On sum of powers of the Laplacian and signless Laplacian eigenvalues of graphs

Saieed Akbari [1,2] Ebrahim Ghorbani [1,2] Jacobus H. Koolen [3,4]
Mohammad Reza Oboudi [1,2]

[1]Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11155-9415, Tehran, Iran
s_akbari@sharif.edu
e_ghorbani@math.sharif.edu
m_r_oboudi@math.sharif.edu

[2]School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran

[3]Department of Mathematics
Pohang University of Science and Technology (POSTECH)
Pohang 790-785, South Korea
koolen@postech.ac.kr

[4]Pohang Mathematics Institute (PMI)
Pohang University of Science and Technology (POSTECH)
Pohang 790-785, South Korea

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Abstract

Let G be a graph of order n with signless Laplacian eigenvalues $q_1, \ldots, q_n$ and Laplacian eigenvalues $\mu_1, \ldots, \mu_n$. It is proved that for any real number α with $0 < \alpha \leqslant 1$ or $2 \leqslant \alpha < 3$, the inequality $q_1^\alpha + \cdots + q_n^\alpha \geqslant \mu_1^\alpha + \cdots + \mu_n^\alpha$ holds, and for any real number β with $1 < \beta < 2$, the inequality $q_1^\beta + \cdots + q_n^\beta \leqslant \mu_1^\beta + \cdots + \mu_n^\beta$ holds. In both inequalities, the equality is attained (for $\alpha \notin \{1, 2\}$) if and only if G is bipartite.

1 Introduction

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The adjacency matrix of G, $A = (a_{ij})$, is an $n \times n$ matrix such that $a_{ij} = 1$ if v_i and v_j

are adjacent, and otherwise $a_{ij} = 0$. The *incidence matrix* of G, denoted by $X = (x_{ij})$, is the $n \times m$ matrix, whose rows are indexed by the set of vertices of G and whose columns are indexed by the set of edges of G, defined by

$$x_{ij} := \begin{cases} 1, & \text{if } e_j \text{ is incident with } v_i; \\ 0, & \text{otherwise.} \end{cases}$$

If we consider an orientation for G, then in a similar manner as for the incidence matrix, the *directed incidence matrix* of the (oriented) graph G, denoted by $D = (d_{ij})$, is defined as

$$d_{ij} := \begin{cases} +1, & \text{if } e_j \text{ is an incomming edge to } v_i; \\ -1, & \text{if } e_j \text{ is an outgoinging edge from } v_i; \\ 0, & \text{otherwise.} \end{cases}$$

Let Δ be the diagonal matrix whose entries are vertex degrees of G. The *Laplacian matrix* of G, denoted by $L(G)$, is defined by $L(G) = \Delta - A$, and it is easy to see that $L(G) = DD^\top$ holds. The *signless Laplacian matrix* of G, denoted by $Q(G)$, is defined by $Q(G) = \Delta + A$, and again it is easy to see that $Q(G) = XX^\top$. Since $L(G)$ and $Q(G)$ are symmetric matrices, their eigenvalues are real. We denote the eigenvalues of $L(G)$ and $Q(G)$ by $\mu_1(G) \geqslant \cdots \geqslant \mu_n(G)$ and $q_1(G) \geqslant \cdots \geqslant q_n(G)$, respectively (we drop G when it is clear from the context). We call the multi-set of eigenvalues of $L(G)$ and $Q(G)$, the L-spectrum and Q-spectrum of G, respectively. The matrices L and Q are similar if and only if G is bipartite (see, e.g., [5]). The *incidence energy* $\mathrm{IE}(G)$ of the graph G is defined as the sum of singular values of the incidence matrix [9]. The *directed incidence energy* $\mathrm{DIE}(G)$ is defined as the sum of singular values of the directed incidence matrix [7]. In other words,

$$\mathrm{IE}(G) = \sum_{i=1}^{n} \sqrt{q_i(G)}, \quad \text{and} \quad \mathrm{DIE}(G) = \sum_{i=1}^{n} \sqrt{\mu_i(G)}.$$

The sum of square roots of Laplacian eigenvalues was also defined as *Laplacian-energy like invariant* and denoted by $\mathrm{LEL}(G)$ in [10]. The connection between IE and Laplacian eigenvalues (for bipartite graphs) was first pointed out in [6]. For more information on IE and DIE/LEL, see [7, 14] and the references therein.

In [2], it was conjectured that $\sqrt{q_1} + \cdots + \sqrt{q_n} \geqslant \sqrt{\mu_1} + \cdots + \sqrt{\mu_n}$ or equivalently $\mathrm{IE}(G) \geqslant \mathrm{DIE}(G)$. In [1], it is proved that this conjecture is true by showing that for any real number α with $0 < \alpha \leqslant 1$, the following holds:

$$q_1^\alpha + \cdots + q_n^\alpha \geqslant \mu_1^\alpha + \cdots + \mu_n^\alpha. \tag{1}$$

Let G be a graph of order n. In [1], the authors proved that if $\sum_{i=0}^{n}(-1)^i a_i \lambda^{n-i}$ and $\sum_{i=0}^{n}(-1)^i b_i \lambda^{n-i}$ are the characteristic polynomials of the signless Laplacian and the Laplacian matrices of G, respectively, then $a_i \geqslant b_i$ for $i = 0, 1, \ldots, n$. Then, using an analytical method, they showed that (1) holds for $0 < \alpha \leqslant 1$. But one question was remained open, namely is it true that equality holds in (1), for $\alpha \neq 1$, if and only if G is bipartite? In this note we give a completely different proof for this statement and we show that equality holds if and only if G is bipartite. Moreover, we show that the Inequality (1) holds for any real number α with $2 \leqslant \alpha \leqslant 3$. Furthermore for every $1 \leqslant \alpha \leqslant 2$ the following holds:

$$q_1^\alpha + \cdots + q_n^\alpha \leqslant \mu_1^\alpha + \cdots + \mu_n^\alpha.$$

We recall that for a real number α the quantity $S_\alpha := \mu_1^\alpha + \cdots + \mu_n^\alpha$ has been already studied (see [11, 12, 13]). In [12], some upper and lower bounds have been obtained for S_α. In this paper we establish some new upper and lower bounds for S_α in terms of the signless Laplacian spectrum.

2 Sum of powers of the Laplacian and signless Laplacian eigenvalues

In this section we prove the main result of the paper. Let G be a graph with the adjacency matrix A and Δ be the diagonal matrix whose entries are vertex degrees of G. Note that $\mathrm{tr}(\Delta + A) = \mathrm{tr}(\Delta - A)$ and since $\mathrm{tr}(\Delta A) = 0$, $\mathrm{tr}(\Delta + A)^2 = \mathrm{tr}(\Delta - A)^2$, which implies that $q_1^\alpha + \cdots + q_n^\alpha = \mu_1^\alpha + \cdots + \mu_n^\alpha$, for $\alpha = 1, 2$.

We use the interlacing property of the Laplacian and signless Laplacian eigenvalues which follows from the Courant-Weyl inequalities (see, e.g., [8, Theorem 4.3.7]).

Lemma 1. *Let G be a graph of order n and $e \in E(G)$. Then the Laplacian (and the signless Laplacian) eigenvalues of G and $G' = G - e$ interlace:*

$$\mu_1(G) \geqslant \mu_1(G') \geqslant \mu_2(G) \geqslant \mu_2(G') \geqslant \cdots \geqslant \mu_n(G) = \mu_n(G') = 0.$$

Now, we are in a position to prove the following theorem.

Theorem 2. *Let G be a graph of order n and let α be a real number.*

(i) *If $0 < \alpha \leqslant 1$ or $2 \leqslant \alpha \leqslant 3$, then*

$$q_1^\alpha + \cdots + q_n^\alpha \geqslant \mu_1^\alpha + \cdots + \mu_n^\alpha.$$

(ii) *If $1 \leqslant \alpha \leqslant 2$, then*

$$q_1^\alpha + \cdots + q_n^\alpha \leqslant \mu_1^\alpha + \cdots + \mu_n^\alpha.$$

For $\alpha \in (0,1) \cup (2,3)$, the equality occurs in (i) *if and only if G is a bipartite graph. Moreover, for $\alpha \in (1,2)$, the equality occurs in* (ii) *if and only if G is a bipartite graph.*

Proof. We recall that, for any real number s, the binomial series $\sum_{k=0}^{\infty} \binom{s}{k} x^k$ converges to $(1+x)^s$ if $|x| < 1$. This also remains true for $x = -1$ if $s > 0$ (see, e.g., [3, p. 419]). Let $\ell := 2n$. By Lemma 1, we find that,

$$\mu_1 \leqslant \mu_1(K_n) = n, \quad \text{and} \quad q_1 \leqslant q_1(K_n) = 2n - 2.$$

Hence $\left|\frac{q_i}{\ell}-1\right|<1$ if $q_i>0$ and $\frac{q_i}{\ell}-1=-1$ if $q_i=0$. Therefore,

$$\begin{aligned}\left(\frac{q_1}{\ell}\right)^\alpha+\cdots+\left(\frac{q_n}{\ell}\right)^\alpha &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{q_1}{\ell}-1\right)^k+\cdots+\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{q_n}{\ell}-1\right)^k\\ &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\operatorname{tr}\left(\frac{1}{\ell}(\Delta+A)-I\right)^k.\end{aligned}$$

In a similar manner as above, we obtain that,

$$\left(\frac{\mu_1}{\ell}\right)^\alpha+\cdots+\left(\frac{\mu_n}{\ell}\right)^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}\operatorname{tr}\left(\frac{1}{\ell}(\Delta-A)-I\right)^k.$$

We claim that

$$\begin{aligned}\text{if } k \text{ is even,}\quad & \operatorname{tr}(\Delta+A-\ell I)^k\leqslant\operatorname{tr}(\Delta-A-\ell I)^k;\\ \text{if } k \text{ is odd,}\quad & \operatorname{tr}(\Delta+A-\ell I)^k\geqslant\operatorname{tr}(\Delta-A-\ell I)^k.\end{aligned}$$

If one expands $((\Delta-\ell I)+A)^k$ and $((\Delta-\ell I)-A)^k$ in terms of powers of $\Delta-\ell I$ and A, then the terms appearing in both expansions, regardless their signs, are the same. To prove the claim, we determine the sign of each term in both expansions. In the expansion of $((\Delta-\ell I)+A)^k$, consider the terms in which there are exactly j factors equal to $\Delta-\ell I$, for some $j=0,1,\ldots,k$. As all the entries of $\Delta-\ell I$ are non-positive and those of A are non-negative, the sign of the trace of each such a term is $(-1)^j$ or 0. On the other hand, in the expansion of $((\Delta-\ell I)-A)^k$ in each term all factors are matrices with non-positive entries, so the sign of the trace of each term is $(-1)^k$ or 0. This proves the claim.

Now, note that if $0<\alpha<1$ or $2<\alpha<3$, then the sign of $\binom{\alpha}{k}$ is $(-1)^{k-1}$ except that $\binom{\alpha}{2}>0$, for $2<\alpha<3$. This implies that for $0<\alpha<1$ and every k,

$$\binom{\alpha}{k}\operatorname{tr}(\Delta+A-\ell I)^k\geqslant\binom{\alpha}{k}\operatorname{tr}(\Delta-A-\ell I)^k. \tag{2}$$

This inequality remains true for $2\leqslant\alpha\leqslant 3$ as $\operatorname{tr}(\Delta+A-\ell I)^2=\operatorname{tr}(\Delta-A-\ell I)^2$. Thus, Part (i) is proved. For $1<\alpha<2$, the sign of $\binom{\alpha}{k}$ is $(-1)^k$ with one exception that $\binom{\alpha}{1}>0$. Since $\operatorname{tr}(\Delta+A-\ell I)=\operatorname{tr}(\Delta-A-\ell I)$, Part (ii) is similarly proved.

Now, we consider the case of equality. If G is bipartite, Q and L are similar which implies that the equality holds in both (i) and (ii). If G is not bipartite, then there exists an odd integer r such that $\operatorname{tr}A^r>0$, since for any positive integer i, $\operatorname{tr}A^i$ is equal to the total number of closed walks of length i in G (see [4, Lemma 2.5]). Hence $\operatorname{tr}(\Delta+A-\ell I)^r>\operatorname{tr}(\Delta-A-\ell I)^r$ and so the inequalities in both (i) and (ii) are strict. □

3 The inequality for real powers

In this section we study the behavior of

$$f_G(\alpha):=\sum_{\substack{i=1\\ q_i>0}}^{n}q_i^\alpha-\sum_{\substack{i=1\\ \mu_i>0}}^{n}\mu_i^\alpha$$

as a function of α. In the previous section, we saw that for any graph G, $f_G(\alpha) \geqslant 0$ for $\alpha \in [0,1]$ or $\alpha \in [2,3]$; and $f_G(\alpha) \leqslant 0$ for $\alpha \in (1,2)$. In this section, we show that, for $\alpha \in (-\infty, 0)$ and $\alpha \in (2k-1, 2k)$, for any integer $k \geqslant 2$, the same kind of inequalities do not hold. We do this by comparing f_{K_n}, f_{C_n}, for odd n, where K_n and C_n denote the complete graph and the cycle graph of order n, respectively, and H_{2n} is the graph obtained by attaching two copies of K_n by a new edge.

It can be shown that $f_{K_n}(\alpha) > 0$ for any $\alpha \in \mathbb{R} \setminus [1,2]$ and any integer $n \geqslant 3$. The proof of this fact is rather involved, so we prove the following weaker assertion which is sufficient for our purpose.

Lemma 3. *For every $\alpha < 1$ and each integer $n \geqslant 3$, $f_{K_n}(\alpha) > 0$. Also for every $\alpha > 2$, there exists an integer $n(\alpha)$ such that for every $n \geqslant n(\alpha)$, $f_{K_n}(\alpha) > 0$.*

Proof. We note that the Q-spectrum and L-spectrum of K_n are $\{[2n-2]^1, [n-2]^{n-1}\}$ and $\{[n]^{n-1}, [0]^1\}$, respectively, where the exponents indicate the multiplicities. Therefore,

$$f_{K_n}(\alpha) = (2n-2)^\alpha + (n-1)(n-2)^\alpha - (n-1)n^\alpha.$$

This is clear that $f_{K_n}(\alpha) > 0$ for any $\alpha \leqslant 0$. If $0 < \alpha < 1$, then by Theorem 2, $f_{K_n}(\alpha) > 0$. If $\alpha > 2$, then $f_{K_n}(\alpha) > 0$ if and only if

$$2^\alpha \left(1 - \frac{1}{n}\right)^\alpha + (n-1)\left(1 - \frac{2}{n}\right)^\alpha > n - 1.$$

By Bernoulli's inequality, the left hand side is at least

$$2^\alpha \left(1 - \frac{\alpha}{n}\right) + (n-1)\left(1 - \frac{2\alpha}{n}\right)$$

which is bigger than $n-1$ for large enough n. □

Lemma 4. *For every integer $n \geqslant 3$, there exists $\alpha_n < 0$ with $\lim_{n\to\infty} \alpha_n = 0$ such that for any $\alpha \leqslant \alpha_n$, $f_{H_{2n}}(\alpha) < 0$.*

Proof. First, notice that if G is a connected non-bipartite graph, then $f_G(0) = 1$. So $f_G(\alpha)$ is always positive in a neighbor of the origin. We determine the Q-spectrum and the L-spectrum of H_{2n}. If e is the edge joining two copies of K_n, then $G - e$ is $2K_n$. So, the Q-spectrum of $G - e$ is $\{[2n-2]^2, [n-2]^{2n-2}\}$. By Lemma 1, the Q-spectrum of G contains $2n-2$ and $n-2$ of multiplicities at least 1 and $2n-3$, respectively. Thus, there are only two eigenvalues q_1, q_2, say, which need to be determined. Since $\operatorname{tr} Q = 2m$, and $\operatorname{tr} Q^2 = \operatorname{tr} \Delta^2 + 2m$, where m is the number of edges of H_{2n}, we find that

$$q_1 + q_2 = 3n - 2 \quad \text{and} \quad q_1^2 + q_2^2 = 5n^2 - 8n + 8.$$

This follows that

$$q_{1,2} = \frac{3n}{2} - 1 \pm \frac{\sqrt{n^2 - 4n + 12}}{2}.$$

In a similar manner we see that the L-spectrum of H_{2n} is $\{[\mu_1]^1, [n]^{2n-3}, [\mu_2]^1, [0]^1\}$, in which

$$\mu_{1,2} = \frac{n}{2} + 1 \pm \frac{\sqrt{n^2+4n-4}}{2}.$$

Therefore, it turns out that for any $\alpha < 0$,

$$f_{H_{2n}}(\alpha) = q_1^\alpha + q_2^\alpha + (2n-2)^\alpha + (2n-3)\left((n-2)^\alpha - n^\alpha\right) - \mu_1^\alpha - \mu_2^\alpha < 3 - \mu_2^\alpha.$$

It is seen that $0 < \mu_2 < 2/n$. Therefore,

$$f_{H_{2n}}(\alpha) < 3 - \left(\frac{2}{n}\right)^\alpha.$$

It turns out that if $\alpha \leqslant \alpha_n := \ln 3/(\ln 2 - \ln n)$, then $f_{H_{2n}}(\alpha) < 0$. □

For cycle C_{2n+1}, the sign of $f_{C_{2n+1}}(\alpha)$ alternately changes on the intervals

$$(0,1), (1,2), \ldots, (2n-1, 2n).$$

Lemma 5. *For every integer $n \geqslant 1$, $f_{C_{2n+1}}(\alpha)$ is positive on the intervals $(2i, 2i+1)$, $i = 0, \ldots, n-1$ and is negative on the intervals $(2i-1, 2i)$, $i = 1, \ldots, n$.*

Proof. For every $\alpha \in (2i, 2i+1)$ and each k with $k - 1 \geqslant 2i+1$, we have $\operatorname{sign}\binom{\alpha}{k} = (-1)^{k-1}$. Similarly, for every $\alpha \in (2i-1, 2i)$ and each k with $k-1 \geqslant 2i$, $\operatorname{sign}\binom{\alpha}{k} = (-1)^k$. Therefore, for any $\alpha \in [0, 2n]$ and $k \geqslant 2n+1$, (2) is satisfied. We show that for the remaining values of k, the equality holds in (2). We have

$$\begin{aligned}(\Delta - \ell I + A)^k - (\Delta - \ell I - A)^k &= ((2-\ell)I + A)^k - ((2-\ell)I - A)^k \\ &= \sum_{i=0}^{k} \binom{k}{i} (2-\ell)^{k-i}(1-(-1)^i)A^i.\end{aligned}$$

The summands for even i is zero. For all odd $i \leqslant 2n-1$, since C_{2n+1} has no closed walk of length i, $\operatorname{tr} A^i = 0$. This shows that for $k \leqslant 2n$, the equality holds in (2). Thus the result follows similarly as in the proof of Theorem 2. □

By the above three lemmas the following corollary is immediate:

Corollary 6. *For each $\alpha \in (-\infty, 0) \cup \bigcup_{k \geqslant 2}(2k-1, 2k)$, there are graphs G and G' such that $f_G(\alpha) > 0$ and $f_{G'}(\alpha) < 0$.*

We close this section by posing the following problem:

Problem. Is it true that for any graph G, the function $f_G(\alpha)$ is non-negative for $\alpha \in (2k, 2k+1)$, where $k = 2, 3, \ldots$?

4 The inequality for real sequences

Let n be a positive integer, and let $(a_i)_{0\leqslant i\leqslant n}$ and $(b_i)_{0\leqslant i\leqslant n}$ be two sequences of non-negative real numbers satisfying that for all integer $k \geqslant 1$ one has

$$\sum_{i=0}^{n} a_i^k \geqslant \sum_{i=0}^{n} b_i^k,$$

and equality holds for $k = 1, 2$. One might ask whether

$$\sum_{i=0}^{n} a_i^{1/2} \geqslant \sum_{i=0}^{n} b_i^{1/2}$$

holds. Here we show that this is not the case. Let

$$a_i = i, \quad \text{for } i = 0, 1, \ldots, 2m-1, \quad \text{and}$$

$$b_0 = \cdots = b_{m-1} = m - \frac{1}{2} - \frac{\sqrt{12m^2-3}}{6}, \quad b_m = \cdots = b_{2m-1} = m - \frac{1}{2} + \frac{\sqrt{12m^2-3}}{6}.$$

Computations show that

$$\sum_{i=0}^{2m-1} a_i^k = \sum_{i=0}^{2m-1} b_i^k, \quad \text{for } k = 1, 2, 3.$$

Note that the leading term of $\sum_{i=0}^{2m-1} i^k$ is $\frac{2^{k+1}}{k+1} m^{k+1}$. On the other hand, $\sum_{i=0}^{2m-1} b_i^k = (\alpha^k + \beta^k) m^{k+1} + O(m^k)$, where

$$\alpha = 1 + \frac{\sqrt{3}}{3}, \quad \text{and} \quad \beta = 1 - \frac{\sqrt{3}}{3}.$$

We have $\alpha^k + \beta^k \leqslant \frac{2^{k+1}}{k+1}$ with equality if and only if $k = 1, 2, 3$. Thus, for large enough m, one has

$$\sum_{i=0}^{2m-1} a_i^k > \sum_{i=0}^{2m-1} b_i^k, \quad \text{for } k \geqslant 4.$$

Now, we look at the sum of square roots. We observe that

$$\sum_{i=0}^{2m-1} \sqrt{i} < \int_0^{2m} \sqrt{x}\, \mathrm{d}x = \frac{4\sqrt{2}}{3} m^{3/2}.$$

On the other hand, $\sum_{i=0}^{2m-1} \sqrt{b_i} = (\sqrt{\alpha} + \sqrt{\beta}) m^{3/2} + O(m)$. Since $\sqrt{\alpha} + \sqrt{\beta} > \frac{4\sqrt{2}}{3}$, for large enough m we have

$$\sum_{i=0}^{2m-1} \sqrt{a_i} < \sum_{i=0}^{2m-1} \sqrt{b_i}.$$

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