Cyclic partitions of complete uniform hypergraphs

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Abstract

By $K_n^{(k)}$ we denote the complete k-uniform hypergraph of order n, $1 \leqslant k \leqslant n-1$, i.e. the hypergraph with the set $V_n = \{1, 2, ..., n\}$ of vertices and the set $\binom{V_n}{k}$ of edges. If there exists a permutation σ of the set V_n such that $\{E, \sigma(E), ..., \sigma^{q-1}(E)\}$ is a partition of the set $\binom{V_n}{k}$ then we call it cyclic q-partition of $K_n^{(k)}$ and σ is said to be a (q, k)-complementing.
In the paper, for arbitrary integers k, q and n, we give a necessary and sufficient condition for a permutation to be (q, k)-complementing permutation of $K_n^{(k)}$.
By $\tilde{K}_n$ we denote the hypergraph with the set of vertices V_n and the set of edges $2^{V_n} - \{\emptyset, V_n\}$. If there is a permutation σ of V_n and a set $E \subset 2^{V_n} - \{\emptyset, V_n\}$ such that $\{E, \sigma(E), ..., \sigma^{p-1}(E)\}$ is a p-partition of $2^{V_n} - \{\emptyset, V_n\}$ then we call it a cyclic p-partition of K_n and we say that σ is p-complementing. We prove that $\tilde{K}_n$ has a cyclic p-partition if and only if p is prime and n is a power of p (and $n > p$). Moreover, any p-complementing permutation is cyclic.

1 Preliminaries and results

Throughout the paper we will write $V_n = \{1, \dots, n\}$. For a set X we denote by $\binom{X}{k}$ the set of all k-subsets of X. A hypergraph $H = (V; E)$ is said to be **k-uniform** if $E \subset \binom{V}{k}$ (the cardinality of any edge is equal to k). We shall always assume that the set of vertices V of a hypergraph of order n is equal to V_n. The complete k-uniform hypergraph of order n is denoted by $K_n^{(k)}$, hence $K_n^{(k)} = (V_n; \binom{V_n}{k})$. Let σ be a permutation of the set V_n, let q be a positive integer, and let $E \subset \binom{V_n}{k}$. If $\{E, \sigma(E), \sigma^2(E), \dots, \sigma^{q-1}(E)\}$ is a partition of $\binom{V_n}{k}$ we call it a **cyclic q-partition** and σ is said to be **(q, k)-complementing**. It is

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very easy to prove that then $\sigma^q(E) = E$. Write $E_i = \sigma^i(E)$ for $i = 0, ..., q-1$. It follows easily that $\sigma^t(E_i) = E_{i+t \pmod q}$, for every integer t.

If there is a cyclic 2-partition $\{E, \sigma(E)\}$ of $K_n^{(k)}$, we say that the hypergraph $H = (V_n; E)$ is **self-complementary** and every $(2,k)$-complementing permutation of $K_n^{(k)}$ is called **self-complementing**. In [16] we have given the characterization of self-complementing permutations which, as it turns out, is exactly Theorem 2 of this paper for $p = 2, \alpha = 1$. Self-complementary k-uniform hypergraphs generalize the self-complementary graphs defined in [13] and [14]. The vertex transitive self-complementary k-uniform hypergraphs are the subject of the paper [11] by Potǒcnik and Šajna. Gosselin gave an algorithm to construct some special self-complementary k-uniform hypergraphs in [3]. In [6] and [10] Knor, Potǒcnik and Šajna study the existence of regular self-complementary k-uniform hypergraphs.

The main result of this paper is a necessary and sufficient condition for a permutation σ of V_n to be (q,k)-complementing, where q is a positive integer (Theorem 3). In Theorem 5 we characterize integers n, k, α and primes p such that there exists a cyclic p^α-partition of $K_n^{(k)}$.

Section 2 contains the proofs of Theorems 1, 2 and 3 given below. Section 3 is devoted to cyclic partitions of complete hypergraph $\tilde{K}_n = (V_n; 2^{V_n} - \{\emptyset, V_n\})$ (we call $\tilde{K}_n$ the **general** complete hypergraph of order n, to stress the distinction between complete uniform and complete hypergraphs).

Theorem 1 *Let n and k be integers, $0 < k < n$, let p_1 and p_2 be two relatively prime integers. A permutation σ on the set V_n is (p_1p_2, k)-complementing if and only if σ is a (p_j, k)-complementing for $j = 1, 2$.*

For integers n and d, $d > 0$, by $r(n,d)$ we denote the reminder when n is divided by d. So we have $n \equiv r(n,d) \pmod d$.

For a positive integer k by $C_p(k)$ we denote the maximum integer c such that $k = p^c a$, where $a \in \mathbf{N}$ ($\mathbf{N}$ stands for the sets of naturals, i.e. nonnegative integers). In other words, if $k = \sum_{i \geqslant 0} k_i p^i$, where $0 \leqslant k_i < p$ for every $i \in \{0, 1, \ldots\}$ (k_i are digits with respect to basis p), then $C_p(k) = \min\{i : k_i \neq 0\}$. If A is a finite set, we write $C_p(A)$ instead of $C_p(|A|)$, for short.

Theorem 2 *Let n, p, k and α be positive integers, such that $k < n$ and p is prime. A permutation σ of the set V_n with orbits $O_1, \ldots, O_m$ is (p^α, k)-complementing if and only if there is a non negative integer l such that the following two conditions hold:*

(i) $r(n, p^{l+\alpha}) < r(k, p^{l+1})$, *and*

(ii) $\sum_{i : C_p(O_i) < l+\alpha} |O_i| = r(n, p^{l+\alpha})$.

A condition slightly different from the above has been given (and proved by different method, independently) in [4].

Observe that for any permutation σ of V_n with orbits $O_1, ..., O_m$ we have $\sum_{i:C_p(O_i)<l+\alpha} |O_i| \equiv r(n, p^{l+\alpha}) \pmod{p^{l+\alpha}}$, since $\sum_{i=1}^{m} |O_i| = n$ and $\sum_{i:C_p(O_i)\geqslant l+\alpha} |O_i| \equiv 0 \pmod{p^{l+\alpha}}$. Hence the condition *(ii)* of Theorem 2 could be written equivalently: $\sum_{i:C_p(O_i)<l+\alpha} |O_i| \leqslant r(n, p^{l+\alpha})$.

Theorem 3 *Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_u^{\alpha_u}$, where $p_1, \ldots, p_u$ are mutually different primes and $\alpha_1, \ldots, \alpha_u$ positive integers. A permutation σ of the set V_n with orbits $O_1, \ldots, O_m$ is (q,k)-complementing if and only if for every $j \in \{1, \ldots, u\}$ there is a positive integer l_j such that the following two conditions hold:*

(i) $r(n, p_j^{l_j+\alpha_j}) < r(k, p_j^{l_j+1})$, *and*

(ii) $\sum_{i:C_{p_j}(O_i)<l_j+\alpha_j} |O_i| = r(n, p_j^{l_j+\alpha_j})$.

For the special case of graphs (i.e. 2-uniform hypergraphs) Theorem 2 has been proved in [1].

One may apply Theorem 2 to check that every permutation of V_{89} consisting of two orbits: one of cardinality 64 and the second of cardinality 25 is $(2,40)$-complementing. Every permutation of V_{89} consisting of orbits O_1 and O_2 such that $|O_1| = 81$ and $|O_2| = 8$ is $(9,40)$-complementing. But it is easily seen (applying either Theorem 2 or Theorem 3) that there is no $(18,40)$-complementing permutation of $K_{89}^{(40)}$.

It has been proved in [15] that for given n and k there is a self-complementary k-uniform hypergraph of order n if and only if $\binom{n}{k}$ is even (the corresponding result for graphs was proved first in [13] and [14], independently). The natural question arises: is it true, that if $\binom{n}{k}$ is divisible by q then there is a cyclic q-partition of $K_n^{(k)}$?

The problem of divisibility of $\binom{n}{k}$ was considered in the literature many times, independently. The theorem we give below has been proved in 1852 by Kummer [8], it was rediscovered by Lucas [9] in 1878, then by Glaisher [2] in 1899 and finally, for $p = 2$ and $\alpha = 1$ only, by Kimball et al. [5] (for an elegant proof of Kummer's result and its connections with Last Fermat Theorem see [12]).

Theorem 4 (Kummer) *Let p be a prime and let (n_i) and (k_i) denote the sequences of digits of n and k in base p, so that $n = \sum_{i\geqslant 0} n_i p^i$ and $k = \sum_{i\geqslant 0} k_i p^i$ ($0 \leqslant n_i, k_i \leqslant p-1$ for every i). $C_p(\binom{n}{k})$ is equal to the number of indices i such that either $k_i > n_i$, or there exists an index $j < i$ with $k_j > n_j$ and $k_{j+1} = n_{j+1}, ..., k_i = n_i$.*

Let p be a prime integer, $0 < k < n$, $k = \sum_{i\geqslant 0} k_i p^i$, $n = \sum_{i\geqslant 0} n_i p^i$, where k_i and n_i are digits with respect to the basis p. Note that, by Theorem 2, if there is a cyclic p^α-partition of $K_n^{(k)}$ then there are integers l and m, $0 \leqslant m \leqslant l$, such that $n_m < k_m$, and $n_{l+\alpha-1} = n_{l+\alpha-2} = ... = n_{l+1} = 0$ (if $\alpha > 1$), and $n_i = k_i$ for $m < i \leqslant l$ (if $m < l$).
Conversely, if for indices l and m we have $n_{l+\alpha-1} = n_{l+\alpha-2} = ... = n_{l+1} = 0$ (for $\alpha > 1$), $n_l = k_l, n_{l-1} = k_{l-1}, ..., n_{m+1} = k_{m+1}$ (if $m < l$), and $n_m < k_m$, then any permutation of V_n

which has two orbits O_1 and O_2 such that $|O_1| = \sum_{i\geqslant l+\alpha} n_i p^i$ and $|O_2| = \sum_{i=0}^{l+\alpha-1} n_i p^i = \sum_{i=0}^{l} n_i p^i$ is, by Theorem 2, (p^α, k)-complementing. We are thus led to the following corollary of Theorem 2.

Theorem 5 *Let n, k, p and α be positive integers such that $k < n$ and p is prime. Suppose that $k = \sum_{i\geqslant 0} k_i p^i$, $n = \sum_{i\geqslant 0} n_i p^i$, where k_i and n_i are digits with respect to the basis p. The complete k-uniform hypergraph $K_n^{(k)}$ has a cyclic p^α-partition if and only if there exist nonnegative integers l and m, $m \leqslant l$, such that $n_m < k_m$, $n_i = k_i$ for $m < i \leqslant l$, and $n_{l+1} = n_{l+2} = ... = n_{l+\alpha-1} = 0$ (if $\alpha > 1$).* ∎

It is clear that for $\alpha > 1$ it may happen that n, k and a prime p satisfy the assumption of Theorem 4, but violate the condition (i) of Theorem 2. Hence, in general, it is not true that if p^α divides $\binom{n}{k}$ then there is a cyclic p^α-partition of $K_n^{(k)}$. However, it is very easy to observe that Theorem 4 and Theorem 5 imply the following.

Corollary 6 *Let n, k and p be positive integers such that $k < n$ and p is prime. The complete k-uniform hypergraph $K_n^{(k)}$ has a cyclic p-partition if and only if $p|\binom{n}{k}$.* ∎

The problem whether for positive n, k and q there is a cyclic q-partition of $K_n^{(k)}$ is in general open (unless q is a power of a prime).

2 Proofs

2.1 Lemmas

Lemma 1 *Let k, n, q be positive integers, $k < n$. A permutation σ of the set V_n is (q, k)-complementing if and only if $\sigma^s(e) \neq e$ for any subset $e \subset V_n$ of cardinality k and $s \not\equiv 0$ (mod q).*

Proof. If σ is (q, k)-complementing, then there is a partition $E_0 \cup ... \cup E_{q-1}$ of $\binom{V_n}{k}$ such that $E_i = \sigma(E_{i-1})$ for $i = 0, ..., q-1$ (considered mod q). Since the sets $E_0, ..., E_{q-1}$ are mutually disjoint, for every $e \in \binom{V_n}{k}$ if $\sigma^s(e) = e$ then $s \equiv 0$ (mod q).

Let us now sppose that σ is a permutation of V_n such that $\sigma^s(e) \neq e$ for $s \not\equiv 0$ (mod q). We may apply the following simple algorithm of coloring the edges of $K_n^{(k)}$ with q colors. Suppose that an edge $e \in \binom{V_n}{k}$is not yet colored. We color e with arbitrary color $i_0 \in \{0, 1, ..., q-1\}$ and for every l we color $\sigma^l(e)$ with the color $i_0 + l$ (mod q). When all the edges are colored, denote by E_i the set of edges colored with the color i. It is clear that $E_0 \cup ... \cup E_{q-1}$ is a partition of $\binom{V_n}{k}$ and that $\sigma(E_{i-1}) = E_i$ for $i = 0, 1, ..., q-1$. ∎

Note that by the algorithm given in the proof of Lemma 1 we may obtain all cyclic p-partitions of $K_n^{(k)}$ generated by σ.

The proof of Theorem 1 follows immediately by Lemma 1 and the fact that for relatively prime integers p_1 and p_2 we have $l \equiv 0 \pmod{p_1 p_2}$ if and only if $l \equiv 0 \pmod{p_1}$ and $l \equiv p_2 \pmod{p_2}$. ■

Lemma 2 *Let n, k, p and α be positive integers such that $k < n$ and p is prime. The cyclic permutation $\sigma = (1, 2, \ldots, n)$ is (p^α, k)-complementig if and only if $C_p(n) \geqslant C_p(k) + \alpha$.*

Proof. Assume first that $C_p(n) - C_p(k) \geqslant \alpha$. We shall prove that then the permutation $\sigma = (1, 2, ..., n)$ is (p^α, k)-complementing.
Observe that for any postive integer s every orbit of the permutation σ^s has the same cardinality.
By Lemma 1 it is sufficient to prove that for any edge $e \in \binom{V_n}{k}$ if $\sigma^s(e) = e$ then $s \equiv 0 \pmod{p^\alpha}$. So let us suppose that $\sigma^s(e) = e$, write $\tau = \sigma^s$ and denote by β the cardinality of any orbit of τ. Note that $\tau^\beta = id_{V_n}$ (where id_{V_n} is the identity of the set V_n).
For every vertex $v \in e$ we have clearly $\tau(v) \in e$, hence every orbit of τ containing a vertex of e is contained in e. Therefore $\beta | k$. So there is an integer γ such that $k = \beta\gamma$. We have $\tau^k = (\tau^\beta)^\gamma = id_{V_n}$, hence $\sigma^{sk} = id_{V_n}$ and therefore $sk \equiv 0(\mod n)$. This means that there is an integer δ such that $sk = \delta n$, so $sp^{C_p(k)}k' = \delta p^{C_p(n)} n'$ where $p \not| \, k'$ and $p \not| \, n'$. Since $C_p(n) - C_p(k) \geqslant \alpha$ the equality $sk' = \delta p^\alpha p^{C_p(n) - C_p(k) - \alpha} n'$ implies $s \equiv 0 \pmod{p^\alpha}$.
Let now suppose $C_p(n) < C_p(k) + \alpha$. Using once more Lemma 1, we shall prove that the cyclic permutation $\sigma = (1, 2, ..., n)$ is not (p^α, k)-complementing. We shall consider two cases, in each indicating an edge $e \in \binom{V_n}{k}$ and $s \not\equiv 0 \pmod{p^\alpha}$ such that $\sigma^s(e) = e$.
Let n' and k' be such that $n = p^{C_p(n)} n'$ and $k = p^{C_p(k)} k'$. Note that n' and k' are integers and $k', n' \not\equiv 0 \pmod{p}$.

Case 1: $C_p(n) < C_p(k)$**.** Since $k = p^{C_p(n)}(p^{C_p(k) - C_p(n)} k') < p^{C_p(n)} n' = n$ we have $p^{C_p(k) - C_p(n)} k' < n'$ and thus we may define

$$e = \bigcup_{j=0}^{p^{C_p(n)} - 1} \{jn' + 1, ..., jn' + p^{C_p(k) - C_p(n)} k'\}$$

It is very easy to check that $|e| = k$ and $\sigma^{n'}(e) = e$, but $n' \not\equiv 0 \pmod{p^\alpha}$ since $n' \not\equiv 0 \pmod{p}$.

Case 2: $C_p(n) \geqslant C_p(k)$**.** Since $k < n$ we have $k' < p^{C_p(n) - C_p(k)} n'$ and we may define

$$e = \bigcup_{j=0}^{p^{C_p(k)} - 1} \{jp^{C_p(n) - C_p(k)} + 1, ..., jp^{C_p(n) - C_p(k)} n' + k'\}$$

Again, $|e| = k$ and we have $\sigma^{p^{C_p(n) - C_p(k)} n'}(e) = e$ while $p^{C_p(n) - C_p(k)} n' \not\equiv 0 \pmod{p^\alpha}$ (since $n' \not\equiv 0 \pmod{p}$ and $C_p(n) - C_p(k) < \alpha$). ■

Lemma 3 *Let n, k, p, α be positive integers such that $k < n$, $\alpha \geqslant 1$ and p is prime. A permutation σ be of the set V_n with orbits $O_1, O_2, \ldots, O_m$ is (p^α, k)-complementing if and only if for every decomposition of k in the form*

$$k = h_1 + \ldots + h_m$$

such that $0 \leqslant h_j \leqslant |O_j|$ for $j = 1, \ldots, m$, there is an index j_0, $1 \leqslant j_0 \leqslant m$, such that $h_{j_0} > 0$ and $C_p(O_{j_0}) \geqslant C_p(h_{j_0}) + \alpha$.

Proof.

1. Let us suppose that σ is a permutation of V_n with orbits $O_1, ..., O_m$ and k is an integer $1 \leqslant k < n$, such that for any decomposition $k = h_1 + ... + h_m$ of k such that $0 \leqslant h_j \leqslant |O_j|$ for $j = 1, 2, ..., m$ there is an index j_0 with $h_{j_0} > 0$ and $C_p(O_{j_0}) \geqslant C_p(h_{j_0}) + \alpha$. We shall apply Lemmas 1 and 2 to prove that then σ is (p^α, k)-complementing.
   Let $e \in \binom{V_n}{k}$ and suppose that $\sigma^s(e) = e$ for a positive integer s. Denote by e_j the set $e_j = O_j \cap e$ and by h_j the cardinality of e_j for $j = 1, 2, ..., m$. Let j_0 be such that $h_{j_0} > 0$ and $C_p(O_{j_0}) \geqslant C_p(h_{j_0}) + \alpha$.
   By Lemma 2, σ_{j_0} is a (p^α, h_{j_0})-complementing permutation of the complete h_{j_0}-uniform hypergraph of order $|O_{j_0}|$. Hence, by Lemma 1, we have $s \equiv 0 \pmod{p^\alpha}$ and, again by Lemma 2, σ is a (p^α, k)-complementing of $K_n^{(k)}$.

2. Let now suppose that σ is a (p^α, k)-complementing permutation of $K_n^{(k)}$. Let $O_1, ..., O_m$ be the orbits of σ and suppose that $k = h_1 + ... + h_m$, where $0 \leqslant h_j \leqslant |O_j|$ for $j = 1, ..., m$. Denote by $\sigma_1, ..., \sigma_m$ the cycles of σ corresponding to $O_1, ..., O_m$, respectively. We shall prove that there is $j_0 \in \{1, 2, ..., m\}$ such that $h_{j_0} > 0$ and $C_p(O_{j_0}) \geqslant C_p(h_{j_0}) + \alpha$.
   Suppose, contrary to our claim, that we have $C_p(O_j) < C_p(h_j) + \alpha$ for all $j \in \{1, 2, ..., m\}$ such that $h_j > 0$. By Lemma 2, for every $j \in \{1, 2, ..., m\}$ the cyclic permutation σ_j is not (p^α, k_j)-complementing permutation of the complete k_j-uniform hypergraph of order $|O_{j_0}|$. Hence, by Lemma 1, for every $j \in \{1, 2, ..., m\}$ such that $h_j > 0$ there is a set $e_j \in \binom{O_j}{h_j}$ and $s_j \not\equiv 0 \pmod{p^\alpha}$, such that $\sigma_j^{s_j}(e_j) = e_j$. Let $e = e_1 \cup ... \cup e_m$. We have $|e| = k$. Denote by $l = \text{lcm}(s_1, ..., s_m)$ (the least common multiple of $s_1, ..., s_m$). It is clear that $\sigma^l(e) = e$ and $l \not\equiv 0 \pmod{p^\alpha}$. Hence, by Lemma 1, σ is not (p^α, k)-comlementing, a contradiction. ■

2.2 Proof of Theorem 2

Proof of sufficiency. Let us suppose that a permutation σ of V_n verifies the conditions (i) and (ii) of the theorem, but it is not (p^α, k)-complementing. By Lemma 3, there is a decomposition $k = h_1 + \ldots + h_m$ of k such that $0 \leqslant h_i \leqslant |O_i|$ and $C_p(O_i) < C_p(h_i) + \alpha$ for every $i = 1, \ldots, m$ for which $h_i > 0$.

Note that if, for an integer l and for an index $i \in \{1, \ldots, m\}$, we have $h_i > 0$ and $C_p(h_i) \leqslant l$, then $C_p(O_i) < C_p(h_i) + \alpha \leqslant l + \alpha$. Hence

$$r(k, p^{l+1}) \overset{(\text{mod } p^{l+1})}{\equiv} \sum_{i:C_p(h_i)\leqslant l} h_i \leqslant \sum_{i:C_p(O_i)<l+\alpha} |O_i| = r(n, p^{l+\alpha}) < r(k, p^{l+1}),$$

a contradiction.

Proof of necessity. Let us suppose now that the conditions of the theorem do not hold. Then, for any l such that $k_l \neq 0$ we have either

1. $r(n, p^{l+\alpha}) \geqslant r(k, p^{l+1})$, or
2. $r(n, p^{l+\alpha}) < r(k, p^{l+1})$ and $\sum_{i:C_p(O_i)<l+\alpha} |O_i| > r(n, p^{l+\alpha})$

We shall prove that σ is not a (p^α, k)-complementing permutation of $K_n^{(k)}$.
We begin by proving three claims.

Claim 1 *For every l such that $k_l \neq 0$ we have*

$$\sum_{i:C_p(O_i)<l+\alpha} |O_i| \geqslant r(k, p^{l+1})$$

Proof of Claim 1.
Case 1: $r(n, p^{l+\alpha}) \geqslant r(k, p^{l+1})$**.**
By the definition of $r(n, p^{l+\alpha})$ we know that there is an integer b such that $n = bp^{l+\alpha} + r(n, p^{l+\alpha})$. Hence $\sum_{i:C_p(O_i)\geqslant l+\alpha} |O_i| \leqslant bp^{l+\alpha}$, and therefore

$$\sum_{i:C_p(O_i)<l+\alpha} |O_i| \geqslant r(n, p^{l+\alpha}) \geqslant r(k, p^{l+1})$$

Case 2: $r(n, p^{l+\alpha}) < r(k, p^{l+1})$ **and** $\sum_{i:C_p(O_i)<l+\alpha} |O_i| > r(n, p^{l+\alpha})$**.**
Since $\sum_{i:C_p(O_i)\geqslant l+\alpha} |O_i| \equiv 0 \pmod{p^{l+\alpha}}$, we have

$$n = \sum_{i:C_p(O_i)\geqslant l+\alpha} |O_i| + \sum_{C_p(O_i)<l+\alpha} |O_i| \overset{(\text{mod } p^{l+\alpha})}{\equiv}$$

$$\overset{(\text{mod } p^{l+\alpha})}{\equiv} \sum_{C_p(O_i)<l+\alpha} |O_i| > r(n, p^{l+\alpha}) \equiv n \pmod{p^{l+\alpha}}$$

Hence there is a positive integer d such that

$$\sum_{C_p(O_i)<l+\alpha} |O_i| = dp^{l+\alpha} + r(n, p^{l+\alpha}) \geqslant p^{l+1} > r(k, p^{l+1})$$

This completes the proof of the claim. □

To see that the next claim is true it is sufficient to represent $x \in \mathbf{N}$ in basis p.

Claim 2 *For any nonnegative integers l, l', x and a, such that $l' \leqslant l$, $x < ap^l$, $C_p(x) \geqslant l'$ and $1 \leqslant a < p$ we have $x + p^{l'} \leqslant ap^l$.* □

Claim 3 *Let $u_1, ..., u_q$ be positive integers such that $C_p(u_i) \leqslant l+\alpha-1$ and $\sum_{i=1}^{q} u_i \geqslant ap^l$, $(0 \leqslant a < p)$. Then there exist $v_1, ..., v_q$ such that*

(1) For every $i \in \{1, ..., q\}$ $v_i \leqslant u_i$,

(2) For every $i \in \{1, ..., q\}$ either $C_p(u_i) \leqslant C_p(v_i) + \alpha - 1$ or $v_i = 0$,

(3) $\sum_{i=1}^{q} v_i = ap^l$.

Proof of Claim 3.
Without loss of generality we may suppose that

$$C_p(u_1) \geqslant C_p(u_2) \geqslant ... \geqslant C_p(u_q)$$

For every $i = 1, ..., q$ denote by $l_i = \min\{C_p(u_i), l\}$.
The conditions *(1)-(3)* are satisfied by the following sequence $(v_i)_{i=1}^{q}$.

$$v_1 = c_1 p^{l_1} \text{ where } c_1 = \max\{c \in \mathbf{N} : cp^{l_1} \leqslant u_1 \text{ and } cp^{l_1} \leqslant ap^l\}$$

$$v_2 = c_2 p^{l_2} \text{ where } c_2 = \max\{c \in \mathbf{N} : cp^{l_2} \leqslant u_2 \text{ and } v_1 + cp^{l_2} \leqslant ap^l\}$$

$$\ldots$$

$$v_i = c_i p^{l_i} \text{ where } c_i = \max\{c \in \mathbf{N} : cp^{l_i} \leqslant u_i \text{ and } v_1 + \ldots + v_{i-1} + cp^{l_i} \leqslant ap^l\}$$

$$\ldots$$

In fact,

1. $v_i \leqslant u_i$ by the definition of c_i.

2. Since $l \geqslant C_p(u_i) - \alpha + 1$ we have $C_p(v_i) \geqslant l_i = \min\{C_p(u_i), l\} \geqslant C_p(u_i) - \alpha + 1$, whenever $v_i \neq 0$, thus (2).

3. Suppose that the sequence $(v_i)_{i=1,...,q}$ violates the condition *(3)* of the claim. Then $\sum_{i=1}^{q} v_i < ap^l$ and by consequence there is $j \in \{1, ..., q\}$ such that $v_j < u_j$. By Claim 2 we have $v_j + p^{l_j} = (c_j + 1)p^{l_j} \leqslant u_j$ and $v_1 + ... + (c_j + 1)p^{l_j} \leqslant ap^l$, contrary to the choise of c_j.

The claim is proved. □

We shall indicate now such a decomposition of k in the form $k = h_1 + ... + h_m$ that

(1) $h_1, ..., h_m$ are non negative integers,

(2) $h_i \leqslant |O_i|$ for every $i = 1, ..., m$.

(3) $C_p(O_i) \leqslant C_p(h_i) + \alpha - 1$ or $h_i = 0$ for every $i = 1, ..., m$.

By Lemma 3, this means that σ is not (p^α, k)-complementing.
Let $k = k_{l_t}p^{l_t} + k_{l_{t-1}}p^{l_{t-1}} + ... + k_{l_0}p^{l_0}$, where $0 < k_{l_j} < p$ for $j = 0, ..., t$ and $l_0 < l_1 < ... < l_t$, By Claim 1 we have $\sum_{i:C_p(O_i) \leqslant l_0+\alpha-1} |O_i| \geqslant k_{l_0}p^{l_0}$. Now apply Claim 3 to construct $h_1^{(0)}, ..., h_m^{(0)}$ such that

(1_0) $h_i^{(0)} \leqslant |O_i|$ for $i = 1, ..., m$,

(2_0) $h_i^{(0)} = 0$ if $C_p(O_i) \geqslant l_0 + \alpha$, $i = 1, ..., m$,

(3_0) $C_p(O_i) \leqslant C_p(h_i^{(0)}) + \alpha - 1$ for i such that $h_i^{(0)} > 0$ and $C_p(O_i) < l_0 + \alpha$, $i = 1, ..., m$,

(4_0) $\sum_{i=1}^m h_i^{(0)} = k_{l_0}p^{l_0}$.

If $t = 0$ set $h_i = h_i^{(0)}$ for $i = 1, ..., m$ and the proof is finished. So we assume that $t \geqslant 1$.

Suppose we have constructed the sequences of non negative integers $(h_i^{(j)})_{i=1,...,m}$ for $j = 0, ..., s-1$, $1 \leqslant s \leqslant t$, such that

(1_{s-1}) $h_i^{(0)} + h_i^{(1)} + ... + h_i^{(s-1)} \leqslant |O_i|$ for $i = 1, ..., m$,

(2_{s-1}) $h_i^{(0)} + h_i^{(1)} + ... + h_i^{(s-1)} = 0$ if $C_p(O_i) \geqslant l_{s-1} + \alpha$, $i = 1, ..., m$,

(3_{s-1}) $C_p(O_i) \leqslant C_p(h_i^{(j)}) + \alpha - 1$ if $h_i^{(j)} > 0$, $C_p(O_i) < l_j + \alpha$, $j = 0, ..., s-1$

(4_{s-1}) $\sum_{i=1}^m h_i^{(j)} = k_{l_j}p^{l_j}$ for $j = 0, ..., s-1$.

We shall apply Claims 1 and 3 to construct the sequence $h_1^{(s)}, ..., h_m^{(s)}$ such that

(1_s) $h_i^{(0)} + h_i^{(1)} + ... + h_i^{(s)} \leqslant |O_i|$ for $i = 1, ..., m$,

(2_s) $h_i^{(0)} + h_i^{(1)} + ... + h_i^{(s)} = 0$ if $C_p(O_i) \geqslant l_s + \alpha$, $i = 1, ..., m$,

(3_s) $C_p(O_i) \leqslant C_p(h_i^{(s)}) + \alpha - 1$ whenever $C_p(O_i) < l_s + \alpha$ and $h_i^{(s)} > 0$, $i = 1, ..., m$,

(4_s) $\sum_{i=1}^m h_i^{(s)} = k_{l_s}p^{l_s}$.

By Claim 1, we have $\sum_{i:C_p(O_i) \leqslant l_s+\alpha-1} |O_i| \geqslant r(k, p^{l_s+1}) = k_{l_s}p^{l_s} + k_{l_{s-1}}p^{l_{s-1}} + ... + k_{l_s}p^{l_s}$.
Write $\lambda_i = \min\{C_p(h_i^{(j)}) : h_i^{(j)} > 0, j = 1, ..., s-1\}$, for $i = 1, ..., m$.
We have $h_i^{(0)} + h_i^{(1)} + ... + h_i^{(s-1)} = p^{\lambda_i}a$, where a is an integer, hence

$$C_p(O_i) \leqslant \lambda_i + \alpha - 1 \leqslant C_p(h_i^{(0)} + h_i^{(1)} + ... + h_i^{(s-1)}) + \alpha - 1.$$

Set $u_i = |O_i| - \sum_{j=0}^{s-1} h_i^{(j)}$ for $i = 1, ..., m$. We have $\sum_{i=1}^m u_i \geqslant k_{l_s}p^{l_s}$ so, by Claim 3, there exist non negative integers $h_1^{(s)}, ..., h_m^{(s)}$ with desired properties (1_s)-(4_s).

For every $i = 1, ..., m$ write $h_i = \sum_{j=0}^{t} h_i^{(j)}$. It is clear that $h_i \leqslant |O_i|$ for $i = 1, ..., m$ and $\sum_{i=1}^{m} h_i = k$.
Repeating the argument applied above we prove easily the inequalities

$$C_p(O_i) \leqslant C_p(h_i) + \alpha - 1$$

whenever $h_i \neq 0$, $i = 1, ..., m$. This proves that the sequence $(h_i)_{i=1}^{m}$ gives the desired decomposition of k. ■

2.3 Proof of Theorem 3

The proof of Theorem 3 follows by Theorem 2 and the following lemma.

Lemma 4 *Let $k, n, p_1, ..., p_u, \alpha_1, ..., \alpha_u$ be positive integers such that $k < n$ and $p_1, ..., p_u$ are primes. Write $q = p_1^{\alpha_1} \cdot ... \cdot p_u^{\alpha_u}$.*
A permutation σ of V_n is (q, k)-complementing if and only if σ is $(p_i^{\alpha_i}, k)$-complementing for $i = 1, ..., u$.

Proof. By Lemma 1, a permutation $\sigma : V_n \to V_n$ is (q, k)-complementing if and only if for every $e \in \binom{V_n}{k}$ $\sigma^s(e) = e$ implies $s \equiv 0 \pmod q$. But $s \equiv 0 \pmod q$ if and only if $s \equiv 0 \pmod{p_i^{\alpha_i}}$ for every $i \in \{1, ..., u\}$. The lemma follows. ■

3 Cyclic partitions of general complete hypergraphs

By $\tilde{K}_n$ we denote the **complete hypergraph** on the set of vertices V_n, i.e. the hypergraph with the set of edges consisting of all non trivial subsets of V_n ($\tilde{K}_n = (V_n; 2^{V_n} - \{\emptyset, V_n\})$). To stress the distinction between $\tilde{K}_n$ and $K_n^{(k)}$ we shall call $\tilde{K}_n$ the **general complete hypergraph**. Let σ be a permutation of V_n. If there is a p-partition $\{E, \sigma(E), ..., \sigma^{p-1}(E)\}$ of $2^{V_n} - \{\emptyset, V_n\}$ then we call it **cyclic p-partition of $\tilde{K}_n$** and permutation σ is then called **p-complementing**. In [18] Zwonek proved that a cyclic 2-partition of the complete general hypergraph $\tilde{K}_n$ exists if and only if n is a power of 2 and every 2-complementing permutation is cyclic (i.e. has exactly one orbit). Note that every partition of $\tilde{K}_n$ (and of $K_n^{(k)}$ as well) into two isomorphic parts is necessarily *cyclic* 2-partition.

Theorem 7 *The general complete hypergraph $\tilde{K}_n$ has a cyclic p-partition if and only if p is prime and n is a power of p ($p < n$). Moreover, every p-complementing permutation is cyclic.*

Proof. Note first that the general complete hypergraph $\tilde{K}_n$ has a cyclic p-partition if and only if every k-uniform complete hypergraph $K_n^{(k)}$ has a cyclic p-partition for $1 \leqslant k \leqslant n-1$. Let us suppose first that $\tilde{K}_n$ has a cyclic p-partition and σ is its p-complementing permutation.
The permutation σ is cyclic. In fact, suppose that $(a_{i_1}, ..., a_{i_k})$ is a cycle of σ, where $1 \leqslant k \leqslant n-1$. Then $\sigma(\{a_{i_1}, ..., a_{i_k}\}) = \{a_{i_1}, ..., a_{i_k}\}$, which is impossible.

Suppose now that p_1 is a prime divisor of p. Let us denote $k = \frac{p}{p_1}$ and $e = \{p_1, 2p_1, ..., kp_1\}$. We have $\sigma^{p_1}(e) = e$ hence, by Lemma 1, $p_1 \equiv 0 \pmod{p}$. Since p_1 is a divisor of p we obtain $p = p_1$.
It remains to prove that n is a power of p. Write $\beta = \max\{\gamma \in \mathbf{N} : p^\gamma \leqslant n\}$. Suppose that $p^\beta < n$. We shall apply Theorem 2 to prove that there is no cyclic p-partition of $K_n^{(p^\beta)}$. Since $p^{\beta+1} > n$ we have $r(n, p^{\beta+1}) = n > r(p^\beta, p^{\beta+1}) = p^\beta$ contradicting the condition (i) in Theorem 2 (for $\alpha = 1$ and $k = p^\beta$).

Let us suppose now that p is prime, $n = p^\beta$ where β is a positive integer. We shall prove that for any integer k, $0 < k < n$, the permutation $\sigma = (1, 2, ..., n)$ is (p, k)-complementing. Let us write $k = k_l p^l + k_{l-1} p^{l-1} + ... + k_o$, where $0 \leqslant k_i < p$ and $k_l \neq 0$. We shall again apply Theorem 2, for $\alpha = 1$. In fact, note that since $r(k, p^{l+1}) = k > r(p^\beta, p^{l+1}) = 0$ and $C_p(n) = \beta \geqslant l+1$ there is no orbit O_i of σ with $C_p(O_i) < l+1$. Hence the both conditions of Theorem 2 are verified and the proof is complete. ■

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References

[1] L. Adamus, B. Orchel, A. Szymański, A.P. Wojda and M. Zwonek, A note on t-complementing permutations for graphs, Information Processing Letters **110** (2009) 44-45.

[2] J.W.L. Glaisher, *On the residue of a binomial coefficient with respect to a prime modulus*, Quarterly Journal of Mathematics **30** (1899) 150-156.

[3] S. Gosselin, Generating self-complementary uniform hypergraphs, Discrete Math. **301** (2010) 1366-1372.

[4] S. Gosselin, Cyclically t-complementary uniform hypergraphs, to appear in the European Journal of Combinatorics.

[5] S.H. Kimball, T.R. Hatcher, J.A. Riley and L. Moser, *Solution to problem E1288: Odd binomial coefficients*. Amer. Math. Monthly **65** (1958) 368-369.

[6] M. Knor and P. Potočnik, A note on 2-subset-regular self-complementary 3-uniform hypergraphs, preprint.

[7] W. Kocay, *Reconstructing graphs as subsumed graphs of hypergraphs, and some self-complementary triple systems*. Graphs and Combinatorics **8** (1992) 259-276.

[8] E.E. Kummer, *Über die Ergrentzungssätze zu den allgemeinen Reziprozitätsgesetzen*, J. Reine Angew. Math., **44** (1852) 93-146.

[9] E. Lucas, *Sur les congruences des nombres eulériens et des coefficients differentiels*, Bull. Soc. Math. France **6** (1878) 49-54.

[10] P. Potočnik and M. Šajna, Regular self-comlementary uniform hypergraphs, preprint.

[11] P. Potŏcnik and Šajna, Vertex-transitive self-complementary uniform hypergraphs, European J. Combin. **30** (2009) 327-337.

[12] P. Ribenboim, *Fermat's Last Theorem for Amateurs*, Springer Verlag 1999.

[13] G. Ringel, *Selbstkomplementäre Graphen*, Arch. Math. **14** (1963) 354-358.

[14] H. Sachs, *Über selbstkomplementäre Graphen.* Publ. Math. Debrecen **9** (1962) 270-288.

[15] A. Szymański and A.P. Wojda, *A note on k-uniform self-complementary hypergraphs of given order*, Discuss. Math. Graph Theory **29** (2009) 199-202.

[16] A. Szymański and A.P. Wojda, *Self-complementing permutations of k-uniform hypergraphs*, Discrete Mathematics and Theoretical Computer Science **11:1** (2009) 117-124.

[17] A.P. Wojda, *Self-complementary hypergraphs*, Discuss. Math. Graph Theory **26** (2006) 217-224.

[18] M. Zwonek, *A note on self-complementary hypergraphs*, Opuscula Mathematica **25/2** (2005) 351-354.