An interesting new Mahonian permutation statistic

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Abstract

The standard algorithm for generating a random permutation gives rise to an obvious permutation statistic DIS that is readily seen to be Mahonian. We give evidence showing that it is not equal to any previously published statistic. Nor does its joint distribution with the standard Eulerian statistics des and exc appear to coincide with any known Euler-Mahonian pair.

A general construction of Skandera yields an Eulerian partner eul such that $(\mathrm{eul}, \mathrm{DIS})$ is equidistributed with $(\mathrm{des}, \mathrm{MAJ})$. However eul itself appears not to be a known Eulerian statistic.

Several ideas for further research on this topic are listed.

1 The statistic

1.1 Random permutations

The standard algorithm [Knu81, 3.4.2, Algorithm P] for uniformly generating a random permutation of $[n] := \{1, \ldots, n\}$ is as follows. Start with the identity permutation $\iota = 1 \ldots n$ in the symmetric group S_n. There are n steps labelled $n, n-1, \ldots, 1$ (the last step can be omitted, but it makes our notation easier to include it here). At step i a random position j_i is chosen uniformly from $[i]$ and the current element in position j_i is swapped with the element at position i.

Example 1. *The permutation* $25413 \in \mathrm{S}_5$ *is formed by choosing* $j_5 = 3, j_4 = 1, j_3 = 1, j_2 = 1, j_1 = 1$. *Its inverse* 41532 *is formed by choosing* $j_5 = 2, j_4 = 3, j_3 = 2, j_2 = 1, j_1 = 1$.

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In terms of multiplication in S_n, π is a product of "transpositions" $\prod_{i \leqslant n}(ij_i)$. Any of these "transpositions" may be the identity permutation. This representation as a "triangular product" gives a bijection between S_n and the set of sequences $(j_1, \ldots, j_n)$ that satisfy $1 \leqslant j_i \leqslant i$ for all i.

Knuth attributes this algorithm to R. A. Fisher and F. Yates [FY38], and a computer implementation was given by Durstenfeld [Dur64]. Recently [Wil09] the present author and others have studied the distribution of various quantities associated with the algorithm.

1.2 The statistic

For each $n \geqslant 1$, there is a map $\mathrm{S}_n \to \mathrm{S}_{n+1}$ that maps π to the permutation $\pi^{\uparrow}$ that fixes $n+1$ and agrees on $1, \ldots, n$ with π. We let S be the direct limit of sets induced by these maps. If we think of each $^{\uparrow}$ as an inclusion map, as is common, then S is simply the union of all S_n. For our purposes a *permutation statistic* is simply a function $T : \mathrm{S} \to \mathbb{N}$.

Of course it is always possible to construct a statistic T by for each n making it equal to a given statistic T_n on S_n. However unless the values of T cohere for different values of n this is not useful. We define a statistic on S to be *coherent* if it satisfies the following property. To be coherent, the identity $T(\pi) = T(\pi^{\uparrow})$ must hold for all n and $\pi \in \mathrm{S}_n$.

We now define a (coherent) permutation statistic, which we denote by DIS, as follows.

Definition 2. At step i of the algorithm described above, one symbol moves rightward a distance $d_i = i - j_i$ (possibly zero), and one symbol moves leftward the same distance. We define $\mathrm{DIS}(\pi) = \sum_i d_i$, the total distance moved rightward by all elements.

There is an alternative interpretation of DIS. The sequence of moves that formed π starting from the identity will take π^{-1} to the identity, and the moves are the same as selection sort. The algorithm then sorts π^{-1} via selection sort. We can think of DIS as a measure of the work done by selection sort when comparisons have zero cost. This model might be useful in analysing, for example, physical rearrangement of very heavy distinct objects.

In view of the last paragraph it makes sense also to consider the statistic IDIS given by $\mathrm{IDIS}(\pi) = \mathrm{DIS}(\pi^{-1})$.

Example 3. *For our running example* 25413*, the value of* DIS *is* $(1+2+3+4+5) - (1+1+1+1+3) = 8$*, while for* IDIS *the value is* 6*. In terms of* $\pi^{-1} = 41532$*, the swaps used to create* π *yield successively* $41235, 31245, 21345, 12345$*.*

Given a permutation for which we do not already know the j_i, we can find these easily.

Example 4. *Given* $\pi = 25413$ *as above, we can read off* $j_5 = 3$ *from* π*. Thus multiplying* π *on the right by the transposition* (35) *leads to* 23415*. We have now reduced to* $\pi = 2341$*. We now read off* $j_4 = 1$ *and reduce to* $\pi = 231$*. Continuing in this way we obtain* $j_3 = 1, j_2 = 1, j_4 = 1$*.*

At first sight it may appear that we must search to find the position of symbol i at step i, leading to a quadratic time algorithm for the procedure of the last example. However this is not the case, provided we compute DIS and IDIS simultaneously, and the entire computation can be done in linear time (note that computing π^{-1} from π is a linear time operation). Note that, for example, it is still unknown whether the number of inversions INV of a permutation can be computed in linear time.

Example 5. *In the running example* $\pi = 25413, \pi^{-1} = 41352$*, we read off* $j_5(\pi) = 3, j_5(\pi^{-1}) = 2$*. To multiply* π *on the right by the transposition* (35) *we need not scan all of* π*, because we know the location of the symbol* 5*, namely* $j_5(\pi^{-1})$*. Thus the multiplication takes constant time. We can either multiply* π^{-1} *on the left by* (35) *or on the right by* (25)*. Each leads to the same answer, namely* 41235*, and this is the inverse of the updated* π*. Continuing in this way we obtain the result of the last example.*

It will be helpful to know the values of DIS on some special permutations.

Example 6. *We define*

$$\begin{aligned}\pi_0 &= n(n-1)\dots 1\\ \pi_1 &= 2\dots n1\\ \pi_1^{-1} &= n12\dots(n-1)\end{aligned}$$

Note that π_0 *is created by the algorithm by choosing* $j_i = n+1-i$ *provided* $n+1-i < i$*, whereupon all later swaps are trivial. Also* π_1 *is created by choosing* $j_i = 1$ *for all* i*, while* π_1^{-1} *is formed by choosing* $j_i = i-1$ *for* $i \geqslant 2$*. Thus*

$$\begin{aligned}\mathrm{DIS}(\iota) &= 0\\ \mathrm{DIS}(\pi_0) = \lfloor n^2/4 \rfloor &= \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;}\\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.}\end{cases}\\ \mathrm{DIS}(\pi_1) &= n(n-1)/2\\ \mathrm{DIS}(\pi_1^{-1}) &= n-1\end{aligned}$$

The maximum value of DIS on S_n is $n(n-1)/2$, corresponding uniquely to the n-cycle π_1. The minimum value of DIS on S_n is 0, corresponding uniquely to the identity ι.

As a random variable, the restriction DIS_n of DIS to S_n is the sum of DIS_{n-1} and a random variable U_n that is uniform on $[0, 1, \dots, n-1]$. Thus, iterating this recurrence, we see that DIS_n has probability generating function $F_n(q) := \prod_{i=1}^{n} \frac{1-q^i}{1-q}$. This is the definition of a *Mahonian statistic* on S_n. Note that $n(n-1)/2 - \mathrm{DIS} = \sum_i j_i$ is also Mahonian by the symmetry of the Mahonian distribution.

2 DIS is not trivially equal to a known statistic

Tabulating numerical values makes it clear that DIS is not equal to any of the most well-known Mahonian statistics. Table 1 gives the values of DIS and several other Mahonian statistics when $n = 4$ (it is amusing to note that they all coincide on the element 2134 - the obvious conjecture that they always coincide on $2134 \dots n$ is in fact correct). These statistics are INV, MAJ, DEN, MAD, MAK, HAG. We recall the unified definition of these statistics given in [CSZ97]. We first require some partial statistics.

Table 1: Values of some permutation statistics for $n = 4$.

π	DIS	INV	MAJ	DEN	MAD	MAK	HAG
1234	0	0	0	0	0	0	0
1243	1	1	3	3	1	3	3
1324	1	1	2	2	1	2	2
1342	3	2	3	5	1	4	5
1423	2	2	2	2	2	2	3
1432	2	3	5	3	2	5	2
2134	1	1	1	1	1	1	1
2143	2	2	4	4	2	4	4
2314	3	2	2	3	1	3	3
2341	6	3	3	6	1	5	6
2413	4	3	2	3	2	3	4
2431	4	4	5	4	2	6	3
3124	2	2	1	1	2	1	2
3142	4	3	4	4	3	5	5
3214	2	3	3	2	2	3	1
3241	5	4	4	5	2	6	4
3412	4	4	2	3	2	3	4
3421	5	5	5	4	2	6	3
4123	3	3	1	1	3	1	3
4132	3	4	4	2	4	4	2
4213	3	4	3	2	3	3	2
4231	3	5	4	3	3	5	1
4312	5	5	3	4	3	3	5
4321	4	6	6	5	3	6	4

Definition 7. A *descent* is an occurrence of the event $\pi(i) > \pi(i+1)$. The index i is the *descent bottom* and $\pi(i)$ is the corresponding *descent top*.

Each π can be uniquely decomposed into *descent blocks* (maximal descending subwords). Denote the first and last letter of each block of length at least 2 by $c(B), o(B)$. The *right embracing number* of $\pi(i)$ is the number of descent blocks strictly to the right

of the block containing $\pi(i)$ and for which $c(B) > \pi(i) > o(B)$. The sum of all right embracing numbers is denoted by $\mathrm{Res}(\pi)$.

Example 8. *For π_1 the descent blocks are all of length 1 except for the last one, $n1$. The right embracing number of each letter $2, \ldots, n-1$ is 1 and the right embracing number of n and of 1 are each 0. For π_1^{-1} there is again a single nontrivial descent block, namely $n1$, and all right embracing numbers are 0. For π_0 there is a single descent block of length n and all right embracing numbers are 0.*

Definition 9. An *excedance* is an occurrence of the event $\pi(i) > i$. The index i is the *excedance bottom* and $\pi(i)$ is the corresponding *excedance top*. The sum of all descent/excedance tops/bottoms of π we denote by $\mathrm{Dtop}(\pi), \mathrm{Etop}(\pi), \mathrm{Dbot}(\pi), \mathrm{Ebot}\,\pi$. The differences $\mathrm{Ddif}(\pi)$ and $\mathrm{Edif}(\pi)$ are given by $\mathrm{Ddif} = \mathrm{Dtop} - \mathrm{Dbot}, \mathrm{Edif} = \mathrm{Etop} - \mathrm{Ebot}$.

There is a unique decomposition π into π_E and π_N, where π_E is the subsequence formed by excedances and π_N the subsequence formed by nonexcedances. For our running example $\pi = 25413$, we have $\pi_E = 25$ and $\pi_N = 413$. For the inverse 41532 we have respectively 45 and 132. We define $\mathrm{Ine}(\pi) = \mathrm{INV}(\pi_E) + \mathrm{INV}(\pi_N)$.

For each excedance bottom i we define $\mathrm{L}(i)$ to be the number of indices k such that $k < i$ and $a_k \leqslant i$; let L be the sum over all such i.

Example 10. *Note that $(\pi_1)_E = 23\ldots n$ and $(\pi_1)_N = 1$. Similarly $(\pi_1^{-1})_E = n$ and $(\pi_1^{-1})_N = 12\ldots(n-1)$. Also $(\pi_0)_E = n(n-1)\ldots t+1$ and $(\pi_0)_N = t\ldots 1$, where $t = \lfloor n/2 \rfloor$.*

The values of the partial statistics defined above are tabulated in Table 2.

Proposition 11 ([CSZ97])**.** *We have*

$$\begin{aligned}
\mathrm{MAK} &= \mathrm{Dbot} + \mathrm{Res} \\
\mathrm{MAD} &= \mathrm{Ddif} + \mathrm{Res} \\
\mathrm{DEN} &= \mathrm{Ebot} + \mathrm{Ine} \\
\mathrm{INV} &= \mathrm{Edif} + \mathrm{Ine} \\
\mathrm{HAG} &= \mathrm{Edif} + \mathrm{INV}(\pi_E) - \mathrm{INV}(\pi_N) + \mathrm{L}
\end{aligned}$$

In addition MAJ *is the sum of indices corresponding to descent tops.*

2.1 Trivial bijections

To show that statistics T and T' are different, it suffices to find some n and some $\pi \in \mathrm{S}_n$ for which $T(\pi) \neq T'(\pi)$. However it may be the case that T and T' agree on S_n for some larger values of n. If both T and T' are coherent, this possibility cannot occur.

Note that DIS and IDIS, along with all statistics from previous literature with which we compare them here, are coherent. Thus simply computing values for small n, as in the

Table 2: Values of partial statistics on special permutations ($t = \lfloor n/2 \rfloor$)

perm	Ebot	Edif	Dbot	Ddif	Res	Ine	L
π_0	$t(t+1)/2$	$\lfloor n^2/4 \rfloor$	$n(n-1)/2$	$n-1$	0	$\lfloor (n-1)^2/4 \rfloor$	0
π_1	$n(n-1)/2$	$n-1$	1	$n-1$	$n-2$	0	$(n-1)(n-2)/2$
π_1^{-1}	1	$n-1$	1	$n-1$	0	0	0

previous section, is usually enough to distinguish the statistics. However we can often give a general construction of permutations for which a given pair of statistics differs greatly.

Although DIS does not equal any of the well-known statistics of the previous section, it is possible *a priori* that DIS has the form $S \circ g$ where S is a known Mahonian statistic and g is a filtered bijection of S (a bijection of S that bijectively takes S_n to S_n for each n).

In this section we consider the so-called "trivial" involutions of S_n (there is a nontrivial bijection Φ of S_n introduced in [CSZ97]; we give more details in Section 2.2.) These involutions are inversion (group-theoretic inverse), reversal (reverse the order of the letters) and complementation (subtract each letter from $n+1$). Then in the obvious notation R and C commute and $IR = CI, IC = RI$. Thus I, R, C generate a group G isomorphic to the dihedral group of order 8.

For example we have

$$\begin{aligned}(25413)^I &= 41532\\ (25413)^R &= 31452\\ (25413)^C &= 41253\\ (25413)^{IC} &= 25134\\ (25413)^{IR} &= 23514\\ (25413)^{RC} &= 35214\\ (25413)^{IRC} &= 41532\end{aligned}$$

We shall show that DIS is not trivially equivalent to any well-known statistic. In the absence of a standardized database of permutation statistics, we define "well-known" to mean "mentioned in at least one of the papers [CSZ97, BS00]". We define Σ to be the set consisting of well-known Mahonian statistics.

In [BS00] it is shown how all known "descent-based" Mahonian statistics can be written in terms of "Mahonian d-functions" for some $d \leqslant 4$. Each such d-function simply computes the numbers of occurrences of a certain generalized permutation pattern of length at most d, then sums this process over a finite number of such patterns. In particular in Table 1 of the above article, all 14 Mahonian 3-functions (up to trivial bijections) are given. In [CSZ97] the images of these statistics under a bijection Φ were also considered. We consider this bijection in Section 2.2.

Theorem 12. *There do not exist $S \in \Sigma$ and $g \in G$ such that* $\mathrm{DIS} = S \circ g$.

Proof. Note that $\pi_0^R = \iota = \pi_0^C$ while π_0 is a product of $\lfloor n/2 \rfloor$ disjoint transpositions, and hence $\pi_0^I = \pi_0$. Hence the orbit of π_0 under G is the set $\{\iota, \pi_0\}$ and this is also the orbit of ι. The orbit of π_1 under G is disjoint from that of π_0 and ι. It consists of $\pi_1, \pi_1^{-1}, \pi_1^R = \pi_1^C = 1n \dots 2, \pi_1^{-R} = (n-1) \dots 1n$.

It follows that if $S \circ g = \mathrm{DIS}$ for some permutation statistic S and element $g \in G$, then $S(\pi_0)$ must equal zero or $\mathrm{DIS}(\pi_0)$. However it is readily seen by comparing with Table 2 that none of the statistics in [CSZ97] satisfy this property. This includes those mentioned in passing, such as LAG and SIST.

Now consider the statistics in [BS00, Table 1], given in terms of permutation pattern counts. Any pattern that is not strictly descending does not occur in π_0, so we need only count occurrences of $ba, cba, cb-a, c-ba$. Again, none of these lead to zero or $\mathrm{DIS}(\pi_0)$, since the number of occurrences of these four patterns in π_0 is respectively $n-1, n-2, (n-1)(n-2)/2, (n-1)(n-2)/2$.

Finally we consider Haglund's statistic HAG and a descent-based variant DAG as defined in [BS00]. The statistic DAG can be dealt with by counting pattern occurrences in π_0 as above. However it is not as easy to differentiate HAG from DIS by using our special permutations. In fact when n is even, HAG and DIS take the same value on π_0 (they coincide with Edif). When n is odd, HAG is smaller than DIS by $(n-1)/2$. We instead use the permutation $\pi_2 = n2 \dots (n-1)1$ formed from ι by a single transposition. Its orbit under G consists of itself and its reverse $1(n-1) \dots 2n$, and DIS takes the values $n-1$ and $\lceil (n-2)^2/4 \rceil$ respectively on these two elements. However, $\mathrm{HAG}(\pi_2) = 1$. □

2.2 Euler-Mahonian pairs and nontrivial bijections

In [CSZ97] a bijection Φ of S_n was given and it was shown that Φ had appeared (somewhat disguised) in several previous papers. The key property of Φ is that it takes $(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res})$ to $(\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})$. This then gives access to equidistribution results for Euler-Mahonian pairs. The term *Euler-Mahonian* refers in the literature to a bistatistic (e, M) such that e is Eulerian, M is Mahonian, and the joint distribution of (e, M) is the same as that of another well-known pair (e', M'). Originally the term was used only for $(e', M') = (\mathrm{des}, \mathrm{MAJ})$. Other authors, for example [BS00, CSZ97] allow more possibilities for (e', M'), and aim to classify these bistatistics up to equidistribution.

In [BS00, Table 2] seven equivalence classes (under equidistribution) of Euler-Mahonian pairs (des, T) were given for $n = 5$ (note that the second matrix, corresponding to MAJ, has an error: in the row indexed by des = 2, the entries listed as 14 should be 16). This corresponds to 14 Mahonian statistics T. It is easy to see that DIS does not occur in this table, because its maximum value occurs on π_1 and $\mathrm{des}(\pi_1) = 1$, yet none of the seven distributions has a nonzero entry in the $(1, 10)$ position. We can also check easily that $(\mathrm{exc}, \mathrm{DIS})$ has a different distribution from all the entries in the table. Thus if T' is the image of such a T under Φ, then $T' \neq \mathrm{DIS}$.

1	0	0	0	0	0	0	0	0	0	0
0	4	6	8	8	0	0	0	0	0	0
0	0	3	7	10	22	15	9	0	0	0
0	0	0	0	2	0	5	6	9	4	0
0	0	0	0	0	0	0	0	0	0	1

1	0	0	0	0	0	0	0	0	0	0
0	4	3	5	3	3	3	2	1	1	1
0	0	6	6	13	12	9	9	8	3	0
0	0	0	4	3	7	8	4	0	0	0
0	0	0	0	1	0	0	0	0	0	0

Table 3: Joint distributions $(\mathrm{exc}, \mathrm{DIS})$ and $(\mathrm{exc}, \mathrm{IDIS})$ for $n = 5$.

1	0	0	0	0	0	0	0	0	0	0
0	4	3	5	5	2	3	2	1	0	1
0	0	6	8	12	14	11	7	5	3	0
0	0	0	2	3	6	5	6	3	1	0
0	0	0	0	0	0	1	0	0	0	0

1	0	0	0	0	0	0	0	0	0	0
0	4	3	5	5	2	2	3	1	0	1
0	0	6	8	12	15	11	6	5	3	0
0	0	0	2	3	5	6	6	3	1	0
0	0	0	0	0	0	1	0	0	0	0

Table 4: Joint distributions $(\mathrm{des}, \mathrm{DIS})$ and $(\mathrm{des}, \mathrm{IDIS})$ for $n = 5$.

We still need to check HAG. By direct computation we can show readily that

$$\Phi(\pi_0) = \pi_1$$
$$\Phi(\pi_1) = (\pi_1)^{-1}$$

Suppose that $T = \mathrm{DIS} \circ \Phi$ for some $T \in \Sigma$. Then $T(\pi_0) = n(n-1)/2$ and $T(\pi_1) = n-1$. Clearly HAG fails this test. If $T \circ \Phi = \mathrm{DIS}$ then $T(\pi_1) = \lceil n^2/4 \rceil$ which again HAG fails.

3 An Eulerian partner for DIS

Skandera [Ska02] gave a general procedure for associating to each Mahonian statistic M another statistic e that is Eulerian and such that the pair (e, M) is Euler-Mahonian

(equidistributed with (des, MAJ)). Of course, such an Eulerian statistic may not be known or particularly interesting.

Applying this procedure to DIS yields an Eulerian statistic eul. Concretely, $\mathrm{eul}(\pi)$ is obtained from the numbers d_i by listing them in order, and counting each time we encounter a number larger than the current record (the record being initialized to zero). For example, for our running example 25413 we have $d = (0, 1, 2, 3, 2)$ and so eul takes the value 3. Also note that $\mathrm{eul}(\pi_0) = \lceil (n-1)/2 \rceil$ while $\mathrm{eul}(\iota) = 0$ and $\mathrm{eul}(\pi_1) = n - 1$.

A well-known Eulerian statistic is the number of excedances exc. Now exc agrees with eul on π_0 and π_1. Also, eul and exc are equal when $n = 3$. Nevertheless, eul is not equal to exc in general, nor does it equal des.

Eulerian statistics in the literature are less commonly found than Mahonian ones. As far as I am aware, eul is itself new, but this is based on much less evidence than the corresponding claim about DIS.

4 Further comments

The current paper gives substantial evidence that the statistic DIS is really new. In order to check thoroughly whether a permutation statistic is new to the literature, one would ideally check a database of such statistics. I have not found such a database. I propose that as a minimum, tables of values for $n = 4$, along with the joint distribution with exc and des for $n = 5$, be included in all papers dealing with this topic, to allow easy comparison. It would then be much easier to show that the entire group Γ generated by G and Φ does not have any element g with $T \circ g = \mathrm{DIS}$ for some known Mahonian T, since all such T of which I am aware are consistent.

It may be desirable to find a "static" description of DIS and IDIS, which have been defined "dynamically". I do not know a systematic way to do this (one possible idea is to find linear combinations of the above partial statistics that fit the values for small n). A related question is to determine whether DIS can be written as a Mahonian d-function for some d.

The statistic IDIS should extend to words via the selection sort interpretation. Whether this statistic is Mahonian on words should be investigated and I intend to do this in future work. Dennis White has informed me that a general machine explained by him and Jennifer Galovich in [GW96] will construct a bijection between (nearly) any Mahonian statistic and INV. The machine works in the context of words, and this connection may be worth pursuing.

Note: As I was preparing this article I was made aware of completely independent recent work by T. Kyle Petersen [Pet10] that also discusses the statistic DIS and some generalizations. The intersection between the topics of these two papers is small, and the reader should consult both articles for a fuller picture.

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