The Planarity Theorems of MacLane and Whitney for Graph-like Continua

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Abstract

The planarity theorems of MacLane and Whitney are extended to compact graph-like spaces. This generalizes recent results of Bruhn and Stein (MacLane's Theorem for the Freudenthal compactification of a locally finite graph) and of Bruhn and Diestel (Whitney's Theorem for an identification space obtained from a graph in which no two vertices are joined by infinitely many edge-disjoint paths).

1 Introduction

The theorems of MacLane and Whitney characterize planarity of a graph G in different ways. The former characterizes planar graphs by the existence of a basis for the cycle space of G in which every edge appears at most twice, while the latter characterization is in terms of the circuit matroid of G having a graphic dual matroid. Naturally, these are closely connected to Kuratowski's Theorem: G is planar if and only if G does not contain a subdivision of either $K_{3,3}$ or K_5.

Recent work has treated extensions of these theorems to infinite graphs. Bruhn and Diestel [1] generalize Whitney's Theorem in the case of an infinite graph in which no two vertices are joined by infinitely many edge-disjoint paths. Bruhn and Stein [2] provide MacLane's Theorem in the case of locally finite graphs. In both cases, the theorem is proved for a topological space obtained from the graph by adding its ends and, in the former case, identifying a vertex with each end that is not separated from the vertex by any finite set of edges. In the central case of 2-connected graphs, these spaces are compact, connected, and Hausdorff.

Thomassen and Vella [12] have recently introduced the notion of a graph-like space. A *graph-like space* is a metric space G that contains a subset V so that (i) for any two elements u and w of V, there is a separation (U, W) of V so that $u \in U$ and $w \in W$, and

(ii) $G - V$ consists of disjoint open subsets of G, each homeomorphic to the real line and having precisely two limit points in V. The elements of V are the *vertices* of G and the components of $G - V$ are the *edges* of G. A *graph-like continuum* is a compact graph-like space; Thomassen and Vella show graph-like continua are locally connected Peano spaces, and that they satisfy a form of Menger's Theorem.

As a remark, compactness, at least of the blocks of the graph-like space, seems to be an important ingredient in these theorems. There is no theory yet developed concerning the cycle space of a 2-connected graph-like space that is not compact. It is certainly not true that a non-compact graph-like space necessarily has spanning trees that satisfy the basis-exchange axiom of a matroid. Consider the graph-like space in Figure 1. It is easy to verify that the set I consisting of all the edges incident with v plus the edge e is a maximal set that does not contain any circle. The same is true for the set J consisting of all the edges in the outer circle except e, plus the edge f. If we try to remove e from I, the exchange axiom indicates we should be able to add an edge from $J \setminus I = J \setminus \{f\}$ to $I \setminus \{e\}$ to get another circle-free set of edges. As this is obviously false, the circle-free sets do not satisfy the exchange axiom. (This is now a standard example. The earliest record we can find of it is in Higgs [5], who attributes it to Bean.)

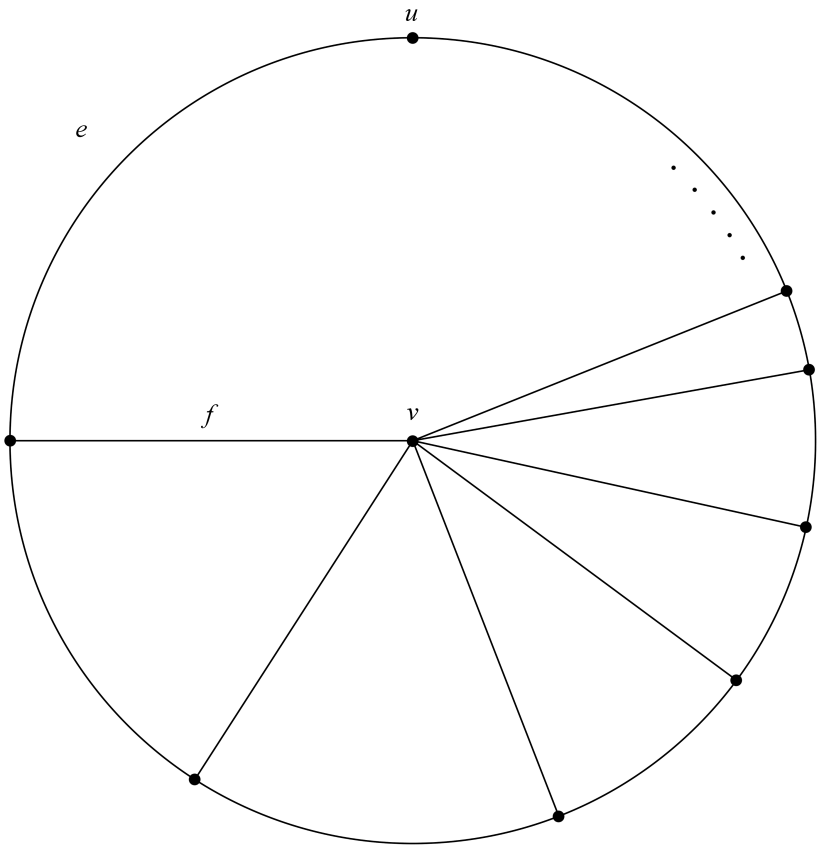

Figure 1: Noncompact graph-like space with non-matroidal spanning trees

Graph-like continua appeal as an object of study because they extend many of the combinatorial properties of finite graphs while still being quite general. The class of graph-like continua contains the class of finite graphs, the compactifications of infinite graphs considered by Casteels, Richter and Vella [3, 13], and the spaces considered by Bruhn, Diestel and Stein in [1, 2]. The graph-like continuum depicted in Figure 2, in which there are two points that have degree 1 but also are the ends of a ladder, is an example of a graph-like continuum that does not fit into any of the aforementioned categories.

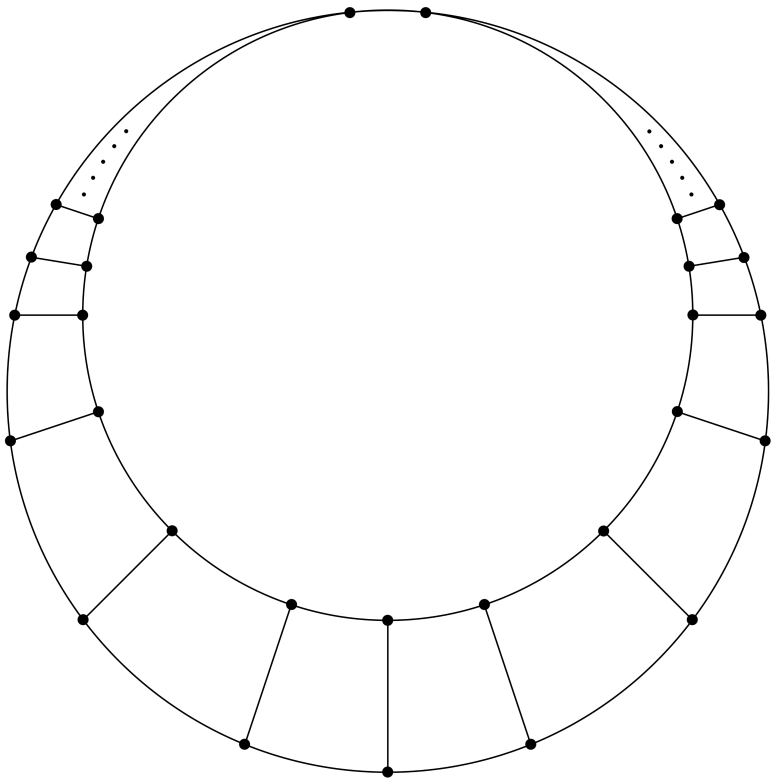

Figure 2: Graph-like continuum formed by joining the two ends of the Freudenthal compactification of the doubly infinite ladder with an edge.

For finite graphs, the theorems of MacLane and Whitney can be easily derived from Kuratowski's Theorem. Fortunately, graph-like continua satisfy the following deep version of Kuratowski's Theorem, due to Thomassen [11] and, in 1934, Claytor [4].

Theorem 1.1 *Let X be a compact, 2-connected, locally connected metric space. Then X is homeomorphic to a subset of the sphere if and only if X contains no subspace homeomorphic to either $K_{3,3}$ or K_5.*

We will use this result to prove MacLane's and Whitney's Theorems for graph-like continua. We note the assumption here that X is *2-connected*. This means X is connected and, for each $x \in X$, $X - x$ is also connected. Thomassen mentions that the thumbtack

space, consisting of a disc together with a compact interval with one end identified with the centre of the disc, is not planar. It satisfies all the hypotheses of Theorem 1.1 except 2-connection, and yet does not contain either $K_{3,3}$ or K_5. There are graph-like continua that have this same property, so we cannot hope to have planarity be the characterization in our generalizations of MacLane's and Whitney's Theorems. Rather, our versions of MacLane's and Whitney's Theorems will characterize the graph-like continua with no subspace homeomorphic to either $K_{3,3}$ or K_5. In both instances, the 2-connected spaces are the interesting ones and this is, by Theorem 1.1, the same as planarity.

2 Blocks, the cycle space and B-matroids

In both proofs the first step is a reduction to the 2-connected case, which allows us to apply Theorem 1.1 for one direction of the characterization. Thomassen and Vella [12] prove that each component of a graph-like continuum is locally connected and has only countably many edges.

A *cut point* in a topological space X is a point u of X so that, if K is the component of X containing u, $K-u$ is not connected. A *block of* X is a maximal non-empty connected subset Y of X so that Y has no cut point. The space X is *2-connected* if it is connected and has no cut point.

To prove the existence of blocks of X, it suffices to make the easy observation that if K and L are connected subsets of X each with no cut point and $|K \cap L| \geqslant 2$, then $K \cup L$ has no cut point; the existence of blocks now follows from Zorn's Lemma. Furthermore, it is an easy exercise to show that a maximal connected subset of X with no cut point is closed.

Let G be a connected graph-like continuum. If e is an edge of G so that $G-e$ is not connected, then every point of $\mathrm{cl}(e)$ is a cut point of G. However, we would like $\mathrm{cl}(e)$ to be a block of G. This is achieved by restricting the consideration of cut points to elements of the vertex set V of G. The application of Zorn's Lemma is just as simple.

A graph-like continum H contained in G is a *subgraph* of G if, for each edge e of G, either $\mathrm{cl}(e) \subseteq H$ or $e \cap H = \varnothing$. A *cycle* in G is a connected subgraph C of G containing at least one edge so that, if e is any edge of C, then $C-e$ is connected, while if e and f are distinct edges of C, then $C-(e \cup f)$ is not connected.

Moreover, the connected closed subsets of a graph-like continuum are arc-wise connected [12]. From this it follows that, in fact, the cycles of G are precisely the homeomorphic images of circles in G [13, Thm. 35]. Therefore, a block of G has at least two edges if and only if it contains a cycle.

The notion of a cycle space for a topological space has been introduced by Vella and Richter [13]. Their results are for spaces more general still than graph-like continua and so, in particular, a graph-like space has a cycle space. We begin by summarizing the relevant results, as applied to graph-like continua.

Let $\mathcal{E}$ be a collection of sets of edges of G. Then $\mathcal{E}$ is *thin* if every edge of G is in at most finitely many of the sets in $\mathcal{E}$. The point is that if $\mathcal{E}$ is thin, then the symmetric

difference of the sets in $\mathcal{E}$ is well defined: it consists of those edges that are in an odd number of elements of $\mathcal{E}$. This is the *thin sum* of the elements of $\mathcal{E}$.

The *cycle space* $\mathcal{Z}_G$ of G is the smallest subset of the power set 2^E of E that contains all edge sets of cycles and is closed under thin sums. From this definition and the earlier remarks, it is easy to see that $\mathcal{Z}_G$ is the direct sum of the $\mathcal{Z}_B$, over all the blocks B of G.

A *bond* of G is a minimal set of edges B such that $G - B$ is not connected. All of the bonds of a graph-like continuum are finite [13].

A natural way to state Whitney's Theorem is that the circuit matroid of a graph has a graphic dual if and only if the graph is planar. We will prove this form of Whitney's Theorem for graph-like continua. To do so, we must introduce B-matroids [6].

A *B-matroid* is a pair $(X, \mathcal{I})$ consisting of a set X and a set $\mathcal{I}$ of subsets of X so that:

1. $\mathcal{I} \neq \varnothing$;
2. if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;
3. for each $Y \subseteq X$, there is a maximal element of $\mathcal{I}$ contained in Y; and
4. for each $Y \subseteq X$, if I and J are two maximal elements of $\mathcal{I}$ contained in Y and $x \in I \setminus J$, then there is a $y \in J \setminus I$ so that $(I \cup \{y\}) \setminus \{x\} \in \mathcal{I}$.

Essentially, B-matroids are one way of generalizing matroids to infinite sets. The last axiom corresponds to the basis exchange axiom for finite matroids. Every B-matroid M has a dual B-matroid M^* whose bases are the complements of the bases of M.

If G is a graph-like continuum or if G is a graph, there are two natural B-matroids to associate with G. The *circuit matroid* has as its independent sets the sets of edges that do not contain the edge set of a circuit, while the *bond matroid* has as its independent sets the sets of edges that do not contain a bond. For both graphs and graph-like continua it is easy to check that the circuit matroid is dual to the bond matroid — the bonds are the minimal sets of edges that meet every basis of the circuit matroid. In the case of graphs, it is easy to see that the circuit matroid is a B-matroid: since all of the circuits are finite, Zorn's Lemma easily implies the existence of maximal independent sets. The bond matroid is also a B-matroid, because it is the dual of the circuit matroid. For graph-like continua the situation is reversed: since all of the bonds are finite it is easy to see that the bond matroid is a B-matroid, and the circuit matroid is its dual.

For a finite matroid M it is well-known (although not trivial) that the relation $\sim$ on the elements of M, defined by $e \sim f$ if and only if e and f are in a circuit together, is an equivalence relation. The matroid M is connected when $\sim$ has just one equivalence class. This is also true for B-matroids (Bruhn and Wollan, personal communication); fortunately we do not need it, as it holds true in an obvious way for graph-like continua.

The B-matroid M associated with a graph-like continuum G is the direct sum of the B-matroids associated to each of the blocks of G. This implies that the dual M^* is also a direct sum of the B-matroids of the duals of the blocks of G.

3 MacLane's Theorem

A *2-basis* for the cycle space $\mathcal{Z}_G$ is a set B of elements of $\mathcal{Z}_G$ so that each element of E is in at most 2 elements of B and every element of $\mathcal{Z}_G$ is a symmetric difference of some (possibly infinite) subset of B. Notice that every 2-basis is thin. (The term 2-basis is fairly standard in this context; the reader should have no trouble distinguishing it from a basis of a B-matroid.)

The following is MacLane's Theorem for graph-like continua. This result implies the version of MacLane's Theorem in [2], but seems to neither easily imply nor easily be implied by Thomassen's version in [10].

Theorem 3.1 *Let G be a graph-like continuum. Then G has a 2-basis if and only if G does not contain a homeomorph of either K_5 or $K_{3,3}$.*

Proof. It suffices to prove the theorem in the case G is 2-connected. Suppose first that G has no homeomorph of either $K_{3,3}$ or K_5. By Theorem 1.1, G is planar. By [8], we know that the face boundaries of any embedding of G in the sphere are circles. We take these circles to be the elements of B and note that every edge of G is in at most 2 elements of B.

Let $\mathcal{B}$ be the smallest subset of 2^E containing the elements of B and closed under thin sums. As $\mathcal{Z}_G$ is such a set, $\mathcal{B} \subseteq \mathcal{Z}_G$. On the other hand, if C is a cycle in G, then C is a circle in the sphere and so C is the symmetric difference of the elements of B on one side of C. Therefore, $C \in \mathcal{B}$, so every cycle of G is in $\mathcal{B}$. We conclude that $\mathcal{Z}_G \subseteq \mathcal{B}$, whence $\mathcal{Z}_G = \mathcal{B}$.

It remains to show that every element of $\mathcal{Z}_G$ is a thin sum of the elements of B. To do this, we shall require the following lemma (which is also required for Whitney's Theorem), whose proof is given below. Note that this lemma implies the planar dual is connected.

Lemma 3.2 *Let G be a 2-connected graph-like continuum, with vertex set V, embedded in the sphere $\mathbb{S}^2$ and let C be any circle in G. If F and F' are distinct faces of G, then there is an arc α in $\mathbb{S}^2 \setminus V$ having one end in F, one end in F' and such that $\alpha \cap G$ is finite. Moreover, if F and F' are in the same component of $\mathbb{S}^2 \setminus C$, then α may be chosen so that $\alpha \cap C = \emptyset$.*

Let $z \in \mathcal{Z}_G$ and partition z into edge-disjoint cycles [13, Thm. 14]; let $\{C_\pi\}$ be the corresponding circles, that is, the edge sets of the C_π partition z. Pick any points u, v of $\mathbb{S}^2 \setminus G$; by Lemma 3.2, there is an arc in $\mathbb{S}^2 \setminus V$ joining u and v intersecting G finitely often.

We claim that, for any two such uv-arcs α and α', $|\alpha \cap z| \equiv |\alpha' \cap z| \pmod 2$. To see this, note that, for each π, if u and v are on the same side of C_π, then $|\alpha \cap C_\pi|$ and $|\alpha' \cap C_\pi|$ are both even, while if u and v are on different sides of C_π, then $|\alpha \cap C_\pi|$ and $|\alpha' \cap C_\pi|$ are both odd. Clearly, $|\alpha \cap z| = \sum_\pi |\alpha \cap C_\pi|$, and the latter sum has only finitely many non-zero terms (α has only finitely many intersections with G). Of course the same holds for $\alpha' \cap z$.

It follows that we can partition the faces of G into those that have odd (with respect to intersection with z) parity paths from u and those that have even parity paths from u. Adjacent faces have different parities if and only if they are separated by an edge of z, so it is easy to see that z is the sum of the boundaries of all the faces in one of these two sets, as required.

Conversely, suppose G has a 2-basis B. We use (without significant change) the proof of [2] to show that G has no subdivision of either $K_{3,3}$ or K_5. We show that every finite 2-connected graph that has a homeomorph H in G also has a 2-basis. Since neither $K_{3,3}$ nor K_5 has a 2-basis, G does not contain a homeomorph of either of these, as claimed.

Since H is 2-connected, its cycle space $\mathcal{Z}_H$ has non-empty elements and each of these is a symmetric difference of the elements of a subset of B. For each element z of $\mathcal{Z}_H$, let B_z denote such a subset of B. Now, among the finitely many elements of $\mathcal{Z}_H$, let $z_1, z_2, \ldots, z_k$ be those so that the B_{z_i} are all the inclusion-wise minimal elements of $\{B_z \mid z \in \mathcal{Z}_H\}$.

Let $z \in \mathcal{Z}_H$ be such that, for some i, $B_z \cap B_{z_i} \neq \varnothing$. We claim that $B_{z_i} \subseteq B_z$.

To see this, let z' be $\sum_{b \in B_z \cap B_{z_i}} b$. If e is an edge of G not in H, then e is in an even number of the elements of each of B_z and B_{z_i}. Since e is in at most two elements of B, it follows that e is in either no element of $B_z \cap B_{z_i}$ or two elements of $B_z \cap B_{z_i}$. In either case, e is not in z'. Therefore, $z' \subseteq E(H)$ and we deduce $z' \in \mathcal{Z}_H$. But $B_z \cap B_{z_i} \subseteq B_{z_i}$ and B_{z_i} is minimal. Thus, $B_{z_i} \subseteq B_z$, as required.

Notice that this claim proves that $B_{z_1}, B_{z_2}, \ldots, B_{z_k}$ are pairwise disjoint. Thus, every edge of H appears in at most two of the z_i. We claim that $z_1, \ldots, z_k$ spans $\mathcal{Z}_H$; this will complete the proof.

Let $z \in \mathcal{Z}_H$ and let I be the set of indices i for which $B_z \cap B_{z_i} \neq \varnothing$. By the preceding remarks, the B_{z_i}, for $i \in I$, are pairwise disjoint and each is contained in B_z.

Let $B' = B_z \setminus \left(\bigcup_{i \in I} B_{z_i}\right)$ and consider $z' = \sum_{b \in B'} b$. Then $z' = z + \sum_{i \in I} z_i \in \mathcal{Z}_H$. If $z' \neq \varnothing$, then there is a $j \in \{1, 2, \ldots, k\}$ so that $B_{z_j} \subseteq B_{z'}$. But then $j \in I$, a contradiction. So $B' = \varnothing$ and $z = \sum_{i \in I} z_i$. □

Proof of Lemma 3.2. We prove the moreover version, since the other is an easy consequence. Let R be the component of $\mathbb{S}^2 \setminus C$ containing both F and F'. Note that R and $R \setminus V$ are both open sets; the former is connected by definition. We claim that $R \setminus V$ is also connected.

If we identify all the points of C to a single point c, then the space $R \cup C$ quotients to the sphere. Now $\{c\} \cup (R \cap V)$ is a closed, totally disconnected subset V' of the sphere; therefore, $\mathbb{S}^2 \setminus V'$ is connected. (This is not at all trivial; one way to prove it is to use the classification of non-compact surfaces [7].)

Now each point of $\mathbb{S}^2 \setminus V'$ has a disc neighbourhood so that any two points in the neighbourhood are joined by an arc in $\mathbb{S}^2 \setminus V'$ that meets $G \cap R$ only finitely often (in fact at most twice and twice only if both points are in G). It follows that the set of points of $\mathbb{S}^2 \setminus V'$ joined to a specific point u by an arc that meets G only finitely often is open, as is its complement. If both were non-empty, then $\mathbb{S}^2 \setminus V'$ would not be connected. Since the points joined by u by such arcs include u, it must be that each point of $\mathbb{S}^2 \setminus V'$ is joined to u by an arc in $\mathbb{S}^2 \setminus V'$ meeting G only finitely often. □

4 Whitney's Theorem

In this section we prove Whitney's Theorem for graph-like continua. Recall that the bonds of a graph-like continuum are finite, so any dual of a graph-like continuum will have only finite circuits. On the other hand, a graph-like continuum can have infinite circuits and so its dual can have infinite cocircuits.

Thinking next about planar duality, if we embed a connected graph-like continuum into the sphere, then the faces are open discs, the planar dual has these as vertices, and each edge of the graph-like continuum has a dual edge joining the two dual vertices on either side of the primal edge. Thus, the planar dual is simply a (possibly infinite) graph.

We can also define abstract duality for graph-like continua. The graph H is an *abstract dual* of the graph-like continuum G if there is a bijection between the edge sets of G and H so that a set of edges is the edge set of a circuit of G if and only if (its image) is a bond of H. In other words, H is an abstract dual of G if the circuit matroid of G is the bond matroid of H.

We are now ready for Whitney's Theorem for graph-like continua.

Theorem 4.1 *Let G be a graph-like continuum. Then G has an abstract dual graph if and only if G contains no homeomorph of either $K_{3,3}$ or K_5.*

Proof. We first assume G is 2-connected. If G contains no homeomorph of either $K_{3,3}$ or K_5, then Theorem 1.1 implies G has an embedding in the sphere $\mathbb{S}^2$. In this case, it suffices to prove that the planar dual has the right B-matroid.

Let H be the planar dual of G and let C be a cycle of G. Lemma 3.2 implies that the subgraphs of H on either side of C are connected and, therefore, the edges of H dual to the edges of C form a bond of H.

To complete the proof that H has the right B-matroid, we must also show that every bond of H is dual to a cycle of G. Let U be a subset of $V(H)$ so that the set $\delta(U)$ of edges of H with precisely one end in U is a bond. For each $u \in U$, let b_u denote the edge set of the cycle of G bounding the face of G containing u. Then $\sum_{u\in U} b_u$ is in the cycle space of G and is equal to the set $(\delta(U))^*$ of edges in G dual to the edges of $\delta(U)$. Since it is in the cycle space of G, it partitions into the edge sets of cycles of G. If C is one of these edge sets, then the preceding paragraph proves that the set C^* of edges of H dual to the edges of C is a bond. Since $C^* \subseteq \delta(U)$, we deduce that $C^* = \delta(U)$ and, therefore $(\delta(U))^*$ is the edge set of a cycle of G, as required.

Conversely, suppose that H is an abstract dual of G. It is easy to see that every finite minor of the circuit matroid of H is graphic, so in particular is not the bond matroid of either $K_{3,3}$ or K_5. Since the circuit matroid of G is dual to that of H, it contains no minor isomorphic to the circuit matroids of $K_{3,3}$ or K_5.

Now, suppose K is a subspace of G that is homeomorphic to a finite graph F. Let D be the set of edges of G not in K, and let C contain all but one edge from each of the arcs of K that correspond to edges of F. It is easy to check that if we delete D and contract C from the circuit matroid of G we will obtain the circuit matroid of F as a minor. Therefore, F cannot be $K_{3,3}$ or K_5.

We conclude by treating the case G is not 2-connected. In this case, the circuit matroid M of G is the direct sum of the circuit matroids of the blocks of G. Each of these is the circuit matroid of a graph-like continuum. If G contains no $K_{3,3}$ or K_5, then neither does any block of G. From the 2-connected case, each block of G has a circuit matroid whose dual is the circuit matroid of a graph. The dual is the direct sum of the circuit matroids of the graphs and so is the circuit matroid of a graph.

On the other hand, suppose the dual M^* of M is the circuit matroid of a graph H. Since M is the direct sum of the circuit matroids of the blocks of G, the dual is the direct sum of the circuit matroids dual to those of the blocks of G. The circuit matroid for the block K of G is obtained by deleting all the edges of G not in K; its dual is obtained from M^* by contracting all the elements of G not in K. Since contractions done in H result in a graph, the circuit matroid of K has the circuit matroid of a graph as its dual. From the 2-connected case, K has no $K_{3,3}$ or K_5 and, therefore, neither does G. □

5 Concluding Remarks

Two of the current authors, together with Thomassen, have proven the following extension of Kuratowski's Theorem [9].

Theorem 5.1 *A compact, locally connected metric space X is not planar if and only if X contains either K_5 or $K_{3,3}$, or a generalized thumbtack, or the disjoint union of the sphere and a point.*

This theorem gives the precise forbidden structure for graph-like continua (and more general spaces) to be planar. (A *generalized thumbtack* is a graph-like continuum that approximates a thumbtack. As shown in [9], there are five canonical generalized thumbtacks, so Theorem 5.1 gives a finite number of obstructions to planar embedding.) In particular, this theorem explains why our versions of the theorems of MacLane and Whitney do not reference planarity.

It is also interesting to contemplate another variation of Whitney's Theorem. Suppose G is an infinite graph with corresponding B-matroid M. Then M^* is the B-matroid of a graph-like continuum if and only if G contains no homeomorph of either $K_{3,3}$ or K_5 (which *is* equivalent to G being planar).

The difficulty with this is that G is not compact and, therefore, it is not so obvious how to make use of a planar embedding to obtain a graph-like continuum planar dual. However, using quite different techniques, the first author has proved this result; this will appear in his doctoral thesis.

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