Cyclic permutations of sequences and uniform partitions

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Abstract

Let $\vec{r}=(r_i)_{i=1}^n$ be a sequence of real numbers of length n with sum s. Let $s_0=0$ and $s_i=r_1+\ldots+r_i$ for every $i\in\{1,2,\ldots,n\}$. Fluctuation theory is the name given to that part of probability theory which deals with the fluctuations of the partial sums s_i. Define $p(\vec{r})$ to be the number of positive sum s_i among $s_1,\ldots,s_n$ and $m(\vec{r})$ to be the smallest index i with $s_i=\max\limits_{0\leqslant k\leqslant n} s_k$. An important problem in fluctuation theory is that of showing that in a random path the number of steps on the positive half-line has the same distribution as the index where the maximum is attained for the first time. In this paper, let $\vec{r}_i=(r_i,\ldots,r_n,r_1,\ldots,r_{i-1})$ be the i-th cyclic permutation of $\vec{r}$. For $s>0$, we give the necessary and sufficient conditions for $\{m(\vec{r}_i)\mid 1\leqslant i\leqslant n\}=\{1,2,\ldots,n\}$ and $\{p(\vec{r}_i)\mid 1\leqslant i\leqslant n\}=\{1,2,\ldots,n\}$; for $s\leqslant 0$, we give the necessary and sufficient conditions for $\{m(\vec{r}_i)\mid 1\leqslant i\leqslant n\}=\{0,1,\ldots,n-1\}$ and $\{p(\vec{r}_i)\mid 1\leqslant i\leqslant n\}=\{0,1,\ldots,n-1\}$. We also give an analogous result for the class of all permutations of $\vec{r}$.

Keywords: Cyclic permutation; Fluctuation theory; Uniform partition

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1 Introduction

Fluctuation theory is the name given to that part of probability theory which deals with the fluctuations of the partial sums $s_i = x_1 + \ldots + x_i$ of a sequence of random variables $x_1, \ldots, x_n$. An important problem in fluctuation theory is that of showing that in a random path the number of steps on the positive half-line has the same distribution as the index where the maximum is attained for the first time. In particular, fix a sequence of real numbers $\vec{r} = (r_i)_{i=1}^n = (r_1, \ldots, r_n)$. Let

$$s_0 = 0, s_1 = r_1, s_2 = r_1 + r_2, \ldots, s_n = r_1 + r_2 + \ldots + r_n.$$

Define $p(\vec{r})$ to be the number of positive sums s_i among $s_1, \ldots, s_n$, i.e., $p(\vec{r}) = |\{i \mid s_i > 0\}|$, and $m(\vec{r})$ to be the smallest index i with $s_i = \max_{0 \leqslant k \leqslant n} s_k$. Let $[n]$ and $[n]-1$ denote the sets $\{1, 2, \ldots, n\}$ and $\{0, 1, \ldots, n-1\}$, respectively. Let $\mathfrak{S}_n$ be the set of all the permutations on the set $[n]$. We write permutations of $\mathfrak{S}_n$ in the form $\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))$. Let $\vec{r}_\sigma = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$ for any $\sigma \in \mathfrak{S}_n$. For any $i \in [n+1]-1$, Let $N(\vec{r}; i)$ (resp. $\Pi(\vec{r}; i)$) be the number of permutations σ in $\mathfrak{S}_n$ such that $p(\vec{r}_\sigma) = i$ (resp. $m(\vec{r}_\sigma) = i$). A basic theorem in fluctuation theory states that $N(\vec{r}; i) = \Pi(\vec{r}; i)$ for any $i \in [n+1]-1$. This result first was proved by Andersen [2]. Feller [10] called this result the Equivalence Principle and gave a simpler proof. This result is mentioned by Spitzer [23]. Baxter [3] obtained this result by bijection method. In [4], Brandt generalized the Equivalence Principle. Hobby and Pyke in [12] and Altschul in [1] gave bijection proofs for the generalization of Brandt.

Given an index $i \in [n]$, let $\vec{r_i} = (r_i, \ldots, r_n, r_1, \ldots, r_{i-1})$. We call $\vec{r_i}$ the i-th cyclic permutation of $\vec{r}$. Let

$$\mathcal{P}(\vec{r}) = \{p(\vec{r_i}) \mid i \in [n]\} \quad \text{and} \quad \mathcal{M}(\vec{r}) = \{m(\vec{r_i}) \mid i \in [n]\}.$$

Spitzer [23] showed implicitly the following specialization of the Equivalence Principle to the case of cyclic permutations.

Lemma 1.1 (Spitzer combinatorial lemma, [23]) *Let $\vec{r}$ be a sequence of real numbers of length n with sum 0 and the partial sums $s_1, \ldots, s_n$ are all distinct. Then $\mathcal{P}(\vec{r}) = \mathcal{M}(\vec{r}) = [n]-1$.*

A set is uniformly partitioned if all partition classes have the same cardinality. Many uniform partitions of combinatorial structures are consequences of Lemma 1.1. A famous example is the Chung-Feller theorem. Let $\mathscr{D}$ be the set of sequences of integers $\vec{r} = (r_i)_{i=1}^{2n}$ such that $s_{2n} = 0$ and $r_i \in \{1, -1\}$ for all $i \in [2n]$. Clearly, $|\mathscr{D}| = \binom{2n}{n}$. The Chung-Feller theorem shows that $n+1$ divides $\binom{2n}{n}$ by uniformly partitioning the set $\mathscr{D}$ into $n+1$ classes.

The Chung-Feller theorem was proved by many different methods. Chung and Feller [7] obtained this result by analytic methods. Narayana [19] showed this theorem by combinatorial methods. Narayana's book [20] introduced a refinement of this theorem.

Mohanty's book [18] devotes an entire section to exploring this theorem. Callan in [5] and Jewett and Ross in [14] gave bijection proofs of this theorem. Callan [6] reviewed and compared combinatorial interpretations of three different expressions for the Catalan number by cycle method.

One also attempted to generalize the Chung-Feller theorem for finding uniformly partitions of other combinatorial structures. Huq [13] developed generalized versions of this theorem for lattice paths. Eu, Liu and Yeh [9] proved this Theorem by using the Taylor expansions of generating functions and gave a refinement of this theorem. In [8], Eu, Fu and Yeh gave a strengthening of this Theorem and a weighted version for Schröder paths.

Suppose $f(x)$ is a generating function for some combinatorial sequences. Let $F(x, y) = \frac{yf(xy)-f(x)}{y-1}$. Liu, Wang and Yeh [15] call $F(x, y)$ the function of Chung-Feller type for $f(x)$. If we can give a combinatorial interpretation for the function $F(x, y)$, then we may uniformly partition the set formed by this combinatorial structure. Ma and Yeh [16] attempted to find combinatorial interpretation of the function of Chung-Feller type for a generating function of three classes of different lattice paths.

Particularly, Narayana [19] showed the following property for cyclic permutations.

Lemma 1.2 (Narayana [19]) *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of integers with sum 1. Then $\mathcal{P}(\vec{r}) = [n]$.*

In [19], Narayana gave a combinatorial proof of the Chung-Feller theorem by Lemma 1.2 and uniformly partition the set $\mathscr{D}$. Lemma 1.2 is derivable as a special case from the Spitzer combinatorial lemma. In [17], Ma and Yeh gave a generalizations of Lemma 1.2 by considering λ-cyclic permutations of a sequence of vectors and uniformly partition sets of many new combinatorial structures.

Based on the rightmost lowest point of a lattice path, Woan [24] presented another new uniform partition of the set $\mathscr{D}$. Let $\mathscr{B}$ be the set of sequences of integers $\vec{r} = (r_i)_{i=1}^{n+1}$ such that $s_{n+1} = 1$ and $r_i \in \{1, 0, -1\}$ for all $i \in [n + 1]$. In [9], Eu, Liu and Yeh proved that there is an uniform partition for the set $\mathscr{B}$, which was found by Shapiro [22]. In [17], Ma and Yeh also proved another interesting property of cyclic permutations as follows.

Lemma 1.3 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of integers with sum 1. Then $\mathcal{M}(\vec{r}) = [n]$.*

Raney [21] discovered a fact: If $\vec{r} = (r_i)_{i=1}^n$ is any sequence of integers whose sum is 1, then exactly one of the cyclic permutations has all of its partial sums positive. Graham and Knuth's book [11] introduced a simple geometric argument of the results obtained by Raney. This geometric argument yields $\mathcal{P}(\vec{r}) = \mathcal{M}(\vec{r}) = [n]$ for integer sequences $\vec{r}$ with sum 1.

Fix a sequence of real numbers $\vec{r} = (r_i)_{i=1}^n$ with sum s. For $s = 0$, Lemma 1.1 give a characterization for $\mathcal{P}(\vec{r}) = [n] - 1$; we note that the conditions in Lemma 1.1 are not necessary for $\mathcal{M}(\vec{r}) = [n] - 1$. For example, let $\vec{r} = (0, 1, -1)$. We have $\mathcal{M}(\vec{r}) = \{0, 1, 2\}$ and $\mathcal{P}(\vec{r}) = \{0, 1\}$. For $s = 1$, Lemmas 1.2 and 1.3 give some sufficient conditions for $\mathcal{P}(\vec{r}) = [n]$ and $\mathcal{M}(\vec{r}) = [n]$ respectively. Note that $\mathcal{M}(\vec{r}) \subseteq [n]$ and $\mathcal{P}(\vec{r}) \subseteq [n]$ if $s > 0$; $\mathcal{M}(\vec{r}) \subseteq [n] - 1$ and $\mathcal{P}(\vec{r}) \subseteq [n] - 1$ if $s \leqslant 0$. Two natural problems arise:

(1) What are necessary and sufficient conditions for $\mathcal{M}(\vec{r}) = [n]$ and $\mathcal{P}(\vec{r}) = [n]$ if $s > 0$?

(2) What are necessary and sufficient conditions for $\mathcal{M}(\vec{r}) = [n] - 1$ and $\mathcal{P}(\vec{r}) = [n] - 1$ if $s \leqslant 0$?

The aim of this paper is to solve these two problems. Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum s and partial sums $(s_i)_{i=0}^n$. We state the main results of this paper as follows.

- Let $s > 0$. Then
  (1) $\mathcal{M}(\vec{r}) = [n]$ if and only if $s_j - s_i \geqslant s$ for all $1 \leqslant i \leqslant j - 1$ with $j = m(\vec{r})$.
  (2) $\mathcal{P}(\vec{r}) = [n]$ if and only if $s_j - s_i \notin (0, s)$ for any $1 \leqslant i < j \leqslant n$, where the notation $(0, s)$ denote the set of all real numbers x satisfying $0 < x < s$.

- Let $s \leqslant 0$. Then
  (1) $\mathcal{M}(\vec{r}) = [n] - 1$ if and only if $s_i - s_j < s$ for all $j + 1 \leqslant i \leqslant n - 1$ with $j = m(\vec{r})$.
  (2) $\mathcal{P}(\vec{r}) = [n] - 1$ if and only if $s_j - s_i \notin [s, 0]$ for all $1 \leqslant i < j \leqslant n$, where the notation $[s, 0]$ denote the set of all real numbers x satisfying $s \leqslant x \leqslant 0$.

The properties of cyclic permutations of the sequence $\vec{r}$ in the main results will be proved in Section 2. Lemmas 1.1, 1.2 and 1.3 are corollaries of the main results.

Recall that $N(\vec{r}; i)$ (resp. $\Pi(\vec{r}; i)$) denotes the number of permutations σ in $\mathfrak{S}_n$ such that $p(\vec{r}_\sigma) = i$ (resp. $m(\vec{r}_\sigma) = i$). Using the main results, we derive the necessary and sufficient conditions of $N(\vec{r}; i) = \Pi(\vec{r}; i) = (n-1)!$ for all $i \in [n]$ (resp. $i \in [n] - 1$) when $s > 0$ (resp. $s \leqslant 0$).

We also consider more general cases. Fix a real number θ. Define $p(\vec{r}; \theta)$ to be the number of sum s_i among $s_1, \ldots, s_n$ such that $s_i > \theta \cdot i$. Let $\mathcal{P}(\vec{r}; \theta) = \{p(\vec{r}_i; \theta) \mid i \in [n]\}$. Define $m(\vec{r}; \theta)$ to be the smallest index i with $s_i - \theta \cdot i = \max\limits_{0 \leqslant k \leqslant n} (s_k - \theta \cdot k)$. Let $\mathcal{M}(\vec{r}; \theta) = \{m(\vec{r}_i; \theta) \mid i \in [n]\}$. Suppose $s > n\theta$. We give the necessary and sufficient conditions for $\mathcal{M}(\vec{r}; \theta) = [n]$ and $\mathcal{P}(\vec{r}; \theta) = [n]$. Suppose $s \leqslant n\theta$. We give the necessary and sufficient conditions for $\mathcal{M}(\vec{r}; \theta) = [n] - 1$ and $\mathcal{P}(\vec{r}; \theta) = [n] - 1$.

We organize this paper as follows. In Section 2, we study properties of cyclic permutations of $\vec{r}$. In Section 3, we consider more general cases.

2 Properties of cyclic permutations of a sequence

In this section, we study properties of cyclic permutations of a sequence $\vec{r}$ with sum s. For $s > 0$, we give the necessary and sufficient conditions for $\mathcal{M}(\vec{r}) = [n]$ and $\mathcal{P}(\vec{r}) = [n]$. For $s \leqslant 0$, we give the necessary and sufficient conditions for $\mathcal{M}(\vec{r}) = [n] - 1$ and $\mathcal{P}(\vec{r}) = [n] - 1$.

Lemma 2.1 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s > 0$. Let $j = m(\vec{r})$. For any $i = j+1, \ldots, n$, let $\vec{r}_i$ be the i-th cyclic permutation of $\vec{r}$. Then $m(\vec{r}_i) = n + j + 1 - i$.*

Proof. It is easy to see $r_i + \ldots + r_n + r_1 + \ldots + r_k < r_i + \ldots + r_n + r_1 + \ldots r_j$ for any $k \in [j] - 1$ and $r_i + \ldots + r_n + r_1 + \ldots r_k \leqslant r_i + \ldots + r_n + r_1 + \ldots r_j$ for any $k \in \{j, j+1, \ldots, i-1\}$. Assume that there is an index $k \in \{i, i+1, \ldots, n-1\}$ such that $r_i + \ldots + r_k \geqslant r_i + \ldots + r_n + r_1 + \ldots + r_j$. Thus $r_{k+1} + \ldots + r_n + r_1 + \ldots + r_j \leqslant 0$. $j = m(\vec{r})$ implies $r_1 + \ldots + r_j \geqslant r_1 + \ldots + r_k$. So $0 \geqslant (r_{k+1} + \ldots + r_n) + r_1 + \ldots + r_j \geqslant r_1 + \ldots + r_k + (r_{k+1} + \ldots + r_n) = s > 0$, a contradiction. We have $r_i + \ldots + r_k < r_i + \ldots + r_n + r_1 + \ldots + r_j$ for any $k \in \{i, i+1, \ldots, n-1\}$. Hence $m(\vec{r_i}) = n + j + 1 - i$. ■

Theorem 2.2 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s > 0$ and partial sums $(s_i)_{i=0}^n$. Let $j = m(\vec{r})$. Then $\mathcal{M}(\vec{r}) = [n]$ if and only if $s_j - s_i \geqslant s$ for all $1 \leqslant i \leqslant j-1$.*

Proof. For any $i \in [n]$, let $\vec{r_i}$ be the i-th cyclic permutation of $\vec{r}$. It is easy to see $m(\vec{r_i}) \neq 0$ since $s > 0$. Lemma 2.1 tells us $m(\vec{r_i}) = n + j + 1 - i$ for any $i \in \{j+1, \ldots, n\}$.

Suppose $s_j - s_i \geqslant s$ for all $1 \leqslant i \leqslant j-1$. Consider the sequence $\vec{r_i} = (r_i, \ldots, r_n, r_1, \ldots, r_{i-1})$ with $i \in [j]$. It is easy to see $r_i + \ldots + r_k < r_i + \ldots + r_j$ for any $k \in \{i, i+1, \ldots, j-1\}$ and $r_i + \ldots + r_k \leqslant r_i + \ldots + r_j$ for any $k \in \{j, j+1, \ldots, n\}$. Assume that there is an index $k \in [i-1]$ such that $r_i + \ldots + r_j < r_i + \ldots + r_n + r_1 + \ldots + r_k$. Thus $s_j - s_k = r_{k+1} + \ldots + r_j < s$, a contradiction. Hence $m(\vec{r_i}) = j + 1 - i$.

Conversely, suppose $\mathcal{M}(\vec{r}) = [n]$. Let $A = \{i \mid s_j - s_i < s,\ 1 \leqslant i \leqslant j-1\}$. Assume $A \neq \emptyset$ and let $i = \min A$. Clearly $i + 1 \leqslant j$. We consider the sequence $\vec{r}_{i+1} = (r_{i+1}, \ldots, r_n, r_1, \ldots, r_i)$. Since $i \in A$, we have $r_{i+1} + \ldots + r_j < s = r_{i+1} + \ldots + r_n + r_1 + \ldots + r_i$. It is easy to see $r_{i+1} + \ldots + r_k < r_{i+1} + \ldots + r_j$ for any $k \in \{i+1, i+2, \ldots, j-1\}$ and $r_{i+1} + \ldots + r_k \leqslant r_{i+1} + \ldots + r_j$ for any $k \in \{j, j+1, \ldots, n\}$. For every $k \in [i-1]$, we have $s_j - s_k = r_{k+1} + \ldots + r_j \geqslant s$ since $k \notin A$. So $r_{i+1} + \ldots + r_j \geqslant r_{i+1} + \ldots + r_n + r_1 + \ldots r_k$. Hence $m(\vec{r}_{i+1}) = n = m(\vec{r}_{j+1})$. So $\mathcal{M}(\vec{r}) \neq [n]$, a contradiction. ■

Lemma 2.3 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s \leqslant 0$. Let $j = m(\vec{r})$. Suppose $j \geqslant 1$. For any $i \in [j]$, let $\vec{r_i}$ be the i-th cyclic permutation of $\vec{r}$. Then $m(\vec{r_i}) = j + 1 - i$.*

Proof. It is easy to see $r_i + \ldots + r_k < r_i + \ldots + r_j$ for any $k \in \{i, i+1, \ldots, j-1\}$ and $r_i + \ldots + r_k \leqslant r_i + \ldots + r_j$ for any $k \in \{j, j+1, \ldots, n\}$. For any $k \in [i-1]$, we have $r_{k+1} + \ldots + r_j > 0 \geqslant s$ since $j = m(\vec{r})$. This implies $0 > r_{j+1} + \ldots + r_n + r_1 + \ldots r_k$ and $r_i + \ldots + r_j > r_i + \ldots + r_n + r_1 + \ldots r_k$. Note that $r_i + \ldots + r_j > 0$ since $j = m(\vec{r})$. Hence $m(\vec{r_i}) = j + 1 - i$. ■

Theorem 2.4 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s \leqslant 0$ and partial sums $(s_i)_{i=0}^n$. Suppose $m(\vec{r}) = j$. Then $\mathcal{M}(\vec{r}) = [n] - 1$ if and only if $s_i - s_j < s$ for all $j + 1 \leqslant i \leqslant n-1$.*

Proof. For any $i \in [n]$, let $\vec{r_i}$ be the i-th cyclic permutation of $\vec{r}$. It is easy to see $m(\vec{r_i}) \neq n$ since $s \leqslant 0$.

Suppose $s_i - s_j < s$ for all $j + 1 \leqslant i \leqslant n-1$. Given an index $i \in \{j+1, j+2, \ldots, n\}$, we consider the sequence $\vec{r_i} = (r_i, \ldots, r_n, r_1, \ldots, r_{i-1})$. It is easy to see $r_i + \ldots + r_n + r_1 + \ldots + r_k < r_i + \ldots + r_n + r_1 + \ldots + r_j$ for any $k \in [j] - 1$ and $r_i + \ldots + r_n + r_1 + \ldots + r_k \leqslant r_i + \ldots + r_n + r_1 + \ldots + r_j$ for any $k \in \{j, j+1, \ldots, i-1\}$. For any $k \in \{i, i+1, \ldots, n-1\}$, since $s_k - s_j = r_{j+1} + \ldots + r_k < s$, we have $r_{k+1} + \ldots + r_n + r_1 + \ldots + r_j > 0$ and $r_i + \ldots + r_k < r_i + \ldots + r_n + r_1 + \ldots + r_j$.

For $i \geqslant j+2$, note that $r_i + \ldots + r_n + r_1 + \ldots + r_j > 0$ since $j = m(\vec{r})$. Clearly, $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_j = s$. Hence $m(\vec{r}_i) = n + j + 1 - i$ for $i = j+2, \ldots, n$ and $m(\vec{r}_{j+1}) = 0$. When $j \geqslant 1$, Lemma 2.3 tells us $m(\vec{r}_i) = j + 1 - i$ for any $i \in [j]$. Thus we have $\mathcal{M}(\vec{r}) = [n] - 1$.

Conversely, suppose $\mathcal{M}(\vec{r}) = [n] - 1$. Let $A = \{i \mid s_i - s_j \geqslant s,\ j+1 \leqslant i \leqslant n\}$. Note that $n \notin A$ if $j \geqslant 1$; otherwise $n \in A$. So, assume $A \setminus \{n\} \neq \emptyset$ and let $i = \max A \setminus \{n\}$. Clearly $j+1 \leqslant i \leqslant n-1$. We consider the sequence $\vec{r}_{i+1} = (r_{i+1}, \ldots, r_n, r_1, \ldots, r_i)$. It is easy to see $r_{i+1} + \ldots + r_n + r_1 + \ldots + r_k < r_{i+1} + \ldots + r_n + r_1 + \ldots + r_j$ for any $k \in [j] - 1$ and $r_{i+1} + \ldots + r_n + r_1 + \ldots + r_k \leqslant r_{i+1} + \ldots + r_n + r_1 + \ldots + r_j$ for any $k \in \{j, j+1, \ldots, i\}$. For any $k \in \{i+1, i+2, \ldots, n-1\}$, we have $s_k - s_j = r_{j+1} + \ldots + r_k < s$ since $k \notin A$ and $r_{i+1} + \ldots + r_k < r_{i+1} + \ldots + r_n + r_1 + \ldots + r_j$. Since $i \in A$, we have $r_{i+1} + \ldots + r_n + r_1 + \ldots + r_j \leqslant 0$. Hence $m(\vec{r}_{i+1}) = 0 = m(\vec{r}_{j+1})$ and $\mathcal{M}(\vec{r}) \neq [n] - 1$, a contradiction. ∎

For any sequence of real numbers $\vec{r} = (r_i)_{i=1}^n$ with partial sums $(s_i)_{i=1}^n$, we define a linear order $\prec_{\vec{r}}$ on the set $[n]$ by the following rules:

for any $i, j \in [n]$, $i \prec_{\vec{r}} j$ if either (1) $s_i < s_j$ or (2) $s_i = s_j$ and $i > j$.

The sequence formed by writing elements in the set $[n]$ in the increasing order with respect to $\prec_{\vec{r}}$ is denoted by $\pi(\vec{r}) = (\pi_1, \pi_2, \ldots, \pi_n)$. Note that $\pi(\vec{r})$ also can be viewed as a bijection from the set $[n]$ to itself.

Lemma 2.5 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s > 0$. Let $\pi(\vec{r})$ be the linear order on the set $[n]$ with respect to $\prec_{\vec{r}}$. Given an index $j \in [n]$, let $\vec{r}_{j+1} = (r_{j+1}, \ldots, r_n, r_1, \ldots, r_j)$. Then*

(1) for any $j \prec_{\vec{r}} i$ we have $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_i > 0$ if $i < j$; $r_{j+1} + \ldots + r_i > 0$ if $i > j$.

(2) Suppose $\pi(k) = j$ for some $k \in [n]$. We have $p(\vec{r}_{j+1}) \geqslant n - k + 1$.

Proof. (1) $j \prec_{\vec{r}} i$ implies either (I) $s_j < s_i$ or (II) $s_j = s_i$ and $j > i$. Hence, we consider two cases as follows.

Case I. $s_j < s_i$. For $i > j$, it is easy to see $r_{j+1} + \ldots + r_i > 0$. For $i < j$, we have $r_{i+1} + \ldots + r_j < 0$. Hence $r_{j+1} + \ldots + r_n + r_1 + \ldots r_i = s - r_{i+1} - \ldots - r_j > s > 0$.

Case II. $s_j = s_i$ and $j > i$. We have $r_{i+1} + \ldots + r_j = 0$ and $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_i = s > 0$.

(2) Note that $r_{j+1} + \ldots + r_n + r_1 \ldots + r_j = s > 0$. Hence $p(\vec{r}_{j+1}) \geqslant n - k + 1$. ∎

Lemma 2.6 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s > 0$ and partial sums $(s_i)_{i=1}^n$. Let $\pi(\vec{r})$ be the linear order on the set $[n]$ with respect to $\prec_{\vec{r}}$. Let $j \in [n]$ and $\vec{r}_{j+1}$ be the $(j+1)$-th cyclic permutation of $\vec{r}$. Suppose $s_j - s_i \notin (0, s)$ for all $1 \leqslant i \leqslant j-1$ and $\pi(k) = j$ for some $k \in [n]$. Then $p(\vec{r}_{j+1}) = n - k + 1$.*

Proof. For any $i \prec_{\vec{r}} j$, we discuss the following two case.

Case 1. $s_i < s_j$. For $i > j$, it is easy to see $r_{j+1} + \ldots + r_i < 0$. For $i < j$, we have $s_j - s_i = r_{i+1} + \ldots + r_j \geqslant s$ since $s_j - s_i > 0$ and $s_j - s_i \notin (0, s)$. Hence $r_{j+1} + \ldots + r_n + r_1 + \ldots r_i = s - r_{i+1} - \ldots - r_j \leqslant 0$.

Case 2. $s_i = s_j$ and $i > j$. Clearly, we have $r_{j+1} + \ldots + r_i = 0$.

By Lemma 2.5, we have $p(\vec{r}_{j+1}) = n + 1 - k$ since $\pi(k) = j$. ∎

Theorem 2.7 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s > 0$ and partial sums $(s_i)_{i=1}^n$. Then $\mathcal{P}(\vec{r}) = [n]$ if and only if $s_j - s_i \notin (0, s)$ for any $1 \leqslant i < j \leqslant n$.*

Proof. Let $\pi(\vec{r})$ be the linear order on the set $[n]$ with respect to $\prec_{\vec{r}}$. Suppose $s_j - s_i \notin (0, s)$ for any $1 \leqslant i < j \leqslant n$. Lemma 2.6 implies $p(\vec{r}_{\pi(k)+1}) = n + 1 - k$ for all $k \in [n]$. Hence $\mathcal{P}(\vec{r}) = [n]$.

Conversely, suppose $\mathcal{P}(\vec{r}) = [n]$. Lemma 2.5 tells us $p(\vec{r}_{\pi(k)+1}) \geqslant n - k + 1$ for all $k \in [n]$. Let $A_k = \{i \mid 0 < s_{\pi(k)} - s_i < s,\ 1 \leqslant i < \pi(k)\}$ for any $k \in [n]$. Assume that $A_k \neq \emptyset$ for some $k \in [n]$. Let $\bar{k} = \min\{k \mid A_k \neq \emptyset\}$. By Lemma 2.6, we have $p(\vec{r}_{\pi(k)+1}) = n - k + 1$ for any $k < \bar{k}$. Suppose $\pi(\bar{k}) = j$. We consider the sequence $\vec{r}_{j+1} = (r_{j+1}, \ldots, r_n, r_1, \ldots, r_j)$. Let $i \in A_{\bar{k}}$. Since $s_j - s_i > 0$, we have $s_j > s_i$. Thus $i \prec_{\vec{r}} j$ and $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_i = s - r_{i+1} - \ldots - r_j > 0$ since $s_j - s_i < s$. By Lemma 2.5, we get $p(\vec{r}_{\pi(\bar{k})+1}) \geqslant n - \bar{k} + 2$. Hence $n - \bar{k} + 1 \notin \mathcal{P}(\vec{r})$, a contradiction. ∎

Lemma 2.8 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s \leqslant 0$ and partial sums $(s_i)_{i=1}^n$. Let $\pi(\vec{r})$ be the linear order on the set $[n]$ with respect to $\prec_{\vec{r}}$. Given an index $j \in [n]$, let $\vec{r}_{j+1} = (r_{j+1}, \ldots, r_n, r_1, \ldots, r_j)$. Then*

(1) for any $i \prec_{\vec{r}} j$, we have $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_i \leqslant 0$ if $i < j$; $r_{j+1} + \ldots + r_i \leqslant 0$ if $i > j$.

(2) Suppose $\pi(k) = j$ for some $k \in [n]$. We have $p(\vec{r}_{j+1}) \leqslant n - k$.

Proof. (1) $i \prec_{\vec{r}} j$ implies either (I) $s_i < s_j$ or (II) $s_i = s_j$ and $i > j$. Hence, we consider two cases as follows.

Case I. $s_i < s_j$. For $i > j$, it is easy to see $r_{j+1} + \ldots + r_i < 0$. For $i < j$, we have $r_{i+1} + \ldots + r_j > 0$. Hence $r_{j+1} + \ldots + r_n + r_1 + \ldots r_i = s - r_{i+1} - \ldots - r_j < 0$.

Case II. $s_i = s_j$ and $i > j$. We have $r_{j+1} + \ldots + r_i = 0$.

(2) Note that $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_j = s \leqslant 0$. Hence $p(\vec{r}_{j+1}) \leqslant n - k$. ∎

Lemma 2.9 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s \leqslant 0$ and partial sums $(s_i)_{i=1}^n$. Let $\pi(\vec{r})$ be the linear order on the set $[n]$ with respect to $\prec_{\vec{r}}$. Let $j \in [n]$ and $\vec{r}_{j+1}$ be the $(j+1)$-th cyclic permutation of $\vec{r}$. Suppose $s_j - s_i \notin [s, 0]$ for all $1 \leqslant i \leqslant j - 1$ and $\pi(k) = j$ for some $k \in [n]$. Then $p(\vec{r}_{j+1}) = n - k$.*

Proof. Clearly, $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_j = s \leqslant 0$. For any $j \prec_{\vec{r}} i$, we claim $s_i > s_j$. Otherwise $s_i = s_j$, then $i < j$ and $s_j - s_i = 0$, a contradiction.

For $i > j$, it is easy to see $r_{j+1} + \ldots + r_i > 0$. For $i < j$, we have $s_j - s_i < s$ since $s_j - s_i < 0$ and $s_j - s_i \notin [s, 0]$. So $r_{j+1} + \ldots + r_n + r_1 + \ldots r_i = s - r_{i+1} - \ldots - r_j > 0$. By Lemma 2.5, we have $p(\vec{r}_{j+1}) = n - k$. ∎

Theorem 2.10 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum $s \leqslant 0$ and partial sums $(s_i)_{i=1}^n$. Then $\mathcal{P}(\vec{r}) = [n] - 1$ if and only if $s_j - s_i \notin [s, 0]$ for all $1 \leqslant i < j \leqslant n$.*

Proof. Let $\pi(\vec{r})$ be the linear order on the set $[n]$ with respect to $\prec_{\vec{r}}$. Suppose $s_j - s_i \notin [s, 0]$ for all $1 \leqslant i < j \leqslant n$. Lemma 2.9 implies $p(\vec{r}_{\pi(k)+1}) = n - k$ for all $k \in [n]$. Hence $\mathcal{P}(\vec{r}) = [n] - 1$.

Conversely, suppose $\mathcal{P}(\vec{r}) = [n]$. Lemma 2.8 tells us $p(\vec{r}_{\pi(k)+1}) \leqslant n - k$ for all $k \in [n]$. Let $A_k = \{i \mid s \leqslant s_{\pi(k)} - s_i \leqslant 0,\ 1 \leqslant i \leqslant \pi(k) - 1\}$ for any $k \in [n]$. Assume that $A_k \neq \emptyset$ for some $k \in [n]$. Let $\bar{k} = \max\{k \mid A_k \neq \emptyset\}$. By Lemma 2.9, we have $p(\vec{r}_{\pi(k)+1}) = n - k$ for any $k > \bar{k}$. Suppose $\pi(\bar{k}) = j$. We consider the sequence $\vec{r}_{j+1} = (r_{j+1}, \ldots, r_n, r_1, \ldots, r_j)$. Let $i \in A_{\bar{k}}$. Since $s_j - s_i \leqslant 0$, we have $s_j \leqslant s_i$. Thus $j \prec_{\vec{r}} i$ and $r_{j+1} + \ldots + r_n + r_1 + \ldots + r_i = s - r_{i+1} - \ldots - r_j \leqslant 0$ since $s_j - s_i \geqslant s$. By Lemma 2.8, we get $p(\vec{r}_{j+1}) \leqslant n - \bar{k} - 1$. Hence $n - \bar{k} \notin \mathcal{P}(\vec{r})$, a contradiction. ∎

Now, we consider integer sequences. Taking $s = 1$ in Theorems 2.2 and 2.7, we immediately obtain the following results.

Corollary 2.11 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of integers with sum 1. Then $\mathcal{M}(\vec{r}) = \mathcal{P}(\vec{r}) = [n]$.*

Taking $s = 0$ in Theorems 2.4 and 2.10, we have the following corollary.

Corollary 2.12 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of integers with sum 0 and the partial sums are all distinct. Then $\mathcal{M}(\vec{r}) = \mathcal{P}(\vec{r}) = [n] - 1$.*

Given a sequence $\vec{r} = (r_1, \ldots, r_n)$, recall that $\vec{r}_\sigma = (r_{\sigma(1)}, \ldots, r_{\sigma(n)})$ for any $\sigma \in \mathfrak{S}_n$. For any $i \in [n+1] - 1$, $N(\vec{r}; i)$ (resp. $\Pi(\vec{r}; i)$) denotes the number of permutations σ in $\mathfrak{S}_n$ such that $p(\vec{r}_\sigma) = i$ (resp. $m(\vec{r}_\sigma) = i$).

Corollary 2.13 *Let $\vec{r} = (r_i)_{i=1}^n$ be a sequence of real numbers with sum s.*

(1) Suppose $s > 0$. Then $\Pi(\vec{r}; i) = N(\vec{r}; i) = (n-1)!$ for all $i \in [n]$, if and only if $\sum_{k \in I} r_k \notin (0, s)$ for all $\emptyset \neq I \subseteq [n]$.

(2) Suppose $s \leqslant 0$. Then $\Pi(\vec{r}; i) = N(\vec{r}; i) = (n-1)!$ for all $i \in [n] - 1$ if and only if $\sum_{k \in I} r_k \notin [s, 0]$ for all $\emptyset \neq I \subset [n]$.

Proof. (1) Let σ and τ be two permutations in $\mathfrak{S}_n$. We say σ and τ are cyclicly equivalent, denoted by $\sigma \sim \tau$, if there is an index $i \in [n]$ such that $\tau = (\sigma(i), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(i-1))$. Hence, given a permutation $\sigma \in \mathfrak{S}_n$, we define a set $EQ(\sigma)$ as $EQ(\sigma) = \{\tau \in \mathfrak{S}_n \mid \tau \sim \sigma\}$. We say the set $EQ(\sigma)$ is an equivalence class of the set $\mathfrak{S}_n$. Clearly $|EQ(\sigma)| = n$ for any $\sigma \in \mathfrak{S}_n$.

Suppose $\sum_{k \in I} r_k \notin (0, s)$ for all $\emptyset \neq I \subseteq [n]$. For any $1 \leqslant i \leqslant n$, by Theorems 2.2(resp. Theorem 2.7), every equivalence class contains exactly one permutation σ such that $m(\vec{r}_\sigma) = i$ (resp. $p(\vec{r}_\sigma) = i$). Hence, $\Pi(\vec{r}; i) = N(\vec{r}; i) = \frac{n!}{n} = (n-1)!$.

Fix a permutation $\sigma \in \mathfrak{S}_n$. Let $\bar{s}_0 = 0, \bar{s}_1 = r_{\sigma(1)}, \bar{s}_2 = r_{\sigma(1)} + r_{\sigma(2)}, \ldots, \bar{s}_n = r_{\sigma(1)} + r_{\sigma(2)} + \ldots + r_{\sigma(n)}$. Let j to be the largest index i with $\bar{s}_i = \min_{0 \leqslant k \leqslant n} \bar{s}_k$. Consider the permutation $\tau = (\sigma(j+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(j))$. Then $\tau \in EQ(\sigma)$ and $p(\vec{r}_\tau) = n$. Thus there is at least one element $\tau \in EQ(\sigma)$ such that $p(\vec{r}_\tau) = n$ and $N(\vec{r}; n) \geqslant (n-1)!$. Let j' to be the smallest index i with $\bar{s}_i = \max_{0 \leqslant k \leqslant n} \bar{s}_k$. Consider the permutation $\tau' = (\sigma(j'+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(j'))$. Then $\tau' \in EQ(\sigma)$ and $m(\vec{r}_{\tau'}) = n$. Thus there is at least one element $\tau' \in EQ(\sigma)$ such that $m(\vec{r}_{\tau'}) = n$ and $\Pi(\vec{r}; n) \geqslant (n-1)!$.

Suppose $\Pi(\vec{r}; i) = N(\vec{r}; i) = (n-1)!$ for any $i \in [n]$. Particularly, $\Pi(\vec{r}; n) = N(\vec{r}; n) = (n-1)!$. Assume that there exists a proper subset I of $[n]$ such that $0 < \sum_{k \in I} r_k < s$. Let $A = \{k \in I \mid r_k \leqslant 0\}$, $a = |A|$ and $j = |I|$. Suppose $I = \{i_1, \ldots, i_a, i_{a+1} \ldots, i_j\}$, where $i_k \in A$ for every $k \in [1, a]$. Let $J = [n] \backslash I$, $B = \{k \in J \mid r_k \leqslant 0\}$ and $b = |B|$. Suppose $J = \{i_{j+1}, \ldots, i_{j+b}, i_{j+b+1}, \ldots, i_n\}$, where $i_{j+k} \in B$ for every $k \in [1, b]$. Let σ be a permutation in $\mathfrak{S}_n$ such that $\sigma(k) = i_k$ for any $k \in [n]$. Note that $0 < \sum_{k=1}^{j} r_{\sigma(k)} = \sum_{k \in I} r_k < s$. Thus we have $m(\vec{r}_\sigma) = n$. Consider another permutation $\tau = (\sigma(j+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(j))$. It is easy to see $\sigma \sim \tau$ and $m(\vec{r}_\tau) = n$. Hence $\Pi(\vec{r}; n) > (n-1)!$, a contradiction. Let $\sigma' = (\sigma(n), \sigma(n-1), \ldots, \sigma(1))$ and $\tau' = (\tau(n), \tau(n-1), \ldots, \tau(1))$. Then $\sigma' \sim \tau'$ and $p(\vec{r}_{\sigma'}) = p(\vec{r}_{\tau'}) = n$. Hence $N(\vec{r}; n) > (n-1)!$, a contradiction.

(2) Suppose $\sum_{k \in I} r_k \notin [s, 0]$ for all $\emptyset \neq I \subset [n]$. Similar to the proof of Corollary 2.13 (1), we can obtain the results as desired.

Fix a permutation $\sigma \in \mathfrak{S}_n$. Let $\bar{s}_0 = 0, \bar{s}_1 = r_{\sigma(1)}, \bar{s}_2 = r_{\sigma(1)} + r_{\sigma(2)}, \ldots, \bar{s}_n = r_{\sigma(1)} + r_{\sigma(2)} + \ldots + r_{\sigma(n)}$. Let j to be the largest index i with $\bar{s}_i = \max_{0 \leqslant k \leqslant n} \bar{s}_k$. Consider the permutation $\tau = (\sigma(j+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(j))$. Clearly, $\tau \in EQ(\sigma)$ and $m(\vec{r}_\tau) = p(\vec{r}_\tau) = 0$. So there is at least one element $\tau \in EQ(\sigma)$ such that $m(\vec{r}_\tau) = p(\vec{r}_\tau) = 0$. Thus $N(\vec{r}; 0) \geqslant (n-1)!$ and $\Pi(\vec{r}; 0) \geqslant (n-1)!$.

Suppose $\Pi(\vec{r}; i) = N(\vec{r}; i) = (n-1)!$ for any $i \in [n] - 1$. Particularly, $\Pi(\vec{r}; 0) = N(\vec{r}; 0) = (n-1)!$. Assume that there exists a proper subset I of $[n]$ such that $s \leqslant \sum_{k \in I} r_k \leqslant 0$. Let $A = \{k \in I \mid r_k \leqslant 0\}$, $a = |A|$ and $j = |I|$. Suppose $I = \{i_1, \ldots, i_a, i_{a+1} \ldots, i_j\}$, where $i_k \in A$ for every $k \in [1, a]$. Let $J = [n] \setminus I$, $B = \{k \in J \mid r_k \leqslant 0\}$ and $b = |B|$. Suppose $J = \{i_{j+1}, \ldots, i_{j+b}, i_{j+b+1}, \ldots, i_n\}$, where $i_{j+k} \in B$ for every $k \in [1, b]$. Let σ be a permutation in $\mathfrak{S}_n$ such that $\sigma(k) = i_k$ for any $k \in [n]$. Note that $\sum_{k=1}^{j} r_{\sigma(k)} = \sum_{k \in I} r_k \leqslant 0$. Thus we have $m(\vec{r}_\sigma) = 0$. Consider another permutation $\tau = (\sigma(j+1), \ldots, \sigma(n), \sigma(1), \ldots, \sigma(j))$. Then $\sum_{k=1}^{n-j} r_{\tau(k)} = s - \sum_{k \in I} r_k \leqslant 0$ since $\sum_{k \in I} r_k \geqslant s$. So $m(\vec{r}_\tau) = 0$. Note that $\sigma \sim \tau$. Hence $\Pi(\vec{r}; 0) > (n-1)!$, a contradiction. It is easy to see $p(\vec{r}_\sigma) = p(\vec{r}_\tau) = 0$. Hence $N(\vec{r}; 0) > (n-1)!$, a contradiction. ∎

3 More general cases

In this section, we consider more general cases and study furthermore generalizations for properties of cyclic permutations of a sequence $\vec{r} = (r_i)_{i=1}^n$.

Theorem 3.1 *Let θ be a real number and $\vec{r} = (r_i)_{i=1}^n$ a sequence of real numbers with sum $s > n\theta$ and partial sums $(s_i)_{i=0}^n$. Then*

(1) $\mathcal{M}(\vec{r};\theta) = [n]$ if and only if $s_j - s_i \geqslant s - (n-j+i)\theta$ for all $1 \leqslant i \leqslant j-1$, where $j = m(\vec{r};\theta)$;

(2) $\mathcal{P}(\vec{r};\theta) = [n]$ if and only if $s_j - s_i \notin ((j-i)\theta, s-(n+i-j)\theta)$ for all $1 \leqslant i < j \leqslant n$, where the notation $((j-i)\theta, s-(n+i-j)\theta)$ denote the set of all real numbers x satisfying $(j-i)\theta < x < s-(n+i-j)\theta$.

Proof. (1) Consider the sequence $\vec{v} = (r_1-\theta, \ldots, r_n-\theta)$. It is easy to see that (I) $\sum_{i=1}^{n} \vec{v}_i = s - n\theta > 0$; (II) $j = m(\vec{r};\theta)$ if and only if $j = m(\vec{v})$; (III) $(s_j - j\theta) - (s_i - i\theta) \geqslant s - n\theta > 0$ for all $1 \leqslant i \leqslant j-1$. By Theorem 2.2, we obtain the results as desired.

(2) Similar to the proof of Theorem 3.1(1), we can obtain the results in Theorem 3.1(2). ∎

Similarly, considering $s \leqslant n\theta$, we can obtain the following results.

Theorem 3.2 *Let θ be a real number and $\vec{r} = (r_i)_{i=1}^n$ a sequence of real numbers with sum $s \leqslant n\theta$ and partial sums $(s_i)_{i=0}^n$. Then*

(1) $\mathcal{M}(\vec{r};\theta) = [n]-1$ if and only if $s_i - s_j < s - (n+j-i)\theta$ for all $j+1 \leqslant i \leqslant n-1$, where $j = m(\vec{r};\theta)$;

(2) $\mathcal{P}(\vec{r};\theta) = [n]-1$ if and only if $s_j - s_i \notin [s-(n+i-j)\theta, (j-i)\theta]$ for any $1 \leqslant i < j \leqslant n$, where the notation $[s-(n+i-j)\theta, (j-i)\theta]$ denote the set of all real numbers x satisfying $s-(n+i-j)\theta \leqslant x \leqslant (j-i)\theta$.

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