Automorphism groups of a graph and a vertex-deleted subgraph

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Abstract

Understanding the structure of a graph along with the structure of its subgraphs is important for several problems in graph theory. Two examples are the Reconstruction Conjecture and isomorph-free generation. This paper raises the question of which pairs of groups can be represented as the automorphism groups of a graph and a vertex-deleted subgraph. This, and more surprisingly the analogous question for edge-deleted subgraphs, are answered in the most positive sense using concrete constructions.

1 Introduction

The Reconstruction Conjecture of Ulam and Kelley famously states that the isomorphism class of all graphs on three or more vertices is determined by the isomorphism classes of its vertex-deleted subgraphs (see [GH69] for a survey of classic results on this problem). A frequent issue when attacking reconstruction problems is that automorphisms of the substructures lead to ambiguity when producing the larger structure.

This paper considers the relation between the automorphism group of a graph and the automorphism groups of the vertex-deleted subgraphs and edge-deleted subgraphs. If a group Γ_1 is the automorphism group of a graph G, and another group Γ_2 is the

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automorphism group of $G-v$ for some vertex v, then we say Γ_1 *deletes to* Γ_2. This relation is denoted $\Gamma_1 \to \Gamma_2$. A corresponding definition for edge deletions is also developed. Our main result is that any two groups delete to each other, with vertices or edges.

These relations also appear in McKay's isomorph-free generation algorithm [McK98], which is frequently used to enumerate all graph isomorphism classes. After generating a graph G of order n, graphs of order $n+1$ are created by adding vertices and considering each $G+v$. To prune the search tree, the canonical labeling of $G+v$ is computed, usually by `nauty`, McKay's canonical labeling algorithm [McK06,HR09]. Finding a canonical labeling of a graph reveals its automorphism group. Since G was generated by this process, its automorphism group is known but is not used while computing the automorphism group of $G+v$. If some groups could not delete to the automorphism group of G, then they certainly cannot appear as the automorphism group of $G+v$ which may allow for some improvement to the canonical labeling algorithm. The current lack of such optimizations hints that no such restrictions exist, but this notion has not been formalized before this paper.

One reason why this problem has not been answered is that the study of graph symmetry is very restricted, mostly to forms of symmetry requiring vertex transitivity. These forms of symmetry are useless in the study of the Reconstruction Conjecture, as regular graphs are reconstructible. On the opposite end of the spectrum, almost all graphs are *rigid* (have trivial automorphism group) [Bol01]. Graphs with non-trivial, but non-transitive, automorphisms have received less attention.

Graph reconstruction and automorphism concepts have been presented together before [Bab95,LS03]. However, there appears to be no results on which pairs of groups allow the deletion relation. While our result is perhaps unsurprising, it is not trivial. The reader is challenged to produce an example for $\mathbb{Z}_2 \to \mathbb{Z}_3$ before proceeding.

For notation, G always denotes a graph, while Γ refers to a group. The trivial group I consists of only the identity element, ε. All graphs in this paper are finite, simple, and undirected, unless specified otherwise. All groups are finite. The automorphism group of G is denoted $\mathrm{Aut}(G)$ and the stabilizer of a vertex v in a graph G is denoted $\mathrm{Stab}_G(v)$.

2 Definitions and Basic Tools

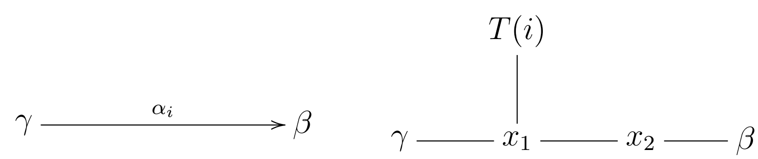
We begin with a formal definition of the deletion relation.

Definition 2.1. Let Γ_1, Γ_2 be finite groups. If there exists a graph G with $|V(G)| \geqslant 3$ and vertex $v \in V(G)$ so that $\mathrm{Aut}(G) \cong \Gamma_1$ and $\mathrm{Aut}(G-v) \cong \Gamma_2$, then Γ_1 (*vertex*) *deletes to* Γ_2, denoted $\Gamma_1 \to \Gamma_2$. Similarly, the group Γ_1 *edge deletes to* Γ_2 if there exists a graph G and edge $e \in E(G)$ so that $\mathrm{Aut}(G) \cong \Gamma_1$ and $\mathrm{Aut}(G-e) \cong \Gamma_2$. If a specific graph G and subobject x give $\mathrm{Aut}(G) \cong \Gamma_1$ and $\mathrm{Aut}(G-x) \cong \Gamma_2$, the deletion relation may be presented as $\Gamma_1 \xrightarrow{G-x} \Gamma_2$.

To determine the automorphism structure of a graph, vertices that are not in the same orbit can be distinguished by means of neighboring structures. A useful gadget to make such distinctions is the rigid tree $T(n)$, where n is an integer at least 2. Build $T(n)$ by starting with a path $u_0, z_1, \ldots, z_n$. For each i, $1 \leqslant i \leqslant n$, add a path $z_i, x_{i,1}, x_{i,2}, \ldots, x_{i,2i}, u_i$ of length $2i+1$. This results in a tree with $n+1$ leaves. Note that each leaf u_i is distance $2i+1$ to a vertex of degree 3 (except for u_n, which is distance $2n+2$). Thus, the leaves are in disjoint orbits and $T(n)$ is rigid. Also, if any leaf u_i is selected with $i \geqslant 1$, $T(n) - u_i$ is rigid. This gives an example of the deletion relation $I \to I$. For notation, let J be a set and $\{T_j\}_{j \in J}$ be disjoint copies of $T(n)$. Then $u_i(T_j)$ designates the copy of u_i in T_j. This is well-defined since there is a unique isomorphism between each T_j and $T(n)$.

For any group Γ, a simple, unlabeled, undirected graph G exists with $\mathrm{Aut}(G) \cong \Gamma$ [Fru39]. The construction is derived from the well-known Cayley graph[1]. Define $C(\Gamma)$ to be a graph with vertex set Γ and complete directed edge set, where the edge (γ, β) is labeled with $\gamma^{-1}\beta$, the element whose right-multiplication on γ results in β. The automorphism group of $C(\Gamma)$ is Γ, and the action on the vertices follows right multiplication by elements in Γ. That is, if $\gamma \in \Gamma$, the permutation σ_γ will take a vertex α to the vertex $\alpha\gamma$.

This directed graph with labeled edge sets is converted to an undirected and unlabeled graph by swapping the labeled edges with gadgets. Specifically, order the elements of $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ so that $\alpha_1 = \varepsilon$. For each edge (γ, β), subdivide the edge labeled $\alpha_i = \gamma^{-1}\beta$ with vertices x_1, x_2, and attach a copy $T_{\gamma,\beta}$ of $T(i)$ by identifying $u_0(T_{\gamma,\beta})$ with x_1. Note that $i \geqslant 2$ in these cases, since $\alpha_i \neq \varepsilon$. See Figure 1 for an example of this process.



(a) A directed edge labeled α_i.

(b) An unlabeled undirected gadget.

Figure 1: Converting a labeled directed edge to an undirected unlabeled gadget.

Denote this modified graph $C'(\Gamma)$. We refer to it as *the* Cayley graph of Γ. Note that the automorphisms of $C'(\Gamma)$ are uniquely determined by the permutation of the group elements and preserve the original edge labels, since the trees $T(i)$ identify the label α_i and have a unique isomorphism between copies. Hence, $\mathrm{Aut}(C'(\Gamma)) \cong \mathrm{Aut}(C(\Gamma)) \cong \Gamma$.

Lemma 2.2. *Let Γ be a group and $G = C'(\Gamma)$. Then the stabilizer of the identity element ε (as a vertex in G) is trivial. That is,* $\mathrm{Stab}_G(\varepsilon) \cong I$.

Proof. Every automorphism of G is represented by right-multiplication of Γ. Hence, every automorphism except the identity map will displace ε. $\square$

[1] In most uses of the Cayley graph, a generating set is specified. For simplicity, we use the entire group.

3 Deletion Relations with the Trivial Group

Now that sufficient tools are available, we prove some basic properties.

Proposition 3.1. *(The Reflexive Property) For any group* Γ, $\Gamma \to \Gamma$.

Proof. Let Γ be non-trivial, as the trivial case has been handled by the rigid tree $T(n)$. Let G be the Cayley graph $C'(\Gamma)$. Create a supergraph G' by adding a dominating vertex v with a pendant vertex u. Now, u is the only vertex of degree 1, and v is the only vertex adjacent to u. Hence, these two vertices are distinguished in G' from the vertices of G. Removing v leaves G and the isolated vertex u. Thus, Γ is the automorphism group for both G' and $G' - v$. $\square$

A key part of our final proof relies on the trivial group deleting to any group.

Lemma 3.2. *For all groups* Γ*,* $I \to \Gamma$.

Proof. Let $G = C'(\Gamma)$. Let $n = |\Gamma|$. Order the group elements of Γ as $\alpha_1, \dots, \alpha_n$. Create a supergraph, G', by adding vertices as follows: For each α_i, create a copy T_{α_i} of $T(2n)$ and identify $u_0(T_{\alpha_i})$ with the vertex α_i in G (Here, $2n$ is used to distinguish these copies from the edge gadgets). Add a vertex v that is adjacent to $u_i(T_{\alpha_i})$ for each i. For each α_i, the leaf of T_{α_i} adjacent to v distinguishes α_i. Hence, no automorphisms exist in G'. However, $G' - v$ restores all automorphisms π from $\mathrm{Aut}(G)$ by mapping T_{α_i} to $T_{\pi(\alpha_i)}$ through the unique isomorphism. $\square$

Note that this proof uses a very special vertex that enforces all vertices to be distinguished. Before producing examples where deleting a vertex removes symmetry, it may be useful to remark that such a distinguished vertex cannot be used.

Lemma 3.3. *Let* G *be a graph and* $v \in V(G)$*. Then, automorphisms in* G *that stabilize* v *form a subgroup in the automorphism group of* $G - v$*. That is,* $\mathrm{Stab}_G(v) \leqslant \mathrm{Aut}(G - v)$.

Proof. Let $\pi \in \mathrm{Stab}_G(v)$. The restriction map $\pi|_{G-v}$ is an automorphism of $G - v$. $\square$

The implication of this lemma is removing a vertex with a trivial orbit cannot remove automorphisms. However, we can remove all symmetry in a graph using a single vertex deletion.

Lemma 3.4. *For any group* Γ*,* $\Gamma \to I$.

Proof. Assume $\Gamma \ncong I$, since the reflexive property handles this case. Let $G = C'(\Gamma)$ and $n = |\Gamma|$.

Let G_1, G_2 be copies of G with isomorphisms $f_1 \colon G \to G_1$ and $f_2 \colon G \to G_2$. Create a graph G' from these two copies as follows. For all elements γ in Γ, create a copy T_γ of $T(n)$ and identify $u_0(T_\gamma)$ with $f_1(\gamma)$ and $u_n(T_\gamma)$ with $f_2(\gamma)$. Note that $\mathrm{Aut}(G') \cong \Gamma$, since no vertices from G_1 can map to G_2 from the asymmetry of the T_γ subgraphs, and any automorphism of G_1 extends to exactly one automorphism of G_2.

Any automorphism π of $G' - f_1(\varepsilon)$ must induce an automorphism $\pi|_{G_2}$ of G_2. But the vertices of G_1 must then permute similarly (by the definition $\pi(f_1(x)) = f_1 f_2^{-1} \pi f_2(x)$). Since $f_1(\varepsilon)$ is not in the image of π, π stabilizes $f_2(\varepsilon)$. Lemma 2.2 implies π must be the identity map. Hence, $\mathrm{Aut}(G' - f_1(\varepsilon)) \cong I$. □

4 Deletion Relations Between Any Two Groups

We are sufficiently prepared to construct a graph to reveal the deletion relation for all pairs of groups.

Theorem 4.1. *If Γ_1 and Γ_2 are groups, then $\Gamma_1 \to \Gamma_2$.*

Proof. Assume both groups are non-trivial, since Lemmas 3.2 and 3.4 cover these cases. Let $G_1 = C'(\Gamma_1)$. Then identify $v_1 \in V(G_1)$ as the vertex corresponding to $\varepsilon \in \Gamma_1$. Note that $\mathrm{Stab}_{G_1}(v_1) \cong I$ as in Lemma 2.2. Also by Lemma 3.2, there exists a graph G_2 and vertex v_2 so that $I \xrightarrow{G_2 - v_2} \Gamma_2$. Define $n_i = |\Gamma_i|$. Order the elements of Γ_1 as $\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n_1}$ so that $\alpha_{1,1} = \varepsilon = v_1$.

We collect the necessary properties of G_1, G_2, v_1, v_2 before continuing. First, G_1 has automorphisms $\mathrm{Aut}(G_1) \cong \Gamma_1$ and v_1 is trivially stabilized ($\mathrm{Stab}_{G_1}(v_1) \cong I$). Second, G_2 is rigid ($\mathrm{Aut}(G_2) \cong I$) but $G_2 - v_2$ has automorphisms $\mathrm{Aut}(G_2 - v_2) \cong \Gamma_2$. The following construction only depends on these requirements.

Let $H_1, \ldots, H_{n_1}$ be copies of G_2. Construct a graph G by taking the disjoint union of G_1, H_1, $\ldots$, H_{n_1}, and adding edges between $\alpha_{1,i}$ and every vertex of H_i, for $i = 1, \ldots, n_1$. Since $\mathrm{Aut}(H_i) \cong I$, the automorphism group of G cannot permute the vertices within each H_i. However, the vertices of G_1 can permute freely within $\mathrm{Aut}(G_1) \cong \Gamma_1$, since $H_i \cong H_j$ for all i, j. Hence, $\mathrm{Aut}(G) \cong \Gamma_1$.

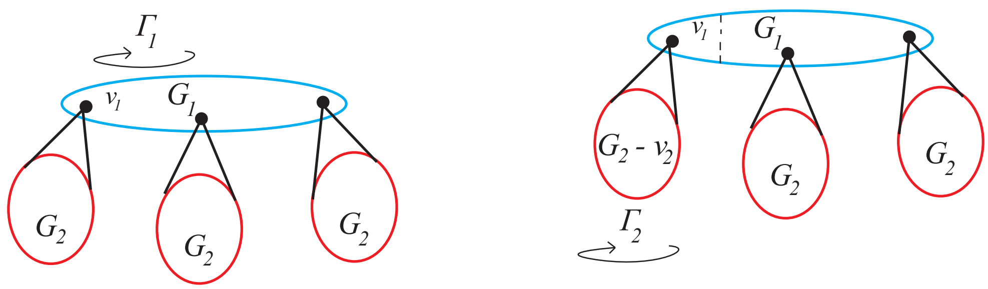
When the copy of v_2 in H_1 is deleted from G, the automorphisms of $H_1 - v_2$ are Γ_2. However, the vertex v_1 of G_1 is now distinguished since it is adjacent to a copy of $G_2 - v_2$, unlike the other elements of Γ_1 in G_1 which are adjacent to a copy of G_2. This means the permutations of G_1 must stabilize v_1. Since $\mathrm{Stab}_{G_1}(v_1) = I$, the only permutation allowed on G_1 is the identity. However, $H_1 - v_2$ has automorphism group Γ_2. Hence, $\mathrm{Aut}(G - v_2) \cong \Gamma_2$. □

Figure 2 presents a visualization of the automorphisms in this construction before and after the deletion. A very similar construction produces this general result for the edge case.

Theorem 4.2. *If Γ_1 and Γ_2 are groups, then there exists a graph G and an edge $e \in E(G)$ so that $\Gamma_1 \xrightarrow{G-e} \Gamma_2$.*

Proof. Set $n_i = |\Gamma_i|$. Let $G_1 = C'(\Gamma_1)$ with v_1 corresponding to $\varepsilon \in \Gamma_1$ and order the elements of Γ_1 similarly to the proof of Theorem 4.1.

Form G_2 by starting with $C'(\Gamma_2)$ and making a copy T_γ of $T(2n_2)$ for each element $\gamma \in \Gamma_2$, identifying $\gamma \in V(C'(\Gamma_2))$ with $u_0(T_\gamma)$. Now, add an edge e between $u_{2n_2}(T_1)$



(a) G with $\operatorname{Aut} G \cong \Gamma_1$. (b) $G - v_2$ with $\operatorname{Aut} G - v_2 \cong \Gamma_2$.

Figure 2: The vertex deletion construction.

and $u_{2n_2-1}(T_1)$. This distinguishes the element ε as a vertex in $C'(\Gamma_2)$ and hence is stabilized. So, $\operatorname{Aut}(G_2) \cong I$ and if e is removed all group elements are symmetric again, so $\operatorname{Aut}(G_2 - e) \cong \Gamma_2$.

Notice that G_1, G_2, v_1, e satisfy the requirements of the construction of G in Theorem 4.1. Hence, the same construction (with e in place of v_2) provides an example of edge deletion from Γ_1 to Γ_2. □

Note that the graph produced for Theorem 4.2 can be used for the proof of Theorem 4.1 by subdividing e and using the resulting vertex as the deletion point.

5 Future Work

While the question posed in this paper is answered completely for the class of all graphs, there remain questions for special cases. For instance, the automorphism groups of trees are fully understood [Ser80]. Let $\mathcal{G}_T$ be the class of groups that are represented by the automorphism groups of trees and $\mathcal{G}_F$ represented by automorphisms of forests[2]. The constructions in this paper are not trees, so new methods will be required to answer the following questions. If we restrict to trees, can any group in $\mathcal{G}_T$ delete to any group in $\mathcal{G}_F$? Or, if we restrict to deleting leaves (and hence stay connected) can all pairs of groups in $\mathcal{G}_T$ delete to each other?

Another interesting aspect of our construction is that the resulting graphs are very large, with the order of the graphs cubic in the size of the groups. Which of these relations can be realized by small graphs? Can we restrict the groups that can appear based on the order of the graph? The current-best upper bound on the order of a graph G with automorphism groups isomorphic to a given group Γ is $|V(G)| \leqslant 2|\Gamma|$ and $\operatorname{Aut}(G) \cong \Gamma$ [Bab74]. This has particular application to McKay's generation algorithm, where only "small" examples are usually computed (for example, all connected graphs up to 11 vertices were computed in [McK97]). To demonstrate that this is not trivial, see Figure 3 for a graph showing $\mathbb{Z}_2 \to \mathbb{Z}_3$.

[2] An elementary proof shows that $\mathcal{G}_T = \mathcal{G}_F$.

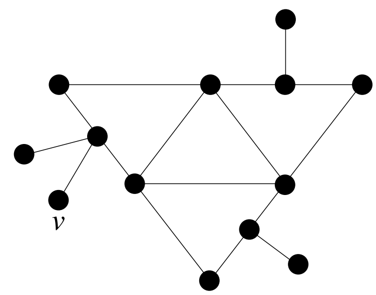


Figure 3: This graph G has $\mathrm{Aut}(G) \cong \mathbb{Z}_2$ and $\mathrm{Aut}(G - v) \cong \mathbb{Z}_3$.

While Theorem 4.1 shows that there exists a graph where *some* vertex can be deleted to demonstrate the deletion relations, our constructions have many other vertices that behave in very different ways when they are deleted. When relating to the Reconstruction Conjecture, this raises questions regarding the combinations of automorphism groups that appear in the vertex-deleted subgraphs. For instance, if the multiset of vertex-deleted automorphism groups is provided, can one reconstruct the automorphism group? This question only gives the groups, but not the vertex-deleted subgraphs. An example is that n copies of S_{n-1} must reconstruct to S_n, but it is unknown whether the graph is K_n or nK_1. Since $\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$, this ambiguity will always naturally arise. Can it arise in other contexts? Is the automorphism group recognizable from a vertex deck?

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