The nonexistence of regular near octagons with parameters $(s, t, t_2, t_3) = (2, 24, 0, 8)$

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Abstract

Let $\mathcal{S}$ be a regular near octagon with $s + 1 = 3$ points per line, let $t + 1$ denote the constant number of lines through a given point of $\mathcal{S}$ and for every two points x and y at distance $i \in \{2, 3\}$ from each other, let $t_i + 1$ denote the constant number of lines through y containing a (necessarily unique) point at distance $i - 1$ from x. It is known, using algebraic combinatorial techniques, that (t_2, t_3, t) must be equal to either $(0, 0, 1)$, $(0, 0, 4)$, $(0, 3, 4)$, $(0, 8, 24)$, $(1, 2, 3)$, $(2, 6, 14)$ or $(4, 20, 84)$. For all but one of these cases, there is a unique example of a regular near octagon known. In this paper, we deal with the existence question for the remaining case. We prove that no regular near octagons with parameters $(s, t, t_2, t_3) = (2, 24, 0, 8)$ can exist.

1 Introduction

A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near polygon* if for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$ there exists a unique point on L nearest to p. Here, distances are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near $2d$-gon*. The near 0-gons are the points and the near 2-gons are the lines. Near quadrangles are usually called generalized quadrangles. Near polygons were introduced 30 years ago by Shult and Yanushka [19].

A near $2d$-gon, $d \geqslant 2$, is called *regular* if there exist constants s, t, t_i ($i \in \{0, 1, \ldots, d\}$) such that every line is incident with precisely $s + 1$ points, every point is incident with precisely $t+1$ lines and for every two points x and y at distance i from each other there are precisely t_i+1 lines through y containing a (necessarily unique) point at distance $i-1$ from x. Clearly, we have $t_0 = -1$, $t_1 = 0$ and $t_d = t$. The numbers s, t, t_i ($i \in \{2, \ldots, d - 1\}$) are called the *parameters* of the regular near polygon. A near $2d$-gon, $d \geqslant 2$, is regular if and only if its collinearity graph is a so-called distance-regular graph. In the book *Distance-regular graphs* [2] by Brouwer, Cohen and Neumaier, (the collinearity graphs of) regular near polygons are regarded as one of the main classes of distance-regular graphs.

The parameters of a regular near polygon must satisfy a number of restrictions, like inequalities and certain numbers (that depend on those parameters) which need to be integral. There are standard techniques for calculating the eigenvalues and corresponding multiplicities of the collinearity graph of a regular near polygon, see e.g. [2]. The fact that all these multiplicities are nonnegative integers, gives severe restrictions on the parameters. Other restriction on the parameters are known, see e.g. Brouwer and Wilbrink [3], Hiraki and Koolen [9, 10, 11], Neumaier [12] and Terwilliger and Weng [16]. There are a number of theorems guaranteeing the existence of sub-near-polygons, see e.g. Shult and Yanushka [19, Proposition 2.5], Brouwer and Wilbrink [3, Theorem 4] and Hiraki [8, Corollary 1.2]. The existence of these subgeometries can be used to derive additional restrictions on the parameters.

In the present paper, we are interested in the case of regular near octagons with 3 points per line ($d = 4$, $s = 2$). The various parameter restrictions imply that there are only a finite number of possibilities for (t_2, t_3, t). Indeed, we have $t_2 \in \{0, 1, 2, 4\}$ since $t_2 \geqslant 1$ implies that every two points at distance 2 are contained in a so-called quad (Shult and Yanushka [19, Proposition 2.5]). The order (s, t_2) of each such quad must be equal to $(2, 1)$, $(2, 2)$ or $(2, 4)$ by Payne and Thas [13, Section 6.1]. By Neumaier [12, Theorem 3.1], we have $t_3 + 1 \leqslant \frac{(s^3+1)(t_2+1+s)}{s+1} \leqslant 21$ and by Brouwer and Wilbrink [3, p. 161], we have $t + 1 \leqslant (s^2 + 1)(t_3 + 1) \leqslant 105$.

In the following table, we list all the possibilities for (t_2, t_3, t) which remain after verifying the various parameter restrictions we have found in the literature. For each possibility of (t_2, t_3, t), we give the number of regular near octagons having these parameters.

(t_2, t_3, t)	Number
$(0, 0, 1)$	1
$(0, 0, 4)$	$\geqslant 1$
$(0, 3, 4)$	1
$(0, 8, 24)$	?
$(1, 2, 3)$	1
$(2, 6, 14)$	1
$(4, 20, 84)$	1

These possibilities already occur in Shad [15, page 82, Theorem 1.5]. In fact, in [15] one more possibility for (t_2, t_3, t) was mentioned, namely $(t_2, t_3, t) = (1, 11, 39)$, but this possibility has been ruled out by Brouwer and Wilbrink [3, page 165].

There is only one regular near octagon with parameters $(s, t_2, t_3, t) = (2, 0, 0, 1)$. It is a generalized octagon which is the point-line dual of the double of the unique generalized quadrangle of order 2. The regular near octagons with parameters $(s, t_2, t_3, t) = (2, 0, 0, 4)$ are precisely the generalized octagons of order $(2, 4)$. Up to now, there is only one such generalized octagon known. It belongs to the family of the so-called Ree-Tits generalized octagons which were first constructed by Tits in [18] using a new family of simple groups discovered by Ree [14]. There exists a unique regular near octagon with parameters $(s,t_2,t_3,t)=(2,0,3,4)$. It is related to the Hall-Janko simple group. It was first constructed in Cohen [5] and its uniqueness was proved in Cohen and Tits [6]. There exists a unique regular near octagon with parameters $(s,t_2,t_3,t)=(2,1,2,3)$, namely the Hamming near octagon with three points on each line. The unique regular near octagons with parameters $(s,t_2,t_3,t)=(2,2,6,14)$ and $(s,t_2,t_3,t)=(2,4,20,84)$ are respectively isomorphic to $DW(7,2)$ and $DH(7,4)$, see Cameron [4] and Brouwer & Wilbrink [3, Lemma 26 and Section (i)]. The regular near octagons $DW(7,2)$ and $DH(7,4)$ are the dual polar spaces (in the sense of Cameron [4]) respectively related to the symplectic polar space $W(7,2)=W_7(2)$ and the Hermitian polar space $H(7,4)$ (Thas [17, Section 9.1]).

There is one possibility, namely $(s,t_2,t_3,t)=(2,0,8,24)$, for which the existence of the corresponding regular near octagons was not yet settled. In this paper, we deal with this remaining case. The following is our main result.

Theorem 1.1 *No regular near octagons exist whose parameters (s,t_2,t_3,t) are equal to $(2,0,8,24)$.*

Remarks. (1) If $\mathcal{S}$ is a regular near octagon with parameters $(s,t_2,t_3,t)=(2,0,8,t)$, then by Neumaier [12, Theorem 3.1], $t+1\geqslant\frac{(s^4-1)(t_3+1-s^2)}{s^2-1}=25$. So, for the regular near octagons under investigation in this paper, this inequality becomes an equality.

(2) As told before, there are some results guaranteeing the existence of sub-near-polygons ([19, Proposition 2.5], [3, Theorem 4] and [8, Corollary 1.2]) and such sub-near-polygons are often helpful for proving the nonexistence of certain near polygons. The necessary conditions for applying these results are however not satisfied here.

(3) If a regular near octagon with parameters $(s,t_2,t_3,t)=(2,0,8,24)$ would have existed, the eigenvalues of its collinearity graph would have been equal to $\lambda_0=s(t+1)=50$, $\lambda_1=13$, $\lambda_2=5$, $\lambda_3=-7$ and $\lambda_4=-(t+1)=-25$. The corresponding multiplicities would have been equal to $m_0=1$, $m_1=2700$, $m_2=14060$, $m_3=14800$ and $m_4=74$.

2 Proof of Theorem 1.1

Let $\mathcal{S}$ be a regular near octagon with parameters $(s,t_2,t_3,t)=(2,0,8,24)$ and let Γ be its collinearity graph. If x is a point of $\mathcal{S}$, then $|\Gamma_0(x)|=1$, $|\Gamma_1(x)|=s(t+1)=50$, $|\Gamma_2(x)|=\frac{s^2(t+1)t}{t_2+1}=2400$, $|\Gamma_3(x)|=\frac{s^3(t+1)t(t-t_2)}{(t_2+1)(t_3+1)}=12800$ and $|\Gamma_4(x)|=\frac{s^4t(t-t_2)(t-t_3)}{(t_2+1)(t_3+1)}=16384$. So, the total number of vertices of $\mathcal{S}$ is equal to 31635.

Let x be a point of $\mathcal{S}$. Then $\mathcal{L}_x$ denotes the set of lines through x and Γ_x denotes the subgraph of Γ induced on the set $\Gamma_3(x)$. We denote by $\mathcal{C}_x$ the set of all connected components of Γ_x. If $y\in\Gamma_3(x)$, then $B(x,y)$ denotes the set of $t_3+1=9$ lines through x which contain a point at distance 2 from y. We define $\mathcal{B}_x:=\{B(x,y)\,|\,y\in\Gamma_3(x)\}$. Let $\mathcal{D}_x=(\mathcal{L}_x,\mathcal{B}_x,\mathrm{I}_x)$ be the point-line geometry with point set $\mathcal{L}_x$, line set $\mathcal{B}_x$ and natural incidence relation I_x.

Let x be a point of $\mathcal{S}$. If y_1 and y_2 are two adjacent vertices of Γ_x, then since $\mathrm{d}(x, y_1) = \mathrm{d}(x, y_2) = 3$, we have $\mathrm{d}(x, z) = 2$ where z is the unique point of the line $y_1 y_2$ distinct from y_1 and y_2. Since $t_2 = 0$, there exists a unique line L through x containing a point collinear with z. We say that the vertices y_1 and y_2 of Γ_x are *L-adjacent*. Clearly, L is contained in $B(x, y_1)$ and $B(x, y_2)$.

Lemma 2.1 *For every point x of Γ, the graph Γ_x has valency 9. More precisely, for every vertex y_1 of Γ_x and every line $L \in B(x, y_1)$, there exists a unique vertex of Γ_x which is L-adjacent with y_1.*

Proof. Let L be a line of $B(x, y_1)$, let u denote the unique point of L at distance 2 from y_1 and let K denote the unique line through y_1 containing a point z at distance 1 from u. Then the unique point y_2 of K distinct from y_1 and z is L-adjacent to y_1. Conversely, if y_2' is a vertex of Γ_x which is L-adjacent to y_1, then the line $y_1 y_2'$ must contain a point collinear with u and hence coincides with K. This implies that $y_2' = y_2$.

So, for each of the nine lines L of $B(x, y_1)$, there exists a unique vertex of Γ_x which is L-adjacent to y_1. Hence, the vertex y_1 of Γ_x has degree 9. ■

Lemma 2.2 *Let x be a point of $\mathcal{S}$ and let $y_1, y_2 \in \Gamma_3(x)$. If y_1 and y_2 belong to the same connected component of Γ_x, then $B(x, y_1) = B(x, y_2)$.*

Proof. It suffices to prove the lemma in the case that y_1 and y_2 are adjacent vertices of Γ_x. By symmetry, it suffices to prove the inclusion $B(x, y_1) \subseteq B(x, y_2)$.

Let L be an arbitrary element of $B(x, y_1)$ and let z denote the unique point on L at distance 2 from y_1. Since y_1 and y_2 are collinear, we have $\mathrm{d}(y_2, z) \leqslant 3$. Since $\mathrm{d}(y_2, x) = 3$, the unique point of L nearest to y_2 lies at distance 2 from y_2, proving that $L \in B(x, y_2)$. Since L was an arbitrary line of $B(x, y_1)$, we have $B(x, y_1) \subseteq B(x, y_2)$ as we needed to prove. ■

Let $\Sigma := \{+, -\}$. Let G be the graph whose vertices are those elements of the cartesian power Σ^9 which contain an odd number of $+$'s, with two vertices adjacent whenever they agree in precisely one position. The graph G is easily seen to be isomorphic to the folded 9-cube discussed in Section 9.2 of Brouwer, Cohen and Neumaier [2]. The following properties of G are clear.

- G has 256 vertices and is a regular graph of diameter 4 and valency 9.
- Two vertices of G agree in an odd number of positions.
- If $m_0 := 9$, $m_1 := 1$, $m_2 := 7$, $m_3 := 3$ and $m_4 := 5$, then two vertices of G lie at distance $i \in \{0, 1, 2, 3, 4\}$ from each other if and only if they agree in precisely m_i positions.

Now, for every two points x and y of $\mathcal{S}$ at distance 3 from each other, a graph $G_{x,y}$ can be defined which is isomorphic to G. Put $\Gamma_1(x) \cap \Gamma_2(y) = \{x_1^+, x_2^+, \ldots, x_9^+\}$ and for every $i \in \{1, 2, \ldots, 9\}$, let x_i^- denote the unique point of the line $x x_i^+$ distinct from x and x_i^+. The vertices of $G_{x,y}$ are the sets of the form $\{x_1^{\epsilon_1}, x_2^{\epsilon_2}, \ldots, x_9^{\epsilon_9}\}$ with $\epsilon_1, \epsilon_2, \ldots, \epsilon_9 \in \{+, -\}$ and $\epsilon_1 \cdot \epsilon_2 \cdot \ldots \cdot \epsilon_9 = +$, with two distinct vertices $\{x_1^{\epsilon_1}, x_2^{\epsilon_2}, \ldots, x_9^{\epsilon_9}\}$

and $\{x_1^{\epsilon_1'}, x_2^{\epsilon_2'}, \ldots, x_9^{\epsilon_9'}\}$ adjacent whenever they have precisely one element in common, or equivalently, if $(\epsilon_1, \epsilon_2, \ldots, \epsilon_9)$ and $(\epsilon_1', \epsilon_2', \ldots, \epsilon_9')$ agree in precisely one position. If two adjacent vertices of $G_{x,y}$ have the element z in common, then we call these vertices *L-adjacent* where L is the unique line through x and z.

Let G_1 and G_2 be two graphs with respective vertex sets $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. For every vertex v of G_i, $i \in \{1, 2\}$, let $v^{\perp_i}$ be the set of vertices of G_i adjacent to v. A surjective map $f : V_1 \to V_2$ is called a *covering map* if for every $v \in V_1$, the restriction of f to $v^{\perp_1}$ is a bijection between $v^{\perp_1}$ and $f(v)^{\perp_2}$. If there exists such a covering map, then G_1 is called a *cover* of G_2. If G_2 is connected and f is a covering map, then there exists an $m \in \mathbb{N} \setminus \{0\}$ such that $|f^{-1}(v)| = m$ for every $v \in V_2$. In this case, G_1 is called an *m-fold cover* of G_2.

Lemma 2.3 *Let x, y_1 and y_2 be three points of $\mathcal{S}$ such that $y_1, y_2 \in \Gamma_3(x)$ belong to the same connected component C of Γ_x. Then $G_{x,y_1} = G_{x,y_2}$. For every $y \in C$, the set $\theta_{x,C}(y) := \Gamma_2(y) \cap \Gamma_1(x)$ is a vertex of $G_{x,y_1} = G_{x,y_2}$. If $L \in B(x, y_1) = B(x, y_2)$ and if z_1 and z_2 are two L-adjacent vertices of C, then $\theta_{x,C}(z_1)$ and $\theta_{x,C}(z_2)$ are L-adjacent vertices of $G_{x,y_1} = G_{x,y_2}$. As a consequence, $\theta_{x,C}$ is a covering map.*

Proof. Suppose z_1 and z_2 are two adjacent vertices of C. Put $\Gamma_2(z_1) \cap \Gamma_1(x) = \{x_1^+, x_2^+, \ldots, x_9^+\}$ and for every $i \in \{1, 2, \ldots, 9\}$, let x_i^- denote the unique point of the line $L_i := xx_i^+$ distinct from x and x_i^+. By Lemma 2.2, $\Gamma_2(z_2) \cap \Gamma_1(x) = \{x_1^{\epsilon_1}, x_2^{\epsilon_2}, \ldots, x_9^{\epsilon_9}\}$ for some $\epsilon_1, \epsilon_2, \ldots, \epsilon_9 \in \{+, -\}$. Now, let z denote the unique point of z_1z_2 distinct from z_1 and z_2. Since $\mathrm{d}(x, z_1) = \mathrm{d}(x, z_2) = 3$, we have $\mathrm{d}(x, z) = 2$ and so x and z have a unique common neighbor. Clearly, $\Gamma_1(x) \cap \Gamma_1(z) = \{x_j^+\}$ for some $j \in \{1, 2, \ldots, 9\}$. We have $x_j^+ \in \Gamma_2(z_2)$ and hence $\epsilon_j = +$. Conversely, suppose that $\epsilon_i = +$ for some $i \in \{1, 2, \ldots, 9\}$. Then since $\mathrm{d}(x_i^+, z_1) = \mathrm{d}(x_i^+, z_2) = 2$, we have $\mathrm{d}(x_i^+, z) = 1$. So, x_i^+ is a common neighbor of x and z and hence $i = j$. So, $\epsilon_i = -$ for every $i \in \{1, 2, \ldots, 9\} \setminus \{j\}$. Notice that the vertices z_1 and z_2 are L_j-adjacent vertices of C and that $\theta_{x,C}(z_1) = \Gamma_2(z_1) \cap \Gamma_1(x)$ and $\theta_{x,C}(z_2) = \Gamma_2(z_2) \cap \Gamma_1(x)$ are L_j-adjacent vertices of G_{x,z_1}.

Now, the vertex set of G_{x,z_1} consists of all sets of the form $\{x_1^{\epsilon_1'}, x_2^{\epsilon_2'}, \ldots, x_9^{\epsilon_9'}\}$ with $\epsilon_1', \epsilon_2', \ldots, \epsilon_9' \in \{+, -\}$ such that $\epsilon_1' \cdot \epsilon_2' \cdot \ldots \cdot \epsilon_9' = +$. The vertex set of G_{x,z_2} on the other hand consists of all sets of the form $\{x_1^{\epsilon_1'}, x_2^{\epsilon_2'}, \ldots, x_9^{\epsilon_9'}\}$ with $\epsilon_1', \epsilon_2', \ldots, \epsilon_9' \in \{+, -\}$ such that $(\epsilon_1'\epsilon_1) \cdot (\epsilon_2'\epsilon_2) \cdot \ldots \cdot (\epsilon_9'\epsilon_9) = +$. Since $\epsilon_1 \cdot \epsilon_2 \cdot \ldots \cdot \epsilon_9 = +$, the vertex sets of G_{x,z_1} and G_{x,z_2} coincide. Hence, also the graphs G_{x,z_1} and G_{x,z_2} coincide.

The lemma now follows from the above discussion and the connectedness of C. ∎

For every point x of $\mathcal{S}$ and every $C \in \mathcal{C}_x$, let $A_{x,C} \in \mathbb{N} \setminus \{0\}$ such that C is an $A_{x,C}$-fold cover of $G_{x,y}$ with associated covering map $\theta_{x,C}$. Here, y is an arbitrary element of C. Clearly, $|C| = 256 \cdot A_{x,C}$.

Lemma 2.4 *For every vertex x of $\mathcal{S}$ and every connected component C of Γ_x, we have $A_{x,C} \geqslant 2$.*

Proof. Let y_1 be an arbitrary point of C and let L_1 and L_2 be two distinct lines of $B(x, y_1)$. Let y_2 be the unique vertex of C which is L_1-adjacent to y_1, let y_3 be the unique vertex of C which is L_2-adjacent to y_2, let y_4 be the unique vertex of C which is L_1-adjacent to y_3 and let y_5 be the unique vertex of C which is L_2-adjacent to y_4. By Lemma 2.3, $\theta_{x,C}(y_1) = \theta_{x,C}(y_5)$. Since $t_2 = 0$, there are no quadrangles in C. Hence, $y_1 \neq y_5$ and $A_{x,C} \geqslant 2$. ■

Lemma 2.5 *For every point x of $\mathcal{S}$, we have $|\mathcal{C}_x| \leqslant 25$ and $\sum_{C \in \mathcal{C}_x} A_{x,C} = 50$. Moreover, if $|\mathcal{C}_x| = 25$, then $A_{x,C} = 2$ and $|C| = 512$ for every $C \in \mathcal{C}_x$.*

Proof. We have $12800 = |\Gamma_3(x)| = \sum_{C \in \mathcal{C}_x} |C| = \sum_{C \in \mathcal{C}_x} 256 \cdot A_{x,C}$. Hence, $\sum_{C \in \mathcal{C}_x} A_{x,C} = 50$. Since $A_{x,C} \geqslant 2$ for every $C \in \mathcal{C}_x$, we have $|\mathcal{C}_x| \leqslant 25$. Clearly, if $|\mathcal{C}_x| = 25$, then $A_{x,C} = 2$ and $|C| = 256 \cdot A_{x,C} = 512$ for every $C \in \mathcal{C}_x$. ■

Lemma 2.6 *Let x be a point of $\mathcal{S}$ and let $(y_1, y_2, \ldots, y_{50})$ be a 50-tuple*[1] *of points of $\Gamma_3(x)$ satisfying the following property: for every $C \in \mathcal{C}_x$, there are precisely $A_{x,C}$ elements $i \in \{1, 2, \ldots, 50\}$ for which $y_i \in C$. Put $B_i := B(x, y_i)$ for every $i \in \{1, 2, \ldots, 50\}$. Then the following holds.*

(1) *For every line $L \in \mathcal{L}_x$, there are precisely 18 elements $i \in \{1, 2, \ldots, 50\}$ for which $L \in B_i$.*

(2) *For every two distinct lines $L_1, L_2 \in \mathcal{L}_x$, there are precisely 6 elements $i \in \{1, 2, \ldots, 50\}$ for which $L_1, L_2 \in B_i$.*

Proof. (1) Let F denote the set of all points y of $\Gamma_3(x)$ for which $L \in B(x, y)$. By Lemma 2.2, F must be the union of some elements of $\mathcal{C}_x$, i.e. $F = \bigcup_{C \in \mathcal{C}} C$ where $\mathcal{C}$ is some suitable subset of $\mathcal{C}_x$. The number of $i \in \{1, 2, \ldots, 50\}$ for which $L \in B_i$ is equal to $\sum_{C \in \mathcal{C}} A_{x,C} = \sum_{C \in \mathcal{C}} \frac{|C|}{256} = \frac{|F|}{256}$. So, it suffices to compute $|F|$. Put $L = \{x, u_1, u_2\}$ and let F_i, $i \in \{1, 2\}$, denote the set of all points $y \in F$ for which $\{u_i\} = L \cap \Gamma_2(y)$. Then $F = F_1 \cup F_2$. A straightforward calculation shows that $|F_1| = |F_2| = \frac{st \cdot s(t - t_2)}{t_2 + 1} = 2304$. Hence, $|F| = 4608$ and $\sum_{C \in \mathcal{C}} A_{x,C} = 18$.

(2) Let F denote the set of all points y of $\Gamma_3(x)$ for which $L_1, L_2 \in B(x, y)$. By Lemma 2.2, F must be the union of some elements of $\mathcal{C}_x$, i.e. $F = \bigcup_{C \in \mathcal{C}} C$ where $\mathcal{C}$ is some suitable subset of $\mathcal{C}_x$. The number of $i \in \{1, 2, \ldots, 50\}$ for which $L_1, L_2 \in B_i$ is equal to $\sum_{C \in \mathcal{C}} A_{x,C} = \sum_{C \in \mathcal{C}} \frac{|C|}{256} = \frac{|F|}{256}$. So, it suffices to compute $|F|$. Put $L_1 = \{x, u_1, u_2\}$ and let F_i, $i \in \{1, 2\}$, denote the set of all $y \in F$ for which $\{u_i\} = L_1 \cap \Gamma_2(y)$. We compute $|F_i|$. Let v be one of the two points of $L_2 \setminus \{x\}$.

Suppose $y \in F_i$. Then y and u_i have a unique common neighbor z. The point z is one of the $st = 48$ points collinear with u_i not contained on the line L_1 and the line zy is one of the $t_3 = 8$ lines through z distinct from zu_i containing a point at distance 2 from v.

Conversely, if z is one of the 48 points collinear with u_i not contained on the line L_1 and the line M is one of the 8 lines through z distinct from zu_i which contain a point at distance 2 from v, then each of the two points of $M \setminus \{z\}$ belongs to F_i.

It follows that $|F_i| = 48 \cdot 8 \cdot 2 = 768$, $|F| = |F_1| + |F_2| = 1536$ and $\sum_{C \in \mathcal{C}} A_{x,C} = 6$. ■

[1] Such a tuple exists by Lemma 2.5.

Lemma 2.7 *Let x be a point of $\mathcal{S}$. Then:*

(1) $|\mathcal{C}_x| = 25$*;*

(2) *every $C \in \mathcal{C}_x$ contains precisely* 512 *vertices;*

(3) $A_{x,C} = 2$ *for every $C \in \mathcal{C}_x$;*

(4) *if $y, y' \in \Gamma_3(x)$ belong to distinct connected components of Γ_x, then $B(x, y) \neq B(x, y')$.*

Proof. Let $(y_1, y_2, \ldots, y_{50})$ and $(B_1, B_2, \ldots, B_{50})$ be as in Lemma 2.6. Let M be the 25×50-matrix over $\mathbb{R}$ whose rows are indexed by the 25 lines of $\mathcal{L}_x$ and whose columns are indexed by the blocks $B_1, B_2, \ldots, B_{50}$ of $\mathcal{B}_x$. The entry of M corresponding to the line $L \in \mathcal{L}_x$ and the block B_i, $i \in \{1, 2, \ldots, 50\}$, of $\mathcal{B}_x$ is equal to 1 if $L \in B_i$ and equal to 0 otherwise. Notice that if y_{i_1} and y_{i_2} ($i_1, i_2 \in \{1, 2, \ldots, 50\}$) are contained in the same connected component of Γ_x, then by Lemma 2.2 the columns of M corresponding to B_{i_1} and B_{i_2} are equal. Hence, $rank(M) \leqslant |\mathcal{C}_x|$. In fact, we can say more. If $rank(M) = |\mathcal{C}_x|$ and $i_1, i_2 \in \{1, 2, \ldots, 50\}$ such that y_{i_1} and y_{i_2} belong to distinct connected components of Γ_x, then $B_{i_1} \neq B_{i_2}$.

By Lemma 2.6, $MM^T = 12 \cdot I + 6 \cdot J$, where I is the 25×25-identity matrix and J is the 25×25 matrix with all entries equal to 1. The matrix $12 \cdot I + 6 \cdot J$ is easily seen to be nonsingular. (E.g., by subtracting the first row from all the remaining rows and subsequently adding to the first column the sum of all the other columns, we obtain a nonsingular upper triangular matrix.)

So, we have $rank(M) = rank(MM^T) = 25$. Hence, $25 = rank(M) \leqslant |\mathcal{C}_x|$. Together with Lemma 2.5, this implies that the conditions (1), (2) and (3) of the lemma hold. Also (4) holds by Lemma 2.2 and the discussion above. Indeed, we have already said that if $rank(M) = |\mathcal{C}_x|$ and $i_1, i_2 \in \{1, 2, \ldots, 50\}$ such that y_{i_1} and y_{i_2} belong to distinct connected components of Γ_x, then $B(x, y_{i_1}) = B_{i_1} \neq B_{i_2} = B(x, y_{i_2})$. ∎

A 2-design is called *symmetric* if it has as many points as blocks. The point-line dual of symmetric 2-design is again a 2-design with the same parameters, see e.g. Beth, Jungnickel and Lenz [1, p. 78, Corollary 3.3]. This fact will be crucial for the remainder of the proof.

Lemma 2.8 *The point-line geometry $\mathcal{D}_x$ is a symmetric* 2-$(25, 9, 3)$*-design for every point x of $\mathcal{S}$. As a consequence, if B_1 and B_2 are two distinct blocks of $\mathcal{D}_x$, then $|B_1 \cap B_2| = 3$.*

Proof. We need to prove that $\mathcal{D}_x$ has precisely 25 points, precisely 9 points in each block and precisely 3 blocks through every two distinct points. The first two claims are trivially fulfilled. The last claim follows from Lemmas 2.2, 2.6(2) and 2.7.

Since $\mathcal{D}_x$ has as many points as blocks, namely 25, it is a symmetric 2-$(25, 9, 3)$-design. This implies that also the point-line dual of $\mathcal{D}_x$ is a 2-$(25, 9, 3)$-design. Hence, every two distinct blocks of $\mathcal{D}_x$ must intersect in precisely 3 points. ∎

Symmetric 2-$(25, 9, 3)$-designs do exist. Denniston [7] classified them and found that there are up to isomorphism 78 of them. We shall not need this classification here.

Lemma 2.9 *Let x be a point of $\mathcal{S}$, let $C \in \mathcal{C}_x$ and let $y \in \Gamma_4(x)$. Then there are at most two lines through y meeting C. Moreover, if y_1 and y_2 are two points of C collinear with y, then $\theta_{x,C}(y_1) = \theta_{x,C}(y_2)$.*

Proof. Suppose L_1, L_2 and L_3 are three not necessarily distinct lines through y meeting C. Put $\{y_i\} = L_i \cap C$, $i \in \{1,2,3\}$. Since y_i is contained on a shortest path between x and y, we have $\theta_{x,C}(y_i) = \Gamma_1(x) \cap \Gamma_2(y_i) = \bigcup_{L \in B(x,y_i)} (L \cap \Gamma_2(y_i)) = \bigcup_{L \in B(x,y_i)} (L \cap \Gamma_3(y))$. Since $B(x,y_1) = B(x,y_2) = B(x,y_3)$, we have $\theta_{x,C}(y_1) = \theta_{x,C}(y_2) = \theta_{x,C}(y_3)$. Since $A_{x,C} = 2$, at least two of the points y_1, y_2, y_3 must coincide. Hence, also at least two of the lines L_1, L_2, L_3 must coincide. This proves the lemma. ∎

Lemma 2.10 *Let x be a point of $\mathcal{S}$, let $y \in \Gamma_4(x)$ and let y_1, y_2 be two distinct points of $\Gamma_1(y) \cap \Gamma_3(x)$. Then $|B(x,y_1) \cap B(x,y_2)| = 3$.*

Proof. Suppose $|B(x,y_1) \cap B(x,y_2)| \neq 3$. Then $B(x,y_1) = B(x,y_2)$ by Lemma 2.8. By Lemma 2.7(4), y_1 and y_2 belong to the same connected component C of Γ_x. By Lemma 2.9, $\theta_{x,C}(y_1) = \theta_{x,C}(y_2)$. Put $\{u_1, u_2, \ldots, u_9\} = \theta_{x,C}(y_1) = \theta_{x,C}(y_2)$. By Lemmas 2.7(4) and 2.9, the set $\{B(y,u_1), B(y,u_2), \ldots, B(y,u_9)\} \subseteq \mathcal{B}_y$ has size at least 5. But each of these blocks of $\mathcal{B}_y$ contains the lines yy_1 and yy_2. This is impossible since there are only three blocks of $\mathcal{B}_y$ through yy_1 and yy_2. ∎

Lemma 2.11 *Let x be a point of $\mathcal{S}$, let $y \in \Gamma_4(x)$ and let $C \in \mathcal{C}_x$. Then precisely one line through y meets C.*

Proof. Put $\Gamma_3(x) \cap \Gamma_1(y) = \{y_1, y_2, \ldots, y_{25}\}$. By Lemma 2.10, the blocks $B(x,y_1)$, $B(x,y_2)$, …, $B(x,y_{25})$ of $\mathcal{D}_x$ are mutually distinct. Since there are only 25 blocks in $\mathcal{D}_x$, these are all the blocks of $\mathcal{D}_x$. Let $y' \in C$ and let i be the unique element of $\{1,2,\ldots,25\}$ such that $B(x,y') = B(x,y_i)$. By Lemmas 2.2 and 2.7(4), y_i is the unique element of $\{y_1, y_2, \ldots, y_{25}\}$ contained in C. Hence, the line yy_i is the unique line through y meeting C. ∎

We are now ready to derive a contradiction. This contradiction implies that there are no regular near octagons with parameters $(s, t_2, t_3, t) = (2, 0, 8, 24)$.

Let x be a point of $\mathcal{S}$ and let B_1 and B_2 be two distinct blocks of $\mathcal{B}_x$. Then $|B_1 \cap B_2| = 3$ by Lemma 2.8. Put $B_1 \cap B_2 = \{L_1, L_2, L_3\}$ and $B_1 = \{L_1, L_2, \ldots, L_9\}$. Let C_i, $i \in \{1,2\}$, be the element of $\mathcal{C}_x$ such that $B_i = B(x, w_i)$ for every $w_i \in C_i$. Let y_1 be an arbitrary point of C_1, let x_i^+, $i \in \{1,2,\ldots,9\}$, denote the unique point of L_i at distance 2 from y_1 and let x_i^- denote the unique point of L_i distinct from x and x_i^+. Now, $\theta_{x,C_1}(y_1) = \{x_1^+, x_2^+, \ldots, x_9^+\}$ is a vertex of G_{x,y_1} and hence also $\{x_1^+, x_2^+, x_3^+, x_4^-, x_5^-, \cdots, x_9^-\}$ is a vertex of G_{x,y_1}. Since $A_{x,C_1} = 2$, there are precisely two points $y_2, y_2' \in C_1$ such that $\theta_{x,C_1}(y_2) = \theta_{x,C_1}(y_2') = \{x_1^+, x_2^+, x_3^+, x_4^-, x_5^-, \ldots, x_9^-\}$.

We prove that the points y_2 and y_2' lie at distance 3 from y_1. Let u denote the unique point of C_1 which is L_1-adjacent to y_1, let v_1 denote the unique point of C_1 which is L_2-adjacent to u, let v_2 denote the unique point of C_1 which is L_3-adjacent to v_1, let v_1'

denote the unique point of C_1 which is L_3-adjacent to u and let v_2' denote the unique point of C_1 which is L_2-adjacent to v_1'. By construction (and the fact that $t_2 = 0$), v_2 and v_2' lie at distance 3 from y_1. If $v_2 = v_2'$, then $u, v_1, v_2 = v_2', v_1', u$ would define a quadrangle in C_1 which is impossible since $t_2 = 0$. Hence, $v_2 \neq v_2'$. One can readily verify that $\theta_{x,C_1}(v_2) = \theta_{x,C_1}(v_2') = \{x_1^+, x_2^+, x_3^+, x_4^-, x_5^-, \ldots, x_9^-\} = \theta_{x,C_1}(y_2) = \theta_{x,C_1}(y_2')$. This implies that $\{v_2, v_2'\} = \{y_2, y_2'\}$. Hence, each of y_2, y_2' lies at distance 3 from y_1. (A reasoning along the above lines can be given to show that each of y_2, y_2' is connected with y_1 by precisely three paths of length 3 which are completely contained in C_1.)

Since each of y_2, y_2' lies at distance 3 from x, there are precisely $2(t_3+1)(t_2+1) = 18$ paths of length 3 which join y_1 with one of y_2, y_2'. We will now construct 32 paths[2] which join y_1 with one of y_2, y_2' leading to our desired contradiction. In fact, we show that for each of the $s(t - t_3) = 32$ points $z \in \Gamma_4(x) \cap \Gamma_1(y_1)$, there exists a path (y_1, z, z', z'') of length 3 with $z'' \in \{y_2, y_2'\}$.

Let M_1 denote the unique line through z meeting C_2 in a point w and let z' denote the unique point of M_1 not contained in $C_2 \cup \{z\}$. Let M_2 denote the unique line through z' meeting C_1 in a point z''.

Let $i \in \{1, 2, 3\}$. Since $L_i \in B(x, y_1) \cap B(x, z'') \cap B(x, w)$, we have $\Gamma_2(y_1) \cap L_i = \Gamma_3(z) \cap L_i = \Gamma_2(w) \cap L_i = \Gamma_3(z') \cap L_i = \Gamma_2(z'') \cap L_i$.

Let $i \in \{4, 5, \ldots, 9\}$. If $\Gamma_3(z) \cap L_i = \Gamma_3(z') \cap L_i = \{v\}$, then since $\mathrm{d}(v, z) = \mathrm{d}(v, z') = 3$, we would have $\mathrm{d}(v, w) = 2$ and hence $L_i \in B(x, w) = B_2$, a contradiction. So, $\Gamma_2(y_1) \cap L_i = \Gamma_3(z) \cap L_i \neq \Gamma_3(z') \cap L_i = \Gamma_2(z'') \cap L_i$.

The two previous paragraphs imply that $\theta_{x,C_1}(z'') = \theta_{x,C_1}(y_2) = \theta_{x,C_1}(y_2')$. Since $A_{x,C_1} = 2$, we have $z'' \in \{y_2, y_2'\}$.

So, we have constructed $32 = |\Gamma_1(y_1) \cap \Gamma_4(x)|$ paths of length 3 which connect y_1 with one of y_2, y_2'. As said before, this is impossible. So, there exists no regular near octagon with parameters $(s, t_2, t_3, t) = (2, 0, 8, 24)$.

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