

Two Simple Proofs of Winkvist's Identity

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Abstract

We give two new proofs of Winkvist's identity. In the first proof, we use basic properties of cube roots of unity and the Jacobi triple product identity. The latter does not use the Jacobi triple product identity.

1 Introduction

Winkvist's identity was discovered by L. Winkvist [6] in 1969. He used it to prove the congruence

$$p(11n + 6) \equiv 0 \pmod{11},$$

where $p(n)$ is the number of partitions of the positive integer n . In 1972, L. Carlitz and M. V. Subbarao [1] gave a simple proof and a generalization of Winkvist's identity. In 1987, M. D. Hirschhorn [3] gave another generalization of Winkvist's identity. In 1997, S.-Y. Kang [4] gave a simple proof using the Jacobi triple product identity, the quintuple product identity and two other identities from Ramanujan's notebooks. In 2003, S. Kongsiriwong and Z.-G. Liu [5] gave a simple proof using the Jacobi triple product identity and some properties of cube roots of unity. In this paper we give two new proofs of Winkvist's identity: for any complex number q with $|q| < 1$, and any nonzero complex numbers a

and b ,

$$\begin{aligned}
& \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{(3m^2+3n^2+3m+n)/2} \\
& \quad (a^{-3m}b^{-3n} - a^{-3m}b^{3n+1} - a^{-3n+1}b^{-3m-1} + a^{3n+2}b^{-3m-1}) \\
& = (q; q)_{\infty}^2 (a; q)_{\infty} (a^{-1}q; q)_{\infty} (b; q)_{\infty} (b^{-1}q; q)_{\infty} \\
& \quad (ab; q)_{\infty} (a^{-1}b^{-1}q; q)_{\infty} (ab^{-1}; q)_{\infty} (a^{-1}bq; q)_{\infty}
\end{aligned} \tag{1.1}$$

where $(a; q)_{\infty}$ denotes $\prod_{n=1}^{\infty} (1 - aq^{n-1})$. In both proofs, we will use the fact about $\omega = \exp(2\pi i/3)$ that, for any complex number a ,

$$(1 - a)(1 - a\omega)(1 - a\omega^2) = 1 - a^3. \tag{1.2}$$

2 First Proof

In this section, we prove Winquist's identity by using some properties of cube roots of unity and the Jacobi triple product identity: for each pair of complex numbers z and q with $z \neq 0$ and $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(n^2-n)/2} z^n = (q; q)_{\infty} (z; q)_{\infty} (z^{-1}q; q)_{\infty}.$$

Let $g(a, b)$ denote the right hand side of (1.1). Since $h(z) := (z; q)_{\infty} (z^{-1}q; q)_{\infty}$ is analytic on $0 < |z| < \infty$, we can write h as a Laurent series

$$h(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Since $g(a, b) = (q; q)_{\infty}^2 h(a)h(b)h(ab)h(ab^{-1})$, we can write g as a double Laurent series

$$g(a, b) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n} a^m b^n = \sum_{m=-\infty}^{\infty} c_m a^m \tag{2.1}$$

where $c_m = \sum_{n=-\infty}^{\infty} c_{m,n} b^n$. From the definition of g , we find that $g(a, b) = -a^3 g(aq, b)$ and $g(a, b) = -a^3 g(a^{-1}, b)$. Thus, from (2.1), we have, for all integers m , $c_m = -q^{m-3} c_{m-3}$ and $c_m = -c_{-m+3}$. The first equation implies that, for each m ,

$$\begin{aligned}
c_{3m} &= (-1)^m q^{(3m^2-3m)/2} c_0, \\
c_{3m+1} &= (-1)^m q^{(3m^2-m)/2} c_1, \\
c_{3m+2} &= (-1)^m q^{(3m^2+m)/2} c_2,
\end{aligned}$$

whereas the second equation implies that $c_1 = -c_2$. By putting all these together, we have

$$\begin{aligned} g(a, b) = & c_0 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-3m)/2} a^{3m} + c_1 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-m)/2} a^{3m+1} \\ & - c_1 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2+m)/2} a^{3m+2}. \end{aligned} \quad (2.2)$$

By putting $a = \omega$ in (2.2), we have

$$g(\omega, b) = c_0 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-3m)/2} + c_1(\omega - \omega^2) \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-m)/2}.$$

Note that, by using the Jacobi triple product identity, we have $\sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-3m)/2} = 0$. Then

$$g(\omega, b) = c_1(\omega - \omega^2) \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-m)/2}. \quad (2.3)$$

From the definition of g and (1.2),

$$g(\omega, b) = -b^{-1}(\omega - \omega^2)(q; q)_{\infty}(q^3; q^3)_{\infty}(b^3; q^3)_{\infty}(b^{-3}q^3; q^3)_{\infty}. \quad (2.4)$$

From (2.3), (2.4), and the Jacobi triple product identity, we obtain

$$c_1 = -b^{-1} \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-3m)/2} b^{3m}. \quad (2.5)$$

By letting $a = b$ in (2.2), we have

$$\begin{aligned} 0 = g(b, b) = & c_0 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-3m)/2} b^{3m} + c_1 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-m)/2} b^{3m+1} \\ & - c_1 \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2+m)/2} b^{3m+2}. \end{aligned}$$

Using (2.5), we have

$$c_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2-m)/2} b^{3m} - \sum_{m=-\infty}^{\infty} (-1)^m q^{(3m^2+m)/2} b^{3m+1}. \quad (2.6)$$

Substituting c_0 and c_1 in (2.2), we obtain Winquist's identity.

3 Second Proof

In this section, we prove Winquist's identity with no use of the Jacobi triple product identity. First, we let $g(a, b, q)$ denote the right hand side of (1.1). From the first proof, we write

$$g(a, b, q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{m,n}(q) a^m b^n.$$

From Kongsiriwong and Liu's proof of Winquist's identity [5], we have

$$\begin{aligned} g(a, b, q) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ (-1)^{m+n} q^{(3m^2-3m+3n^2-n)/2} (1 - bq^n) a^{3m} b^{3n} c_{0,0}(q) \right. \\ \left. + (-1)^{m+n} q^{(3m^2-m+3n^2-3n)/2} (1 - aq^m) a^{3m+1} b^{3n-1} c_{1,-1}(q) \right\}. \end{aligned}$$

Setting $a = b = q^{1/2}$, we obtain

$$\begin{aligned} 0 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ (-1)^{m+n} q^{(3m^2+3n^2+2n)/2} (1 - q^{n+1/2}) c_{0,0}(q) \right. \\ \left. + (-1)^{m+n} q^{(3m^2+2m+3n^2)/2} (1 - q^{m+1/2}) c_{1,-1}(q) \right\}. \end{aligned}$$

It follows that $c_{0,0}(q) = -c_{1,-1}(q)$. Thus we have

$$\begin{aligned} g(a, b, q) = c_{0,0}(q) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{(3m^2-3m+3n^2-n)/2} \\ \left\{ a^{3m} b^{3n} - q^n a^{3m} b^{3n+1} - a^{3n+1} b^{3m-1} + q^n a^{3n+2} b^{3m-1} \right\}. \quad (3.1) \end{aligned}$$

Next, we show that $c_{0,0}(q) = 1$; this part of the proof is similar to Kongsiriwong and Liu's proof of the Jacobi triple product identity [5] and Chan's proof of the quintuple product identity [2]. By putting $a = \omega$ and $b = -\omega$ in (3.1), we have

$$\begin{aligned} g(\omega, -\omega, q) = c_{0,0}(q) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{(3m^2-3m+3n^2-n)/2} \\ \left\{ (-1)^n + (-1)^n q^n \omega + (-1)^m - (-1)^m q^n \omega \right\}. \end{aligned}$$

Since

$$(-1)^n + (-1)^n q^n \omega + (-1)^m - (-1)^m q^n \omega = \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even,} \\ -2q^n \omega & \text{if } m \text{ is even and } n \text{ is odd,} \\ 2q^n \omega & \text{if } m \text{ is odd and } n \text{ is even,} \\ -2 & \text{if } m \text{ and } n \text{ are odd,} \end{cases}$$

we obtain

$$g(\omega, -\omega, q) = 2(1 - \omega)c_{0,0}(q) \left\{ \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{6k^2-3k+6l^2-l} - \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{6k^2-3k+6l^2-5l+1} \right\}. \quad (3.2)$$

By the definition of g and (1.2), we have

$$g(\omega, -\omega, q) = 2(1 - \omega)(q; q)_{\infty}(-q^3; q^3)_{\infty}(q^6; q^6)_{\infty}. \quad (3.3)$$

By (3.2) and (3.3), we have

$$\begin{aligned} & (q; q)_{\infty}(-q^3; q^3)_{\infty}(q^6; q^6)_{\infty} \\ &= c_{0,0}(q) \left\{ \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{6k^2-3k+6l^2-l} - \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} q^{6k^2-3k+6l^2-5l+1} \right\}. \end{aligned} \quad (3.4)$$

Next, we evaluate $g(-\omega^2 q, -\omega q, q^4)$. By (3.1), we have

$$\begin{aligned} g(-\omega^2 q, -\omega q, q^4) &= (1 - \omega)c_{0,0}(q^4) \left\{ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{6m^2-3m+6n^2-n} \right. \\ &\quad \left. - \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{6m^2-3m+6n^2-5n+1} \right\}. \end{aligned} \quad (3.5)$$

Again, we evaluate $g(-\omega^2 q, -\omega q, q^4)$ as an infinite product:

$$\begin{aligned} g(-\omega^2 q, -\omega q, q^4) &= (q^4; q^4)_{\infty}^2(-\omega^2 q; q^4)_{\infty}(-\omega q^3; q^4)_{\infty}(-\omega q; q^4)_{\infty}(-\omega^2 q^3; q^4)_{\infty} \\ &\quad (q^2; q^4)_{\infty}(q^2; q^4)_{\infty}(\omega; q^4)_{\infty}(\omega^2 q^4; q^4)_{\infty} \\ &= \frac{(1 - \omega)(q^{12}; q^{12})_{\infty}(q^4; q^4)_{\infty}(q^2; q^4)_{\infty}^2(-q^3; q^{12})_{\infty}(-q^9; q^{12})_{\infty}}{(-q; q^4)_{\infty}(-q^3; q^4)_{\infty}} \\ &= \frac{(1 - \omega)(q^{12}; q^{12})_{\infty}(q^2; q^2)_{\infty}(q^2; q^4)_{\infty}(-q^3; q^6)_{\infty}}{(-q; q^2)_{\infty}} \\ &= \frac{(1 - \omega)(q^6; q^6)_{\infty}(-q^6; q^6)_{\infty}(q; q)_{\infty}(-q; q)_{\infty}(q^2; q^4)_{\infty}(-q^3; q^6)_{\infty}}{(-q; q^2)_{\infty}} \\ &= (1 - \omega)(q^6; q^6)_{\infty}(-q^3; q^3)_{\infty}(q; q)_{\infty}. \end{aligned} \quad (3.6)$$

Substituting (3.6) in (3.5), we obtain

$$\begin{aligned} (q; q)_{\infty}(-q^3; q^3)_{\infty}(q^6; q^6)_{\infty} &= c_{0,0}(q^4) \left\{ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{6m^2-3m+6n^2-n} \right. \\ &\quad \left. - \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{6m^2-3m+6n^2-5n+1} \right\}. \end{aligned} \quad (3.7)$$

Comparing (3.4) with (3.7), we see that $c_{0,0}(q) = c_{0,0}(q^4)$. It follows that

$$c_{0,0}(q) = c_{0,0}(q^4) = c_{0,0}(q^{16}) = \dots = c_{0,0}(q^{4^k}) = \dots = c_{0,0}(0) = 1.$$

Hence we have proved Winquist's identity.

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