c^2 &= 25\ 1_A + 25b + 22c \\ bc &= cb = 2c \end{aligned}$$

Then one can see that the pair (A, B) is an integral table algebra. By using the orthogonality relation given in Lemma 3.1 part (ii) the character table of (A, B) is as the following:

	1_A	b	c	ζ_{χ_i}
χ_1	1	2	25	1
χ_2	1	2	-3	$\frac{25}{3}$
χ_3	1	-1	0	$\frac{56}{3}$

Table (2)

Thus from Table (2) the standard feasible multiplicities of (A, B) are not integers. This shows that $(A, B) \notin \mathcal{S}$.

In this section we find a necessary and sufficient condition for which a table algebra to originate from an association scheme in the following sense:

Definition 4.4. *We say that a table algebra (A, B) originates from an association scheme, if there are an association scheme (X, G) and a table algebra isomorphism $T : (A, B) \to (\mathbb{C}G, C)$, where $C = \{\sigma_g : g \in G\}$ is the basis of the complex adjacency algebra $\mathbb{C}G$.*

Lemma 4.5. *Let $(A, B) \in \mathcal{S}$ be a table algebra and let D be a matrix representation of A which affords the standard character ζ. Then $(D(A), D(B))$ is a table algebra isomorphic to (A, B). In particular, the structure constants of (A, B) and $(D(A), D(B))$ are the same.*

Proof. Let $B = \{b_0 = 1_A, b_1, \ldots, b_d\}$ and let $\{\lambda_{ijk}\}_{i,j,k}$ be the structure constants of the table algebra (A, B). Let $D : A \to \mathrm{Mat}_n(\mathbb{C})$ be a matrix representation of A affording the standard character ζ. We first show that $D(B) = \{D(1_A), D(b_1), \cdots, D(b_d)\}$ is a basis of the algebra $D(A)$. To do this we need to prove that $D(b_i), i = 0, 1, \ldots, d$, are linearly independent. Suppose that $\sum_{i=0}^{d} \mu_i D(b_i) = 0$ where $\mu_i \in \mathbb{C}$. If $\mu_j \neq 0$, then multiplying both sides of the latter equation by $D(b_j^*)$ will yield

$$\mu_0 D(b_j^*) + \mu_1 D(b_1 b_j^*) + \cdots + \mu_j D(b_j b_j^*) + \cdots + \mu_d D(b_d b_j^*) = 0. \tag{7}$$

If we apply the trace function to both sides of (7) we obtain $\mu_j \lambda_{jj^*1} |B^+| = 0$. It implies that $\mu_j = 0$, a contradiction.

Let $\{\gamma_{ijk}\}_{i,j,k}$ be the structure constants of the algebra $D(A)$ with the basis $D(B)$. Then $D(b_i)D(b_j) = \sum_{k=0}^{d} \gamma_{ijk} D(b_k)$. On the other hand, since D is an algebra homomorphism we have $D(b_i b_j) = \sum_{k=0}^{d} \lambda_{ijk} D(b_k)$. Thus $\gamma_{ijk} = \lambda_{ijk}$, for all i, j, k.

Define $D(b)^* := D(b^*)$ and $|D(b)| := |b|$. It is easy to verify that $*$ is a semilinear involutory anti-automorphism of the algebra $D(A)$ such that

$$D(B)^* = D(B) \quad \text{and} \quad \gamma_{ij0} = \delta_{ij^*}|D(b_i)|$$

and the mapping $D(b_i) \to |D(b_i)|, b_i \in B$ is a one dimensional $*$-linear representation of the algebra $D(A)$. Thus $(D(A), D(B))$ is a table algebra. □

Remark 4.6. *If (A, B) is a table algebra which originates from an association scheme, then from Lemma 3.6 it follows that $(A, B) \in \mathcal{S}$. Therefore, from Lemma 4.5 we conclude that the structure constants of (A, B) are non-negative integers.*

In [5, Theorem 3.28], which is a reformulation of [11, Theorem 1.8], it is shown that a given table algebra originates from an association scheme if and only if it has a maximal irreducible action. In the next theorem and corollary we provide another point of view of this result for table algebras in terms of standard character.

Theorem 4.7. *Let (A, B) be a table algebra. Then (A, B) originates from an association scheme if and only if $(A, B) \in \mathcal{S}$ and a matrix representation D which affords the standard character ζ satisfies the following conditions for any $b \in B$:*

(1) $D(b^*) = D(b)^t$.

(2) $D(b)$ *is a* $\{0, 1\}$*-matrix.*

Proof. Suppose that (A, B) originates from an association scheme (X, G). So there exists a table algebra isomorphism T from A onto $\mathbb{C}G$. Then $T(A) = \mathbb{C}G$ and $T(b^*) = T(b)^t$. It follows that $|b| = |T(b)|$, for $b \in B$. Therefore, T induces a matrix representation D of degree $|B^+|$ and conditions (1) and (2) are valid for D. It shows that the character which is afforded by D has values $|B^+|$ at 1_A and 0 at any $b \in B \setminus \{1_A\}$ and so it is the standard character ζ of (A, B). It means that $(A, B) \in \mathcal{S}$. In particular, from Remark 4.6 we see that (A, B) is an integral table algebra.

Conversely, suppose that $(A, B) \in \mathcal{S}$ and conditions (1) and (2) hold for a matrix representation D of A which affords the standard character ζ. We claim that $(D(A), D(B))$ is a coherent algebra. From Lemma 4.5 $(D(A), D(B))$ is a table algebra isomorphic to (A, B) and its structure constants $\{\lambda_{ijk}\}_{i,j,k}$ are equal to the structure constants of (A, B). Now we prove that the algebra $D(A)$ is closed with respect to the Hadamard multiplication and $\sum_{i=0}^{d} D(b_i) = J_n$, where $n = |B^+|$ and $B = \{b_0, b_1, \ldots, b_d\}$. For $b_i, b_j \in B$ we have

$$D(b_i)D(b_j) = \sum_{k=0}^{d} \lambda_{ijk} D(b_k). \tag{8}$$

Furthermore, for $b_t \in B - \{1_A\}$ we have

$$\mathrm{tr}(D(b_t)) = \zeta(b_t) = 0, \tag{9}$$

and from condition (1) and equation (8) we get

$$D(b_t)D(b_t)^t = D(b_t)D(b_t^*) = |b_t|D(1_A) + \sum_{k=1}^{d} \lambda_{tt^*k} D(b_k). \tag{10}$$

Now since $D(b_t)$ is a $\{0,1\}$-matrix, from (9) and (10) it follows that the diagonal entries of the matrix $D(b_t)$ are 0, $|b_t|$ is an integer and the matrix $D(b_t)$ contains $|b_t|$ ones in each row and each column. On the other hand, from equation (8) it follows that each diagonal entry of the matrix $D(b_i)D(b_j^*)$ is equal to λ_{ij^*0}, for $b_i, b_j \in B$. Hence, if $b_i \neq b_j$, then $D(b_i)$ and $D(b_j)$ have no nonzero common entries. So the Hadamard product of $D(b_i)$ and $D(b_j)$ is equal to $\delta_{ij}D(b_i)$. Thus $\sum_{i=0}^{d} D(b_i) = J_n$. Furthermore, since $D(b_i), b_i \in B$ are $\{0,1\}$-matrices, we conclude that (A,B) is an integral table algebra. This implies that $(D(A), D(B))$ is a coherent algebra and so is a complex adjacency algebra of an association scheme. This completes the proof of the theorem. □

Example 4.8. Let A be a $\mathbb{C}$-linear space with the basis $B = \{1_A, b, c\}$ such that

$$\begin{aligned} b^2 &= 1_A \\ c^2 &= 2\ 1_A + 2b \\ bc &= cb = c \end{aligned} \tag{11}$$

Then one can check that the pair (A,B) is a table algebra with $b^* = b, c^* = c$ and $|b| = 1, |c| = 2$. By using the orthogonality relation given in Lemma 3.1 part (ii) the character table of (A,B) is as the following:

	1_A	b	c	ζ_{χ_i}
χ_1	1	1	2	1
χ_2	1	1	-2	1
χ_3	1	-1	0	2

Table (3)

From Table (3) we conclude that the standard feasible multiplicities of the characters of (A,B) are non-negative integers. This shows that $(A,B) \in \mathcal{S}$. It is easily seen that the map $D : A \to \mathrm{Mat}_4(\mathbb{C})$ defined by

$$D(1_A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad D(c) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

is a matrix representation of A which affords the standard character ζ. Moreover, the representation D satisfies conditions (1) and (2) of Theorem 4.7. Now from Theorem 4.7 we conclude that (A,B) originates from an association scheme.

Apart from Theorem 4.7, it is not hard to see that the constant structures of the adjacency algebra of the association scheme associated with the strongly regular graph with parameters $(n,k,\lambda,\mu) = (4,2,0,2)$ satisfy (11).

Let (A, B) be a C-algebra. The coordinate-wise multiplication $\circ$ with respect to the basis B by $b \circ c = \delta_{bc} b$, for $b, c \in B$ is defined in the sense of [7]. We say that a matrix representation D of A preserves the Hadamard product if $D(b \circ c) = D(b) \circ D(c)$, for $b, c \in B$.

For a matrix C, $\tau(C)$ denotes the sum of all entries C. One can see that for any two square matrices C and D of the same size:

$$\tau(C \circ D) = \mathrm{tr}(CD^t) = \mathrm{tr}(C^t D).$$

Corollary 4.9. *Let $(A, B) \in \mathcal{S}$ be a table algebra and let D be a matrix representation of A which affords the standard character ζ. Then the table algebra $(D(A), D(B))$ is a coherent algebra if and only if D perseveres Hadamard products.*

Proof. The necessity is obvious. For the sufficiency, since $D(b), b \in B$, persevere Hadamard products, each $D(b), b \in B$ is $\{0, 1\}$- matrix. On the other hand,

$$\tau(D(b^*) \circ D(c)^t) = \mathrm{tr}(D(b^*)D(c)) \quad b, c \in B.$$

But $\mathrm{tr}(D(b^*)D(c)) = 0$ if and only if $b \neq c$. Thus $D(b^*) = D(b)^t$. Now the result follows from Theorem 4.7 and we are done. □

In the rest of this section, we suppose that (A, B) is a commutative C-algebra of dimension d with the set of the primitive idempotents $\{\varepsilon_\chi |\ \chi \in \mathrm{Irr}(A)\}$. Then from [4, Section 2.5] there are two matrices $P = (p_b(\chi))$ and $Q = (q_\chi(b))$ in $\mathrm{Mat}_d(\mathbb{C})$, where $b \in B$ and $\chi \in \mathrm{Irr}(A)$ such that $PQ = QP = |B^+|I$, where I is the identity matrix in $\mathrm{Mat}_d(\mathbb{C})$, and

$$b = \sum_{\chi \in \mathrm{Irr}(A)} p_b(\chi)\varepsilon_\chi \quad \text{and} \quad \varepsilon_\chi = \frac{1}{|B^+|} \sum_{b \in B} q_\chi(b) b. \tag{12}$$

Then from Remark (3.2) and (12) we get

$$q_\chi(1_A) = \zeta_\chi \quad \text{and} \quad \chi(b) = p_b(\chi), \tag{13}$$

where $b \in B$ and $\chi \in \mathrm{Irr}(A)$. The dual of (A, B) in the sense of [4] is as follows: with each linear representation $\Delta_\chi : b \mapsto p_b(\chi)$, we associate the linear mapping $\Delta_\chi^* : b \mapsto q_\chi(b)$. Since the matrix $Q = (q_\chi(b))$ is non-singular, it follows that the set $\widehat{B} = \{\Delta_\chi^* : \chi \in \mathrm{Irr}(A)\}$ is a linearly independent and so form a base of the set of all linear mapping $\widehat{A}$ of A into $\mathbb{C}$. From [4, Thorem 5.9] the pair $(\widehat{A}, \widehat{B})$ is a C-algebra with the identity $1_{\widehat{A}} = \Delta_\rho^*$ and involutory automorphism which maps Δ_χ^* to $\Delta_{\overline{\chi}}^*$, where $\overline{\chi}$ is complex conjugate to χ. The C-algebra $(\widehat{A}, \widehat{B})$ is called the *dual C-algebra* of (A, B). Moreover, the structure constants of $(\widehat{A}, \widehat{B})$ which are given in [4, (5.26)] can be written as the following

$$q_{\varphi\psi}^\chi = \frac{\zeta_\varphi \zeta_\psi}{|B^+|} \sum_{b \in B} \frac{1}{|b|^2} p_b(\varphi) p_b(\psi) \overline{p_b(\chi)} \tag{14}$$

which are real numbers, where $\overline{p_b(\chi)}$ is the complex conjugate to $p_b(\chi)$. From (14) and (3) one can see that $q^{\rho}_{\chi,\overline{\chi}} = \zeta_\chi$. Then $|\widehat{B}^+| = \sum_{\chi\in \mathrm{Irr}(A)} \zeta_\chi$. The primitive idempotents $f_b, b \in B$ of $\widehat{A}$ are given by [4, 5.23] as the following

$$f_b = \frac{1}{|\widehat{B}^+|} \sum_{\chi\in \mathrm{Irr}(A)} p_b(\chi)\Delta^*_\chi. \tag{15}$$

Lemma 4.10. *Keeping the notation above, there is a bijection correspondence between the standard feasible multiplicities of the characters of $(\widehat{A}, \widehat{B})$ and the degrees of basis elements B.*

Proof. From (15), one can see that the coefficient of the unit element $1_{\widehat{A}}$ of $\widehat{A}$ in the linear decomposition of $|\widehat{B}^+|f_b$ in terms of the basis elements $\widehat{B}$ is equal to $p_b(\rho)$. On the other hand, from the equation of the right hand side of (13) we get $p_b(\rho) = \rho(b) = |b|$. But from Remark 3.2 any standard feasible multiplicity of the characters of $(\widehat{A}, \widehat{B})$ corresponds to the number $p_b(\rho)$ for some $b \in B$, as desired. □

A C-algebra is called *integral degree* if its all degrees $|b|, b \in B$, are integers.

Corollary 4.11. *Let (A, B) be a C-algebra. Then (A, B) is integral degree and belongs to $\mathcal{S}$ if and only if so is $(\widehat{A}, \widehat{B})$.*

Proof. Let (A, B) be a C-algebra and $(\widehat{A}, \widehat{B})$ be its dual with the standard feasible traces ζ and $\widehat{\zeta}$, respectively. To prove the necessity, since (A, B) is in $\mathcal{S}$ the equality $q^{\rho}_{\chi,\overline{\chi}} = \zeta_\chi$ implies that $(\widehat{A}, \widehat{B})$ is integral degree. Since (A, B) is integral degree, from Lemma 4.10 we conclude that $(\widehat{A}, \widehat{B})$ is in $\mathcal{S}$.

To prove the sufficiency, by the necessity we see that $(\widehat{\widehat{A}}, \widehat{\widehat{B}}) \in \mathcal{S}$ is integral degree. Now the proof follows from Lemma 3.6 and the Duality Theorem [4, Theorem 5.10], i.e., $(A, B) \simeq (\widehat{\widehat{A}}, \widehat{\widehat{B}})$. □

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