Enumeration of Restricted Permutation Triples

Xiaojing Chen
School of Mathematics and Computational Science
China University of Petroleum
Dongying 257061, P. R. China
chenxiaojing1982@yahoo.com.cn

Wenchang Chu*
Institute of Combinatorial Mathematics
Hangzhou Normal University
Hangzhou 310036, P. R. China
chu.wenchang@unisalento.it

Submitted: Aug 22, 2010; Accepted: Sep 20, 2010; Published: Oct 5, 2010
Mathematics Subject Classifications: 05A05, 05A15

Abstract

The counting problem is investigated for the permutation triples of the first n natural numbers with exactly k occurrences of simultaneous "rises". Their recurrence relations and bivariate generating functions are established.

1 Introduction and Motivation

Let $[n]$ stand for the first n natural numbers $\{1, 2, \cdots, n\}$ and $\mathfrak{S}_n$ for the permutations of $[n]$. Given a permutation $\pi = (a_1, a_2, \cdots, a_n) \in \mathfrak{S}_n$, a rise (shortly as "R") at the kth position refers to $a_k < a_{k+1}$, while a fall (shortly as "F") at the same position refers to $a_k > a_{k+1}$, where the position index k runs from 1 to $n-1$. It is classically well–known (cf. Comtet [2, §6.5]) that the number of the permutations of $[n]$ with exactly $k-1$ rises is equal to the Eulerian number $A(n,k)$, which admits the following bivariate generating function

$$1 + \sum_{1 \leqslant k \leqslant n} A(n,k) \frac{y^n}{n!} x^k = \frac{1-x}{1-xe^{(1-x)y}}. \tag{1}$$

*The Corresponding author (Current address: Dipartimento di Matematica, Università del Salento, Lecce–Arnesano P. O. Box 193, 73100 Lecce, Italy)

When $[n]$ is replaced by multiset, the corresponding counting question is called "the problem of Simon Newcomb", which can be found in Riordan [3, Chapter 8].

Carlitz [1] examined permutation pairs $\{\pi, \sigma\}$ of $\mathfrak{S}_n$ with $\sigma = (b_1, b_2, \cdots, b_n)$. Then at the kth position, there are four possibilities "RR", "FF", "RF" and "FR". Denote by $B(n, k)$ the number of the permutations pairs of $[n]$ with exactly k occurrences of "RR". Then Carlitz found the following beautiful result

$$\sum_{0 \leqslant k \leqslant n} B(n,k) \frac{y^n}{(n!)^2} x^k = \frac{1-x}{f((1-x)y)-x} \quad \text{where} \quad f(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{(n!)^2}. \tag{2}$$

In the last double sum, the summation indices n and k run over the triangular domain $0 \leqslant k \leqslant n < \infty$, even though $B(n, n) = 0$ for all the natural numbers $n = 1, 2, \cdots$, except for $B(0, 0) = 1$. The same fact will be assumed also for other two sequences $C(n, k)$ and $D(n, k)$.

In particular, letting $x = 0$ in this equality leads to the generating function for the number of permutation pairs of $[n]$ with "RR" forbidden

$$\sum_{n \geqslant 0} \mathcal{B}_n \frac{y^n}{(n!)^2} = \frac{1}{f(y)} \quad \text{where} \quad \mathcal{B}_n := B(n,0). \tag{3}$$

Reading carefully Carlitz' article [1], we notice that Carlitz' approach can further be employed to investigate permutation triples $\{\pi, \sigma, \tau\}$ of $\mathfrak{S}_n$ with $\tau = (c_1, c_2, \cdots, c_n)$. In this case, there are eight possibilities "RRR", "RRF" "RFR", "FRR", "FFR", "FRF", "RFF" and "FFF" at the kth position. Let $C(n, k)$ be the number of the permutations triples of $[n]$ with exactly k occurrences of "RRR". Then we shall prove the following analogous formula.

Theorem 1 (Bivariate generating function)**.**

$$\sum_{0 \leqslant k \leqslant n} C(n,k) \frac{y^n}{(n!)^3} x^k = \frac{1-x}{g\big((1-x)y\big)-x} \quad \textit{where} \quad g(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{(n!)^3}.$$

When $x = 0$, the last expression becomes the generating function for the number $\mathcal{C}_n$ of permutation triples of $\mathfrak{S}_n$ with "RRR" forbidden.

Corollary 2 (Univariate generating function)**.**

$$\sum_{n \geqslant 0} \mathcal{C}_n \frac{y^n}{(n!)^3} = \frac{1}{g(y)} \quad \textit{where} \quad \mathcal{C}_n := C(n,0).$$

Applying the inverse transformation to a given $\theta = (d_1, d_2, \cdots, d_n) \in \mathfrak{S}_n$

$$d'_k = n - d_k + 1 \quad \text{with} \quad k = 1, 2, \cdots, n$$

we get another permutation $\theta' = (d'_1, d'_2, \cdots, d'_n) \in \mathfrak{S}_n$. Then "R" (rise) or "F" (fall) in each position in θ will be inverted in θ'. Thus the preceding results about permutation triples $\{\pi, \sigma, \tau\}$ with "RRR" forbidden hold also for each of the other seven cases.

2 Proof of the Theorem

In general, a permutation triple $\{\pi, \sigma, \tau\}$ of $\mathfrak{S}_n$ can be represented by

$$\begin{aligned} \pi &= (a_1, a_2, \cdots, a_n), \\ \sigma &= (b_1, b_2, \cdots, b_n), \\ \tau &= (c_1, c_2, \cdots, c_n). \end{aligned}$$

Following Carlitz' approach, denote by $C_{a,b,c}(n,k)$ the number of permutation triples $\{\pi, \sigma, \tau\}$ with exactly k occurrences of "RRR" and the initials $a_1 = a$, $b_1 = b$ and $c_1 = c$. The classification according to the initial letters yields the equation

$$C(n,k) = \sum_{a,b,c=1}^{n} C_{a,b,c}(n,k). \tag{4}$$

For $\theta = (d_1, d_2, \cdots, d_n) \in \mathfrak{S}_n$, define the map ϕ from $\mathfrak{S}_n$ onto $\mathfrak{S}_{n-1}$ by

$$\phi(\theta) = \theta'' = (d''_1, d''_2, \cdots, d''_{n-1}) : \quad d''_{k-1} = \begin{cases} d_k, & d_k < d_1; \\ d_k - 1, & d_k > d_1. \end{cases}$$

Comparing the first two initial letters of permutation triples and then taking into account of the map ϕ, we have

$$\begin{aligned} C_{a,b,c}(n,k) &= \sum_{\alpha<a}\sum_{\beta<b}\sum_{\gamma<c} C_{\alpha,\beta,\gamma}(n-1,k) + \sum_{\alpha<a}\sum_{\beta<b}\sum_{\gamma\geqslant c} C_{\alpha,\beta,\gamma}(n-1,k) \\ &+ \sum_{\alpha<a}\sum_{\beta\geqslant b}\sum_{\gamma<c} C_{\alpha,\beta,\gamma}(n-1,k) + \sum_{\alpha<a}\sum_{\beta\geqslant b}\sum_{\gamma\geqslant c} C_{\alpha,\beta,\gamma}(n-1,k) \\ &+ \sum_{\alpha\geqslant a}\sum_{\beta<b}\sum_{\gamma<c} C_{\alpha,\beta,\gamma}(n-1,k) + \sum_{\alpha\geqslant a}\sum_{\beta<b}\sum_{\gamma\geqslant c} C_{\alpha,\beta,\gamma}(n-1,k) \\ &+ \sum_{\alpha\geqslant a}\sum_{\beta\geqslant b}\sum_{\gamma<c} C_{\alpha,\beta,\gamma}(n-1,k) + \sum_{\alpha\geqslant a}\sum_{\beta\geqslant b}\sum_{\gamma\geqslant c} C_{\alpha,\beta,\gamma}(n-1,k-1) \end{aligned}$$

which can further be simplified into the following interesting relation

$$C_{a,b,c}(n,k) = C(n-1,k) - \sum_{\alpha\geqslant a}\sum_{\beta\geqslant b}\sum_{\gamma\geqslant c} \Big\{ C_{\alpha,\beta,\gamma}(n-1,k) - C_{\alpha,\beta,\gamma}(n-1,k-1) \Big\}. \tag{5}$$

Summing over a, b, c from 1 to n across this equation, we get the equality

$$C(n,k) = n^3 C(n-1,k) - \sum_{\alpha,\beta,\gamma} \alpha\beta\gamma \Big\{ C_{\alpha,\beta,\gamma}(n-1,k) - C_{\alpha,\beta,\gamma}(n-1,k-1) \Big\}. \tag{6}$$

Similarly, multiplying across (5) by abc and then summing over a, b, c, we have another equality

$$\begin{aligned} \sum_{a,b,c} abc\, C_{a,b,c}(n,k) = \binom{n+1}{2}^3 C(n-1,k) - \sum_{\alpha,\beta,\gamma} \binom{\alpha+1}{2}\binom{\beta+1}{2}\binom{\gamma+1}{2} \\ \times \Big\{ C_{\alpha,\beta,\gamma}(n-1,k) - C_{\alpha,\beta,\gamma}(n-1,k-1) \Big\}. \end{aligned} \tag{7}$$

For $\ell \in \mathbb{N}_0$, define the triple sum

$$C^{(\ell)}(n,k) = \sum_{a,b,c} \binom{a+\ell-1}{\ell}\binom{b+\ell-1}{\ell}\binom{c+\ell-1}{\ell} C_{a,b,c}(n,k)$$

which reduces, for $\ell = 0$, to

$$C(n,k) = C^{(0)}(n,k) = \sum_{a,b,c} C_{a,b,c}(n,k).$$

Then (6) and (7) can be restated respectively as

$$C(n,k) = n^3 C(n-1,k) - C^{(1)}(n-1,k) + C^{(1)}(n-1,k-1),$$
$$C^{(1)}(n,k) = \binom{n+1}{2}^3 C(n-1,k) - C^{(2)}(n-1,k) + C^{(2)}(n-1,k-1).$$

Recall the binomial identity

$$\sum_{b \leqslant \beta} \binom{b+\ell-1}{\ell} = \binom{\beta+\ell}{1+\ell}.$$

Multiplying across (5) further by $\binom{a+\ell-1}{\ell}\binom{b+\ell-1}{\ell}\binom{c+\ell-1}{\ell}$ and then summing over a, b, c, we find the following general relation

$$C^{(\ell)}(n,k) = \binom{n+\ell}{\ell+1}^3 C(n-1,k) - C^{(\ell+1)}(n-1,k) + C^{(\ell+1)}(n-1,k-1). \tag{8}$$

By introducing further the polynomials

$$C_n^{(\ell)}(x) = \sum_k C^{(\ell)}(n,k)x^k \quad \text{and} \quad C_n(x) = \sum_k C(n,k)x^k$$

we can translate (8) into the relation

$$C_n^{(\ell)}(x) = \binom{n+\ell}{\ell+1}^3 C_{n-1}(x) + (x-1)C_{n-1}^{(\ell+1)}(x). \tag{9}$$

In particular for the first few values of ℓ, this reads as

$$\begin{aligned}
C_n(x) &= n^3 C_{n-1}(x) + (x-1)C_{n-1}^{(1)}(x),\\
C_{n-1}^{(1)}(x) &= \binom{n}{2}^3 C_{n-2}(x) + (x-1)C_{n-2}^{(2)}(x),\\
C_{n-2}^{(2)}(x) &= \binom{n}{3}^3 C_{n-3}(x) + (x-1)C_{n-3}^{(3)}(x).
\end{aligned}$$

Iterating (9) n-times and keeping in mind the initial condition

$$C_0(x) = C_1(x) = 1$$

we get the equation

$$C_n(x) = \sum_{k=1}^{n} (x-1)^{k-1} \binom{n}{k}^3 C_{n-k}(x)$$

which is equivalent to the recurrence relation

$$xC_n(x) = \sum_{k=0}^{n} (x-1)^{k} \binom{n}{k}^3 C_{n-k}(x) \quad \text{for} \quad n > 0. \tag{10}$$

Finally, we are now ready to compute the bivariate generating function

$$\begin{aligned}
\Omega(x,y) := \sum_{0 \leqslant k \leqslant n} C(n,k) \frac{y^n}{(n!)^3} x^k &= 1 + \sum_{n=1}^{\infty} \frac{y^n}{(n!)^3} C_n(x) \\
&= 1 - \frac{1}{x} + \frac{1}{x} \sum_{n=0}^{\infty} \frac{y^n}{(n!)^3} \sum_{k=0}^{n} (x-1)^k \binom{n}{k}^3 C_{n-k}(x) \\
&= 1 - \frac{1}{x} + \frac{1}{x} \sum_{k=0}^{\infty} \frac{(x-1)^k y^k}{(k!)^3} \sum_{n=k}^{\infty} \frac{y^{n-k} C_{n-k}(x)}{\{(n-k)!\}^3}
\end{aligned}$$

which simplifies into the relation

$$\Omega(x,y) = 1 - \frac{1}{x} + \frac{1}{x} g\big((1-x)y\big)\Omega(x,y).$$

By resolving this equation, we get an expression of Ω in terms of g, which turns to be the generating function displayed in the theorem.

Furthermore, letting $x = 0$ in (10), we deduce that the number of permutation triples of $\mathfrak{S}_n$ with "RRR" forbidden satisfies the following binomial relation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \mathcal{C}_k = 0 \quad \text{with} \quad n > 0. \tag{11}$$

3 Enumeration of m-tuples of Permutations

More generally, we may consider the m-tuples of permutations of $\mathfrak{S}_n$ with exactly k occurrences of "R^m". Denote by $D(n,k)$ the number of such multiple permutations. Then the same approach can further be carried out to establish the following bivariate generating function

$$\sum_{0 \leqslant k \leqslant n} D(n,k) \frac{y^n}{(n!)^m} x^k = \frac{1-x}{h\big((1-x)y\big) - x} \quad \text{where} \quad h(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{(n!)^m}. \tag{12}$$

When $x = 0$, it gives rise to the generating function for the number $\mathcal{D}_n$ of m-tuples of $\mathfrak{S}_n$ with "R^m" forbidden

$$\sum_{n \geqslant 0} \mathcal{D}_n \frac{y^n}{(n!)^m} = \frac{1}{h(y)} \quad \text{where} \quad \mathcal{D}_n := D(n, 0) \tag{13}$$

which is equivalent to the following recurrence relation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^m \mathcal{D}_k = 0 \quad \text{with} \quad n > 0. \tag{14}$$

The details are not produced and left to the interested reader.

References

[1] L. Carlitz – R. Scoville – T. Vaughan, *Enumeration of pairs of permutations,* Discrete Math. 14 (1976), 215–239.

[2] L. Comtet, *Advanced Combinatorics,* Dordrecht–Holland, The Netherlands, 1974.

[3] J. Riordan, *An Introduction to Combinatorial Analysis,* John Wiley, New York, 1958.