Pattern Hypergraphs *

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Abstract

The notion of pattern hypergraph provides a unified view of several previously studied coloring concepts. A pattern hypergraph H is a hypergraph where each edge is assigned a type Π_i that determines which of possible colorings of the edge are proper. A vertex coloring of H is proper if it is proper for every edge. In general, the set of integers k such that H can be properly colored with exactly k colors need not be an interval. We find a simple sufficient and necessary condition on the edge types $\Pi_1, \ldots, \Pi_\lambda$ for the existence of a pattern hypergraph H with edges of types $\Pi_1, \ldots, \Pi_\lambda$ such that the numbers of colors in proper colorings of H do not form an interval of integers.

1 Introduction

Coloring problems are among the most intensively studied combinatorial problems both for the theoretical and the practical reasons. Generalizations of usual graph and hypergraph coloring, e.g., the channel assignment problem, are widely applied in practice. A new general concept of mixed hypergraphs has attracted a lot of attention as witnessed by a recent monograph by Voloshin [29] and an enormous number of papers on the

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subject, e.g., [6, 10, 13, 17–24, 26, 30–33]. The concept generalizes usual colorings of hypergraphs in which it is required that no edge is monochromatic as well as colorings of co-hypergraphs [7,18] in which it is required that each edge contains at least two vertices with the same color. The latter type of hypergraph colorings arises naturally in the classical notion of anti-Ramsey problems [1,12,14,15]. In addition, both types of hypergraph colorings are closely related to face-constrained colorings of embedded graphs [11, 16, 27, 28]. The notion of mixed hypergraphs is powerful enough to model general constraint satisfaction problems, in particular, list colorings, graph homomorphisms, circular colorings, locally surjective, locally bijective and locally injective graph homomorphisms, $L(p, q)$-labelings, the channel assignment problem, T-colorings and generalized T-colorings [19].

A *mixed hypergraph* is a hypergraph with two types of edges, $\mathcal{C}$-edges and $\mathcal{D}$-edges. A coloring of a mixed hypergraph is *proper* if no $\mathcal{C}$-edge is polychromatic (rainbow) and no $\mathcal{D}$-edge is monochromatic. Mixed hypergraphs have some very surprising properties. The most striking results include: for any finite set of integers I with $1 \notin I$, there is a mixed hypergraph which can be colored by precisely k colors if and only if $k \in I$ [13], e.g., there exists a mixed hypergraph on 6 vertices which is 2-colorable and 4-colorable and which is not 3-colorable. An even stronger result holds: for any sequence $s_1, \ldots, s_k$ of integers such that $s_1 = 0$, there exists a mixed hypergraph which has precisely $s_{k'}$ proper colorings using k' colors, $1 \leqslant k' \leqslant k$, and no proper coloring using more than k colors [20]. These results led to a lot of papers describing which subclasses of mixed hypergraphs have such unusual properties [6, 10, 17, 21–24, 26, 30, 32, 33].

Another generalization of mixed hypergraphs are *color-bounded hypergraphs* introduced by Bujtás and Tuza [3, 4]. In this model, every edge of a hypergraph is assigned two numbers s and t, and it is required that the number of colors used to color vertices of that edge is at least s and at most t. An even more general model is considered in [5] where each edge is assigned four numbers s, t, a and b, and it is required that the number of colors used on the edge is between s and t and the largest number of vertices having the same color is between a and b. Clearly, mixed hypergraphs can be viewed as a special type of color-bounded hypergraphs. Like for mixed hypergraphs, the numbers of colors that can be used in a proper coloring of a color-bounded hypergraph need not form an interval and can in fact be almost any set of integers, even for hypergraph with very restricted types of edges.

In this paper, we provide a full characterization of edge types of hypergraphs that can cause this behavior. We introduce a notion of pattern hypergraphs that includes usual (hyper)graph colorings and colorings of co-hypergraphs and mixed hypergraphs. In addition, pattern hypergraphs appear naturally in certain types of constraint satisfaction problems and our characterization yields also interesting results in this area as described later in this section.

1.1 Pattern hypergraphs

An *edge type* is a non-empty set Π of equivalence relations on an ordered set A. The *size of the edge type* Π is $|A|$. A **pattern hypergraph** H consists of a *vertex set* $V(H)$ and

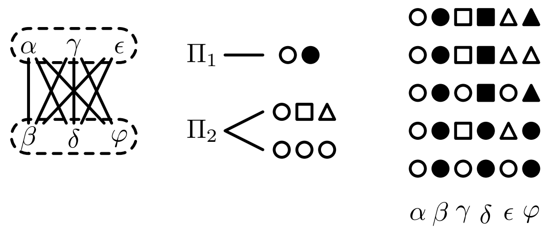

Figure 1: An example of a pattern hypergraph (depicted in the very left part of the figure). The hypergraph consists of edges of sizes two (depicted as segments) and edges of sizes three (dashed-line ovals). The edges of sizes two are of type Π_1 that contains only the trivial equivalence relation. The edges of sizes three are of type Π_2 that contains the trivial and the universal equivalence relation. The feasible set of the pattern hypergraph is $\{2, 4, 5, 6\}$. All distinct proper colorings are shown in the right part of the figure.

an *edge set* $\mathcal{E}(H)$. Each edge E is assigned an edge type whose size matches the size of E. The hypergraph is oriented, i.e. any edge is considered to be an ordered tuple and each vertex appears at most once in it. The vertices of E naturally correspond to the elements of the support set of its edge type. The hypergraph H may contain the same edge several times with distinct edge types assigned as well as edges with the same set of vertices but with different orderings.

An edge of type Π_i is called a Π_i-edge. If H is a pattern hypergraph with edges of types $\Pi_1, \dots, \Pi_\lambda$, then H is a $(\Pi_1, \dots, \Pi_\lambda)$-hypergraph. In case that $\lambda = 1$, H is briefly called a Π_1-hypergraph. An example of a pattern hypergraph can be found in Figure 1.

A *k-coloring* c of a pattern hypergraph H is a mapping of V **onto** a set of k colors. A coloring c is *proper* if for each edge E of type Π, the equivalence relation π of "having the same color" restricted to the vertices of E is contained in Π (under the fixed correspondence between the vertices of E and the elements of the support set of Π). In that case, the equivalence relation $\pi \in \Pi$ is called *consistent* with c on E. The *feasible set* $\mathcal{F}(H)$ of H is the set of all integers k for which there is a proper k-coloring of H.

If $\mathcal{F}(H)$ is non-empty, then H is *colorable*. The least element of $\mathcal{F}(H)$ is called the *chromatic number* of H and denoted by $\chi(H)$. The largest element of $\mathcal{F}(H)$ is called the *upper chromatic number* of H and denoted by $\bar{\chi}(H)$. If $\mathcal{F}(H) = [\chi(H), \bar{\chi}(H)]$ or $\mathcal{F}(H) = \emptyset$, i.e., $\mathcal{F}(H)$ is an interval of integers, the feasible set is said to be *unbroken* or *gap-free*. Otherwise, it is called *broken*.

An equivalence relation is *universal* if it consists of a single class only. It is called *trivial* if all of its classes are singletons. $\mathcal{C}_l$ is the edge type containing all the equivalence relations on l elements except for the trivial one. $\mathcal{D}_l$ is the edge type containing all the equivalence relations on l elements except for the universal one. $\mathcal{D}_2$-hypergraphs are usual graphs and the proper colorings of a $\mathcal{D}_2$-hypergraph are exactly the proper colorings of the corresponding graph. Similarly, $\mathcal{D}_l$-hypergraphs are l-uniform hypergraphs and their proper colorings are exactly the proper colorings of the corresponding hypergraphs.

As an example of the expressive power of pattern hypergraphs, we show how $(\mathcal{D}_2, \mathcal{C}_{l+1})$-

hypergraphs can be used to model list l-colorings (an analogous construction can be found in [25] for mixed hypergraphs). In a list coloring problem we are given a graph $G = (V, E)$ together with a list (of size l) of possible colors $\Lambda(v)$ at each vertex v, the goal is to find a coloring c of its vertices such that $c(v) \in \Lambda(v)$ for each $v \in V$ and $c(u) \neq c(v)$ whenever $uv \in E$. Consider a $(\mathcal{D}_2, \mathcal{C}_{l+1})$-hypergraph H with the vertex set $V \cup \Lambda$ where Λ is the union of $\Lambda(v)$. Each pair of adjacent vertices u and v forms a $\mathcal{D}_2$-edge of H. Similarly each pair of colors of Λ forms a $\mathcal{D}_2$-edge of H. For every $v \in V$, there is a $\mathcal{C}_{l+1}$-edge comprised of the $(l+1)$-tuple $\{v\} \cup \Lambda(v)$. It is easy to check that proper colorings of H correspond to list colorings of G.

There is also a close relation between pattern hypergraphs and certain types of constraint satisfaction problems. A *constraint satisfaction problem (CSP)* consists of variables $x_1, \ldots, x_n$, a domain set U and several types of constraints $P_i \subseteq U^{r_i}$. Each constraint P_i must be satisfied for certain prescribed r_i-tuples of $x_1, \ldots, x_n$, i.e., the r_i-tuple of the values of such variables must be contained in P_i. The goal is to find an assignment $\sigma : \{x_1, \ldots, x_n\} \to U$ that satisfies all the constraints.

An important class of constraint satisfaction problems are those where each constraint can be expressed as a disjunction of conjunctions of equalities and inequalities [2] (so-called equality constrained languages). In addition to finding a solution, the goal is often to minimize the size of the domain of a constructed solution. Constraint satisfaction problems of this type can be easily modeled by pattern hypergraphs. The problem we study in this paper may be reformulated as the following question related to verification of optimality of a constructed solution for a CSP of this type: for which types of constraints can one conclude that there is no solution with domain of size at most $k-1$ from the facts that there is no solution for a domain of size $k-1$ and there is a solution for a domain of size k?

1.2 Notation and our results

An equivalence relation π is *finer* than π', if $x \sim_\pi y$ implies that $x \sim_{\pi'} y$, i.e., the classes of π partition the classes of π'. Conversely, if π is finer than π', then π' is *coarser* than π. If Π is a set of equivalence relations with the same support set, then $\rho(\Pi)$ denotes the equivalence relation such that $x \sim_{\rho(\Pi)} y$ if and only if $x \sim_\pi y$ for all $\pi \in \Pi$. It is easy to check that a relation defined in this way is indeed an equivalence relation. The relation $\rho(\Pi)$ is the (unique) coarsest equivalence relation finer than all the relations of Π. An equivalence relation π' is a *refinement* of π with respect to $\rho(\Pi)$ if π' is coarser than $\rho(\Pi)$ and π' can be obtained from π by splitting one of the equivalence classes into two. The following four closure concepts are considered in this paper (see Figures 2–5 for examples):

- The edge type Π is *simply-closed* if it contains all the equivalence relations π that have at most one equivalence class of size greater than one. In particular, Π contains both the universal and the trivial equivalence relation. The unique inclusion-wise smallest edge type that is simply-closed is denoted by Π_{simple}. Note that Π is simply-closed if and only if $\Pi_{\text{simple}} \subseteq \Pi$.

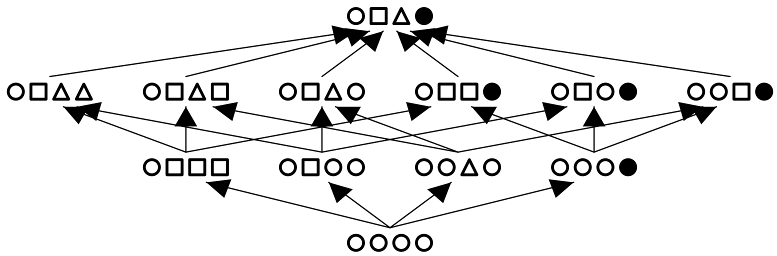

Figure 2: The smallest simply-closed edge type Π_{simple} for the edge size 4. Each 4-tuple represents a single equivalence relation (equivalent elements are drawn using the same geometric object). The number of equivalence classes of the relations grows from the bottom to the top. The arrows lead in the direction from coarser to finer equivalence relations.

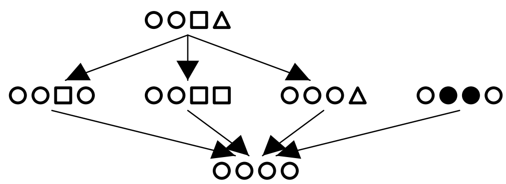

Figure 3: An edge type which is down-closed but which is neither simply-closed, up-closed nor up-group-closed. At each arrow, the relation at the tail forces the presence of the relation at the head in the edge type.

- The edge type Π is *down-closed* if for any $\pi \in \Pi$ the edge type Π contains all equivalence relations π' that are coarser than π. In particular, Π contains the universal equivalence relation.

- The edge type Π is *up-closed* if for any $\pi \in \Pi$ the edge type Π contains all the equivalence relations π' that can be obtained from π by choosing an element x and introducing a new single element class containing only x. In particular, Π contains the trivial equivalence relation.

- The edge type Π is *up-group-closed* if for any $\pi \in \Pi$ the edge type Π contains all the refinements π' of the equivalence relation π with respect to $\rho(\Pi)$. Note that the edge type Π also contains all other equivalence relations that are finer than π and coarser than $\rho(\Pi)$.

If all the edge types $\Pi_1, \ldots, \Pi_\lambda$ are simply-closed, then any $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph has an unbroken feasible set. The same holds, if all the types are down-closed, up-closed or up-group-closed. Our main result is that these sufficient conditions are also necessary. This provides a full characterization of edge types that can cause the feasible set of a pattern hypergraph to be broken.

The paper is structured as follows: we first discuss the relation between the concepts of pattern hypergraphs and mixed hypergraphs in Section 2 and show that our new general results on pattern hypergraphs also provide new results for mixed hypergraphs. The

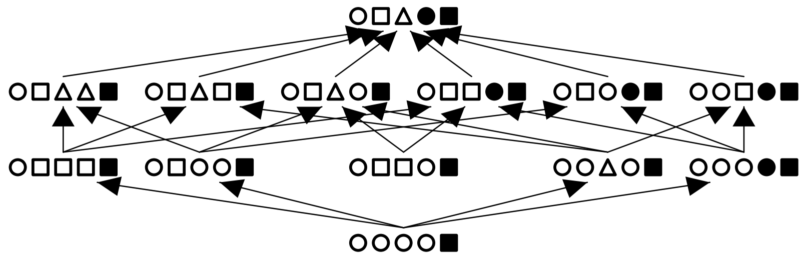

Figure 4: An edge type which is up-closed but which is neither simply-closed, down-closed nor up-group-closed. At each arrow, the relation at the tail forces the presence of the relation at the head in the edge type.

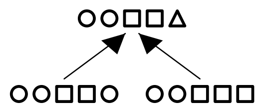

Figure 5: An edge type which is up-group-closed but which is neither simply-closed, down-closed nor up-closed.

sufficiency and necessity of the conditions are studied in Sections 3 and 4. In Section 5, we show that several possible modifications of the definition of pattern hypergraphs do not lead to more general concepts and briefly discuss possible directions for future research.

2 Mixed Hypergraphs

Mixed hypergraphs were introduced in [30, 31]. A mixed hypergraph has two types of edges: $\mathcal{C}$-edges and $\mathcal{D}$-edges. $\mathcal{C}$-edges and $\mathcal{D}$-edges of size l are exactly $\mathcal{C}_l$-edges and $\mathcal{D}_l$-edges in the language of pattern hypergraphs. A mixed hypergraph is a mixed bi-hypergraph if each edge is simultaneously a $\mathcal{C}$-edge and a $\mathcal{D}$-edge. A hypergraph H is spanned by a graph G if $V(G) = V(H)$ and every edge of H induces a connected subgraph of G. The following results on feasible sets of mixed hypergraphs were obtained:

- For any finite integer set I such that $1 \notin I$, there exists a mixed hypergraph H with $\mathcal{F}(H) = I$ [13]. Moreover, there is such a hypergraph H which has only one proper k-coloring for any $k \in I$. A similar result may be obtained for l-uniform mixed bihypergraphs for $l \geqslant 3$.

- Any mixed hypergraph spanned by a path [6], a tree [21, 22], a cycle [32, 33] or a strong cactus [23] has an unbroken feasible set. There are mixed hypergraphs spanned by weak cacti with a broken feasible set [23].

- For any non-planar graph G with at least six vertices, there is a mixed hypergraph H spanned by G with a broken feasible set [23].

- There are planar mixed hypergraphs with broken feasible sets but the gap in such sets may be only for 3 colors [10, 17, 26]. There is no planar mixed bihypergraph with a broken feasible set [8, 10, 17, 26].

Theorem 18 allows us to enhance this list of results:

Theorem 1. *For any $l_1 \geqslant 3$ and $l_2 \geqslant 2$, there exists a mixed hypergraph H with $\mathcal{C}$-edges only of size l_1 and $\mathcal{D}$-edges only of size l_2 such that $\mathcal{F}(H)$ is broken.*

Proof. Since the edge type $\mathcal{C}_{l_1}$ is neither simply-closed, up-closed nor up-group-closed and the edge type $\mathcal{D}_{l_2}$ is not down-closed, Theorem 18 applies. $\square$

In a similar fashion, one may also reprove the following theorem of [13]:

Theorem 2. *For any $l \geqslant 3$, there exists an l-uniform mixed bihypergraph H with a broken feasible set.*

Proof. Since the edge type $\mathcal{C}_l \cap \mathcal{D}_l$ is neither simply-closed, down-closed, up-closed nor up-group-closed, Theorem 18 applies. $\square$

3 Sufficiency of the Conditions

We show the sufficiency of the conditions in this section.

Lemma 3. *If each of edge types $\Pi_1, \ldots, \Pi_\lambda$ is simply-closed, then each $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph has an unbroken feasible set.*

Proof. Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H with n vertices. Let $1 \leqslant k \leqslant n$. Color $k-1$ vertices with mutually different colors and all the remaining vertices with the same color different from the $k-1$ colors. This coloring is a proper k-coloring because all the edge types are simply-closed. Hence, $\mathcal{F}(H) = [1, n]$. $\square$

Lemma 4. *If each of edge types $\Pi_1, \ldots, \Pi_\lambda$ is down-closed, then every $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph has an unbroken feasible set.*

Proof. Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H. If H is uncolorable, then its feasible set is not broken. Otherwise, consider a proper k-coloring c of H with $k = \bar{\chi}(H)$. Since all the edge types are down-closed, the coloring c' defined as $c'(v) := c(v)$ for $c(v) \leqslant \ell$ and $c'(v) := \ell$ for $c(v) > \ell$ is a proper ℓ-coloring for every $\ell \leqslant k$. Hence, $\mathcal{F}(H) = [1, \bar{\chi}(H)]$. $\square$

Lemma 5. *If each edge type $\Pi_1, \ldots, \Pi_\lambda$ is up-closed, then each $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph has an unbroken feasible set.*

Proof. Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H with n vertices. If H is uncolorable, then its feasible set is not broken. Otherwise, let c be a proper k-coloring of H with $k < n$. By symmetry, we can assume that the color k is used to color at least two vertices. Assume that one of them is a vertex w. Since all the edge types are up-closed, the coloring c' equal for $v \neq w$ to c and assigning w a new color is a proper $(k+1)$-coloring. Hence, $\mathcal{F}(H) = [\chi(H), n]$. $\square$

Lemma 6. *If each edge type $\Pi_1, \ldots, \Pi_\lambda$ is up-group-closed, then every $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph has an unbroken feasible set.*

Proof. Fix a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H. If H is uncolorable, then its feasible set is not broken. Otherwise, consider the following relation $\sim'$ on the vertices of H: $v \sim' w$ if $v \sim_{\rho(\Pi_i)} w$ for some Π_i-edge. Let $\sim$ be the equivalence closure of the relation $\sim'$ and k_0 the number of its classes. If c is a proper coloring of H and $v \sim w$, then $c(v) = c(w)$. Hence, $\bar{\chi}(H) \leqslant k_0$. Let c be a k-coloring of H with $k < k_0$. Observe that one of the k colors is used to color at least two different equivalence classes of $\sim$. Let w be a vertex colored with such a color. Consider the coloring c' defined by $c'(v) = c(v)$ for $v \not\sim w$ and assigning a completely new color to each v with $v \sim w$. Since all the edge-types are up-group-closed and $\rho(\Pi_i)$ is finer than $\sim$ on each Π_i-edge, c' is a proper $(k+1)$-coloring. Hence, $\bar{\chi}(H) = k_0$ and $\mathcal{F}(H) = [\chi(H), k_0]$. ☐

4 Necessity of the Conditions

We first consider the case when all the edges of a pattern hypergraph are of the same type. Later we generalize our arguments to pattern hypergraphs with more types of edges. Let us start with several lemmas on edge types that contain the trivial or the universal equivalence relation:

Lemma 7. *If Π is an edge type which contains both the trivial and the universal equivalence relation and which is not simply-closed, then there exists a Π-hypergraph H with a broken feasible set.*

Proof. Let l be the edge size of Π and consider a hypergraph H with $n = l^2$ vertices such that all possible l-tuples form edges of H. Clearly, $1 \in \mathcal{F}(H)$ and $n \in \mathcal{F}(H)$. Assume for the sake of contradiction that $l \in \mathcal{F}(H)$. Let c be a proper l-coloring of H. We can assume without loss of generality that there are l vertices colored with the color 1, say $v_1, \ldots, v_l$. Let u_i for $2 \leqslant i \leqslant l$ be any vertex colored with the color i. Let π be an arbitrary equivalence relation belonging to Π_{simple}. The tuple containing some of vertices $v_1, \ldots, v_l$ in the positions of the largest class of π and some of vertices $u_2, \ldots, u_l$ in the positions of the single-element classes of π is an edge of H and thus $\pi \in \Pi$. Hence, $\Pi_{\text{simple}} \subseteq \Pi$. But this is impossible because Π is not simply-closed. ☐

Lemma 8. *If Π is an edge type that contains the trivial equivalence relation, that does not contain the universal equivalence relation and that is not up-closed, then there exists a Π-hypergraph H with a broken feasible set.*

Proof. Let l be the edge size of Π. We construct a Π-hypergraph H with $n = l^2(l+1)$ vertices v_{ij} for $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant l(l+1)$. Fix an l-coloring c_0 such that $c_0(v_{ij}) = i$. Include to H as edges all l-tuples such that c_0 remains a proper coloring of H. Clearly, $l \in \mathcal{F}(H)$ and $n \in \mathcal{F}(H)$. Assume for the sake of contradiction that $l+1 \in \mathcal{F}(H)$. Let c be a proper $(l+1)$-coloring and let ξ_i be the color used by c to color the largest number of the vertices v_{ij}, $1 \leqslant j \leqslant l(l+1)$. We may assume without loss of generality that $c(v_{i1}) = \ldots = c(v_{il}) = \xi_i$ for each $1 \leqslant i \leqslant l$.

We first prove that $\xi_i \neq \xi_{i'}$ for all $i \neq i'$. Assume that $\xi_i = \xi'_i$. Let π be an equivalence relation of Π such that the size l_0 of the largest equivalence class of π is as large as possible. Note that $l_0 < l$ because Π does not contain the universal equivalence. Consider an edge E of H that contains the vertices $v_{i1}, \ldots, v_{il_0}$ and the vertex $v_{i'1}$ (such an edge exists by the construction of H and the choice of π). If c is a proper coloring, then Π contains an equivalence relation with an equivalence class of size at least $l_0 + 1$ since all the vertices $v_{i1}, \ldots, v_{il_0}$ and $v_{i'1}$ have the same color. This contradicts the choice of π.

Next, we show for contradiction that Π is up-closed. Since c is a proper $(l+1)$-coloring, we may assume without loss of generality that $c(v_{1,l+1}) \neq \xi_i$ for all $1 \leqslant i \leqslant l$. Let π be an equivalence relation of Π that is not the trivial one and π' be an equivalence relation obtained from π by creating a single element class by separating an element w from a class W of π. Consider an edge E of H such that π is consistent with c_0 on E, $v_{1,l+1} \in E$, $v_{1,l+1}$ corresponds to w, the remaining elements of W are some of the vertices $v_{11}, \ldots, v_{1l}$ and other vertices of E are some of the vertices $v_{i1}, \ldots, v_{il}$ with $2 \leqslant i \leqslant l$. Since c is a proper coloring, it follows that $\pi' \in \Pi$. Hence, Π is up-closed, thus contradicting the assumptions of the lemma. $\square$

Lemma 9. *If Π is an edge type that contains the universal equivalence relation, that does not contain the trivial equivalence relation and that is not down-closed, then there exists a Π-hypergraph H with a broken feasible set.*

Proof. Let l be the edge size of Π. We construct a Π-hypergraph H with $n = l^3$ vertices v_{ij} for $1 \leqslant i \leqslant l^2$ and $1 \leqslant j \leqslant l$. Let $L = l^2$. Fix a coloring c_0 such that $c_0(v_{ij}) = i$ for $1 \leqslant i \leqslant L$. Include to H as edges all l-tuples such that c_0 is a proper coloring of the tuple. Clearly, $1 \in \mathcal{F}(H)$ and $L \in \mathcal{F}(H)$. We prove $L - 1 \notin \mathcal{F}(H)$.

Assume for the sake of contradiction that there is a proper $(L-1)$-coloring c of H. We first prove that $c(v_{ij}) = c(v_{ij'})$ for all $1 \leqslant i \leqslant L$ and $1 \leqslant j, j' \leqslant l$. Assume that, e.g., $c(v_{11}) \neq c(v_{12})$. Let π be an equivalence relation contained in Π with the largest number l_0 of equivalence classes. Since Π does not contain the trivial equivalence relation, $l_0 < l$. Since the coloring c uses $l^2 - 1$ colors, there exists a vertex v_{ij} with $i \neq 1$ such that the color of $c(v_{ij})$ is neither $c(v_{11})$ nor $c(v_{12})$. We may assume that v_{21} is such a vertex. Similarly, there exists a vertex $c(v_{ij})$ with $i \neq 1, 2$ such that $c(v_{ij}) \notin \{c(v_{11}), c(v_{12}), c(v_{21})\}$. We may assume that v_{31} is such a vertex. In this way, we conclude that we can assume without loss of generality that the colors of the vertices v_{11}, v_{21}, $\ldots$, v_{l1} and v_{12} are mutually distinct. Consider an edge E of H such that π is consistent with c_0 on E and such that E contains all the vertices v_{11}, v_{21}, $\ldots$, v_{l_01} and v_{12} (such an edge exists by the construction of H). Since c is a proper coloring, Π must contain an equivalence relation consistent with c on E and such an equivalence relation is comprised of at least $l_0 + 1$ equivalence classes. This contradicts the choice of π.

Let ξ_i be the common color of the vertices v_{ij} for $1 \leqslant j \leqslant l$. We can assume without loss of generality that $\xi_1 = \xi_2$ and all the colors ξ_i for $i \geqslant 2$ are mutually different. Consider now an equivalence relation $\pi \in \Pi$ and an equivalence relation π' obtained from

π by an union of two classes of π. Let E be an edge H such that π is consistent with c_0 on E, E contains v_{11} and v_{21} and these two vertices correspond to elements of the two unified equivalence classes of π. Since c is a proper coloring of H, the equivalence relation π' must be contained in Π. Consequently, Π is down-closed, thus contradicting assumptions of the lemma. $\square$

We now focus on edge types avoiding both the universal and the trivial equivalence relations:

Lemma 10. *If Π is an edge type which contains neither the universal nor the trivial equivalence relation, then there exists a Π-hypergraph H_0 that has a unique proper coloring (up to a permutation of colors) and all color classes have the same size.*

Proof. Let l be the edge size of Π. Consider a Π-hypergraph H_0 with $2l^3$ vertices v_{ij} such that $1 \leqslant i \leqslant 2l$ and $1 \leqslant j \leqslant l^2$ and a coloring $c_0(v_{ij}) = i$. The edge set of H_0 consists of all l-tuples of vertices that are consistent with c_0, i.e. H_0 is the Π-hypergraph with the maximum number of edges that has c_0 as a proper coloring. We claim that c_0 is the only proper coloring of H_0.

Consider a proper coloring c of H_0. Let $C_i = \{c(v_{ij}), 1 \leqslant j \leqslant l^2\}$ for $1 \leqslant i \leqslant 2l$ and let I be the set of i's for which $|C_i| \leqslant l$. We first assume that $|I| \leqslant l$. By symmetry, we may also assume that $1, \ldots, l \notin I$. Let π be any equivalence relation contained in Π. Let $A_1, \ldots, A_k$ be the equivalence classes of π. Consider an l-tuple X of vertices such that $|X \cap \{v_{i1}, \ldots, v_{il^2}\}| = |A_i|$ and all the vertices of X are assigned different colors by c (such a tuple exists because $1, \ldots, k \notin I$). The hypergraph H_0 contains an edge E formed by the vertices of X. Since c is proper, Π has to contain the trivial equivalence relation. This excludes the case that $|I| \leqslant l$.

In the rest, we assume that $|I| \geqslant l+1$. By symmetry, we may assume that $[1, l+1] \subseteq I$ and $c(v_{i1}) = \ldots = c(v_{il})$ for each $i \in I$. Let $\xi_i = c(v_{i1})$ for $i \in I$ and $V = \{v_{ij}, c(v_{ij}) = \xi_i, 1 \leqslant i \leqslant l\}$.

We claim that the colors $\xi_i, i \in I$ are mutually different. By symmetry, it is enough to exclude the case $\xi_1 = \xi_2$. Let l_0 be the largest size of the equivalence class of an equivalence relation contained in Π and let $\pi \in \Pi$ be an equivalence relation with an equivalence class of size l_0. Consider an edge E formed by some of the vertices of V such that π is consistent with c_0 on E, the vertices corresponding to the largest equivalence class are some of the vertices $v_{11}, \ldots, v_{1l}$ and E contains the vertex v_{21}. Since c is a proper coloring of H, there exists $\pi' \in \Pi$ consistent with c on E. However, the size of the largest equivalence class of π' is at least $l_0 + 1$, thus contradicting the choice of π and l_0. Observe that we have actually shown that $\xi_1 \neq c(v_{ij})$ for any $i \neq 1$ and arbitrary j.

Next, we show that $c(v_{i_0 j_0}) = c(v_{i_0 j_0'})$ for all i_0 and $j_0 \neq j_0'$. Fix any such i_0, j_0 and j_0'. We may assume that $i_0 > l$ (this includes both the cases that $i_0 \in I$ and $i_0 \notin I$). By the observation at the end of the previous paragraph, $c(v_{i_0 j_0})$ and $c(v_{i_0 j_0'})$ are distinct from all the colors $\xi_1, \ldots, \xi_l$. Consider now an equivalence relation $\pi \in \Pi$ with the largest number l_0 of equivalence classes. Consider an edge E of H such that π is consistent with c_0 on E, E contains both the vertices $v_{i_0 j_0}$ and $v_{i_0 j_0'}$ and the remaining vertices of E form a subset of $\{v_{i,j}, i = 1, \dots, l, i_0 \text{ and } 1 \leqslant j \leqslant l\}$. The vertices of E are colored by c with more than l_0 colors. Since c is a proper coloring, Π contains an equivalence relation with at least l_0+1 equivalence classes which contradicts the choice of l_0 and π. This implies that $I = \{1, \dots, 2l\}$ and V is the set of all the vertices of H. Hence, all the proper colorings of H differ only by a permutation of colors. □

Lemma 11. *If Π is an edge type which contains neither the universal nor the trivial equivalence relation, then there exists a Π-hypergraph with a broken feasible set if and only if there exists a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph with a broken feasible set.*

Proof. It is enough to prove that if there is a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with a broken feasible set, then there is a Π-hypergraph H' with a broken feasible set. The opposite implication is trivial. Let l be the edge size of Π. In addition, let H_0 be the hypergraph from the statement of Lemma 10. Let $V_1, \dots, V_k$ be the color classes of the unique proper coloring of H_0. Note that $|V_1| = \cdots = |V_k| \geqslant 2$. For the remainder of the proof, fix a vertex $v_1 \in V_1$.

Fix a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with a broken feasible set. Let U be the vertex set of H. We construct a Π-hypergraph H' with a broken feasible set. The vertex set of H' is $(U \times V_1) \cup V_2 \cup \cdots \cup V_k$. For every $u \in U$, include a copy of H_0 to H' on the vertex set $(\{u\} \times V_1) \cup V_2 \cup \cdots \cup V_k$ (identifying the corresponding vertices of $\{u\} \times V_1$ and V_1). If E is a Π-edge of H, then we include a Π-edge $E' = \{[u, v_1] | u \in E\}$ in H'. If E is a $\mathcal{D}_2$-edge $\{u_1, u_2\}$ of H, we include a copy of H_0 to H' on the vertex set $(\{u_1\} \times V_1) \cup (\{u_2\} \times V_1) \cup V_3 \cup \cdots \cup V_k$. If E is a $\mathcal{C}_2$-edge $\{u_1, u_2\}$ of H, we include a copy of H_0 to H' on the vertex set $(\{u_1\} \times (V_1 \setminus \{v_1\}) \cup \{[u_2, v_1]\}) \cup V_2 \cup V_3 \cup \cdots \cup V_k$.

Fix a proper coloring c of H. Color the vertices $\{u\} \times V_1$ by $c(u)$ and the vertices of $V_2 \cup V_k$ by $k-1$ mutually distinct colors not used by c. It is straightforward to verify that the obtained coloring is a proper coloring of H'. Hence, $\{x + k - 1 | x \in \mathcal{F}(H)\} \subseteq \mathcal{F}(H')$. Consider now a proper coloring c of H'. We show that c restricted to the vertices $U \times \{v_1\}$ is a proper coloring of H. Clearly, all the constrains imposed by Π-edges of H are satisfied. By our construction, it is easy to verify that if H contains a $\mathcal{D}_2$-edge $\{u_1, u_2\}$, then $c(u_1) \neq c(u_2)$ and that if H contains a $\mathcal{C}_2$-edge $\{u_1, u_2\}$, then $c(u_1) = c(u_2)$. Moreover, the vertices of $V_2 \cup \cdots \cup V_k$ are colored by $k-1$ colors distinct from the colors of the vertices $U \times \{v_1\}$ and the vertices of $U \times V_1$ have only the colors assigned to the vertices of $U \times \{v_1\}$. Hence, $\{x - k + 1 | x \in \mathcal{F}(H')\} \subseteq \mathcal{F}(H)$ and consequently $\mathcal{F}(H') = \{x + k - 1 | x \in \mathcal{F}(H)\}$. Since H has a broken feasible set, the feasible set of H' is also broken. □

We now introduce a notion of projections of edge types. An edge type Π' is a *$\mathcal{C}$-projection* of Π if there exist α and β such that $\Pi' = \{\pi | \pi \in \Pi \wedge \alpha \sim_\pi \beta\}$. An edge type Π' is a *$\mathcal{D}$-projection* of Π if there exist α and β such that $\Pi' = \{\pi | \pi \in \Pi \wedge \alpha \not\sim_\pi \beta\}$. A *projection* of Π is any edge type that can be obtained from Π by a sequence of $\mathcal{C}$-projections and $\mathcal{D}$-projections. Note that Π itself is a trivial projection of Π.

If Π' is a $\mathcal{C}$-projection (or $\mathcal{D}$-projection) of Π, then for every $(\Pi', \mathcal{C}_2, \mathcal{D}_2)$-hypergraph there is a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph with the exactly same proper colorings obtained by adding $\mathcal{C}_2$-edges (or $\mathcal{D}_2$-edges, respectively). Thus the following lemma is an immediate consequence of Lemma 11:

Lemma 12. *Let* Π *be an edge type containing neither the trivial nor the universal equivalence relation. If there exists a projection* Π' *of* Π *and a* $(\Pi', \mathcal{C}_2, \mathcal{D}_2)$*-hypergraph with a broken feasible set, then there exists a* Π*-hypergraph with a broken feasible set.*

Elements α, β, γ and δ (that need not be distinct) of the support set of an edge type Π have the *Property* $P\!\sim$ if the following holds:

- there is $\pi \in \Pi$ such that $\alpha \sim_\pi \beta$,
- $\alpha \not\sim_{\rho(\Pi)} \beta$, $\alpha \not\sim_{\rho(\Pi)} \gamma$, $\alpha \not\sim_{\rho(\Pi)} \delta$, $\beta \not\sim_{\rho(\Pi)} \gamma$ and $\gamma \not\sim_{\rho(\Pi)} \delta$, and
- for each $\pi \in \Pi$, if $\alpha \sim_\pi \beta$, then $\gamma \sim_\pi \delta$,

The elements α, β, γ and δ have the *Property* $P\!\not\sim$ if the last condition of the previous definition is replaced by:

- for each $\pi \in \Pi$, if $\alpha \not\sim_\pi \beta$, then $\gamma \sim_\pi \delta$,

The elements with one of these properties can be used to simulate other edge types and eventually to obtain pattern hypergraphs with broken feasible sets. We state this more precisely in the following series of lemmas.

Lemma 13. *If* Π *is an edge type such that* $\rho(\Pi) \notin \Pi$*, then there is a projection* Π_0 *of* Π *and (not necessarily distinct) elements* α*,* β*,* γ *and* δ *that have Property* $P\!\not\sim$ *for* Π_0*.*

Proof. Let $\pi_0 = \rho(\Pi)$ and α and β be any two elements such that $\alpha \not\sim_{\pi_0} \beta$ and $\alpha \sim_\pi \beta$ for some $\pi \in \Pi$. Consider the $\mathcal{D}$-projection Π' with respect to the pair α and β. Assume first that $\pi_0 \neq \rho(\Pi')$. Then there exist γ and δ such that $\gamma \not\sim_{\pi_0} \delta$ and $\gamma \sim_{\rho(\Pi')} \delta$. If $\alpha \sim_{\pi_0} \gamma$ and $\beta \sim_{\pi_0} \gamma$, then $\alpha \sim_{\pi_0} \beta$ which is impossible by the choice of α and β. Similarly, it cannot hold that $\alpha \sim_{\pi_0} \delta$ and $\beta \sim_{\pi_0} \delta$. An analogous argument excludes that $\alpha \sim_{\pi_0} \gamma$ and simultaneously $\alpha \sim_{\pi_0} \delta$, and $\beta \sim_{\pi_0} \gamma$ and simultaneously $\beta \sim_{\pi_0} \delta$.

If $\alpha \sim_{\pi_0} \gamma$ and $\beta \sim_{\pi_0} \delta$, then $\gamma \sim_{\rho(\Pi')} \delta$ implies $\alpha \sim_{\rho(\Pi')} \beta$ which is impossible by the definition of Π'. A similar argument excludes that $\alpha \sim_{\pi_0} \delta$ and $\beta \sim_{\pi_0} \gamma$. Hence, we may assume without loss of generality that $\alpha \not\sim_{\pi_0} \gamma$, $\alpha \not\sim_{\pi_0} \delta$ and $\beta \not\sim_{\pi_0} \gamma$. Recall that $\alpha \not\sim_{\pi_0} \beta$ and $\gamma \not\sim_{\pi_0} \delta$. Since $\gamma \sim_{\rho(\Pi')} \delta$ we conclude that the elements α, β, γ and δ have the property P$\not\sim$ for $\Pi_0 = \Pi$.

If $\rho(\Pi') = \rho(\Pi)$, we apply the same procedure to Π' by considering the new relation π_0' and new elements α' and β'. Since each time the number of equivalence relations contained in Π is decreased (we have chosen α and β such that $\alpha \sim_\pi \beta$ for a relation $\pi \in \Pi$) the process must eventually terminate. We either end with Π_0, α, β, γ and δ with the property P$\not\sim$ (Π_0 is obtained from Π by a sequence of $\mathcal{D}$-projections) or we end up with a $\mathcal{D}$-projection of Π comprised of a single equivalence relation π_0. In the latter case, $\pi_0 = \rho(\Pi)$ must be contained in Π which contradicts the assumption of the lemma. □

Lemma 14. *Let* Π *be an edge type which contains neither the universal nor the trivial equivalence relation. If there exist elements* α*,* β*,* γ *and* δ *that have Property* $P\!\not\sim$*, then there exists a* $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$*-hypergraph* H *with a broken feasible set.*

Proof. Let l be the edge size of Π (note that $l \geqslant 3$). Observe that α, β and γ are mutually different and δ is either β or different from all of α, β and γ. We construct a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with the vertices v_{ijk} for $1 \leqslant i \leqslant l+2$, $1 \leqslant j \leqslant l+4$ and $1 \leqslant k \leqslant l$. For simplicity, we say that the vertices with the same first index form rows and the vertices with the same second index form columns. Let c_1 be the $(l+2)$-coloring defined as $c_1(v_{ijk}) = i$ and c_2 the $(l+4)$-coloring defined as $c_2(v_{ijk}) = j$. We include to H as Π-edges all the l-tuples such that c_1 and c_2 remain proper colorings of H. Similarly, the pairs of vertices assigned different colors by both c_1 and c_2 form $\mathcal{D}_2$-edges and the pairs of vertices assigned the same colors by both the colorings form $\mathcal{C}_2$ edges. Hence, two vertices v_{ijk} and $v_{i'j'k'}$ form a $\mathcal{D}_2$-edge if and only if $i \neq i'$ and $j \neq j'$ and two vertices v_{ijk} and $v_{i'j'k'}$ form a $\mathcal{C}_2$-edge if and only if $i = i'$ and $j = j'$. Consider any proper coloring c of H and let ξ_{ij} be the common color of the vertices v_{ijk} for $1 \leqslant k \leqslant l$.

Let π_1 be an equivalence relation of Π such that $\gamma \not\sim_{\pi_1} \delta$ (and hence $\alpha \sim_{\pi_1} \beta$) and π_2 be an equivalence relation of Π such that $\alpha \not\sim_{\pi_2} \beta$ (and hence $\gamma \sim_{\pi_2} \delta$). We distinguish several cases with respect to the mutual relation of π_1, π_2, α, β, γ and δ: in each case, we show that either the rows or the columns of c are monochromatic under c and that different rows (columns respectively) use different colors (i.e., that c differs from c_1 or c_2 only by a permutation of colors). Therefore, $\mathcal{F}(H) = \{l+2, l+4\}$ and the feasible set of H is broken.

- $\beta = \delta$

  Assume that there exists i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. By the construction of H and the existence of π_1 and π_2, we can find an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{i'j'1}$ for all $i' \neq i$. Consequently $\xi_{i'j'} = \xi_{ij'}$ for all $i \neq i'$. Similarly, with setting $\alpha = v_{ij'1}$ and $\beta = v_{ij1}$ we conclude that $\xi_{i'j} = \xi_{ij}$ for all $i \neq i'$. Let j'' be any number different from both j and j'. Since $\xi_{ij''} \neq \xi_{ij}$ or $\xi_{ij''} \neq \xi_{ij'}$, we may conclude that $\xi_{ij''} = \xi_{i'j''}$ for all $i \neq i'$. Hence each column is monochromatic in c. In such case, c has to be a proper $(l+4)$-coloring by the presence of $\mathcal{D}_2$-edges.

  On the other hand, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c consists of monochromatic rows and it is a proper $(l+2)$-coloring.

- **$\beta \neq \delta$, either γ or δ is π_1-equivalent to α (and hence to β as well) and either α or β is π_2-equivalent to γ (and hence to δ as well).**

  By symmetry we can assume that $\beta \sim_{\pi_1} \delta$ and $\beta \sim_{\pi_2} \delta$ (otherwise we exchange α and β, or γ and δ). Proceed as in the previous case with the additional setting $\delta = v_{ij2}$ whenever β has been set to be v_{ij1} for some i and j (note that two such vertices are contained in a $\mathcal{C}_2$-edge).

- **$\beta \neq \delta$ and either γ or δ is π_1-equivalent to α (and hence to β as well), but $\alpha \not\sim_{\pi_2} \gamma$ and $\beta \not\sim_{\pi_2} \gamma$.**

  By symmetry we can assume that $\alpha \sim_{\pi_1} \gamma$. Assume that there exist i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. By the construction of H and the existence of π_1 and π_2, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$, $\gamma = v_{ij''1}$ and $\delta = v_{i'j''1}$ for any $i' \neq i$

and $j'' \neq j, j'$. But then $\xi_{ij''} = \xi_{i'j''}$. Hence all the columns are monochromatic with a possible exception for the two columns indexed by j and j'. By the presence of the $\mathcal{D}_2$-edges, there exist k and k', $k, k' \notin \{j, j'\}$ such that $\xi_{ik} \neq \xi_{ik'}$. By an analogous argument, we conclude that the j-th column and the j'-th column are also monochromatic. Consequently c is a proper $(l+4)$-coloring.

On the other hand, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', then the coloring c is a proper $(l+2)$-coloring.

- **$\beta \neq \delta$ and either α or β is π_2-equivalent to γ (and hence to δ as well), but $\alpha \not\sim_{\pi_1} \gamma$ and $\alpha \not\sim_{\pi_1} \delta$.**

  By symmetry we can assume that $\beta \sim_{\pi_2} \gamma$. Assume that there exist i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. By the construction of H and the existence of π_1 and π_2, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$, $\gamma = v_{i'j'1}$ and $\delta = v_{i''j'1}$ for all $i', i'' \neq i$. But then $\xi_{i'j'} = \xi_{i''j'}$ for all $i \neq i', i''$. Similarly as in the previous two cases, we conclude that all the columns except possibly for their elements of the i-th row are monochromatic. Using the same reasoning for another choice of i one can conclude that the columns are completely monochromatic. Hence c is a proper $(l+4)$-coloring.

  Otherwise, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c consists of monochromatic rows.

- **$\beta \neq \delta$, $\alpha \not\sim_{\pi_1} \gamma$, $\alpha \not\sim_{\pi_1} \delta$, $\alpha \not\sim_{\pi_2} \gamma$ and $\beta \not\sim_{\pi_2} \gamma$.**

  Assume that there exist i, j and j' such that $\xi_{ij} \neq \xi_{ij'}$. By the construction of H and the existence of π_1 and π_2, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{i'j''1}$ and $\delta = v_{i''j''1}$ for any $i \neq i', i''$ and $j'' \neq j, j'$. But then $\xi_{i'j''} = \xi_{i''j''}$. Hence, all the columns with a possible exception for the columns indexed by j and j' are monochromatic in all the rows different from the i-th row. Using analogous arguments as in the previous two cases, we infer that all the columns are completely monochromatic. Hence, the coloring c is a proper $(l+4)$-coloring.

  Again, if $\xi_{ij} = \xi_{ij'}$ for all possible triples i, j and j', the coloring c consists of monochromatic rows.

$\square$

Lemma 15. *Let Π be an edge type which contains neither the universal nor the trivial equivalence relation. If there exist (not necessarily distinct) elements α, β, γ and δ that have Property P$\sim$, then there exists a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph H with a broken feasible set.*

Proof. If $\rho(\Pi) \notin \Pi$, then by Lemmas 13 and 14, there exists a $(\Pi_0, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph with a broken feasible set a projection Π_0 of Π. Hence, there is also such a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph. In the rest, we deal with the case $\rho(\Pi) \in \Pi$. Let $\pi_0 = \rho(\Pi)$.

Let H be a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph on $l(l+2)(l+4)$ vertices obtained by the construction from the proof of Lemma 14. Let c be a proper coloring of H and let us keep the notation introduced in the proof of Lemma 14. Observe that the elements α, β and γ from the statement of the lemma are mutually different and δ is either β or different from all of α, β and γ.

We first deal with the case $\beta = \delta$. The assumptions of the Lemma 15 yield that $\alpha \not\sim_{\pi_0} \beta$, $\alpha \not\sim_{\pi_0} \gamma$ and $\beta \not\sim_{\pi_0} \gamma$. Let π be an equivalence relation of Π such that $\alpha \sim_\pi \beta$ (and thus $\beta \sim_\pi \gamma$). Consider i, j and j' such that $\xi_{ij} = \xi_{ij'}$. By the construction of H and the existence of π_0 and π, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$ and $\gamma = v_{ij''1}$ for any $j'' \neq j, j'$. This implies that $\xi_{ij} = \xi_{ij'} = \xi_{ij''}$ for all j''. Hence, each row is either monochromatic or completely polychromatic. If all the rows are monochromatic, then c is a proper $(l+2)$-coloring. If any of the rows is completely polychromatic, then c uses at least $l+4$ colors. We conclude that $\mathcal{F}(H) \subseteq \{l+2\} \cup [l+4, n]$ where $n = (l+2)(l+4)$, and since $\{l+2, l+4\} \subseteq \mathcal{F}(H)$, the feasible set of H is broken.

The remaining case is that all α, β, γ and δ are mutually different. If it holds that $\beta \sim_\pi \delta$ for all $\pi \in \Pi$, apply the same argument as in the previous paragraph setting $\delta = v_{ij'2}$. Again, we conclude that the feasible set of H is broken.

Assume that $\beta \not\sim_\pi \delta$ for some $\pi \in \Pi$. In particular, $\beta \not\sim_{\pi_0} \delta$. Observe that for any pair of elements α, β, γ and δ, there is $\pi \in \Pi$ such that the elements of this pair are not π-equivalent. Hence, any two of α, β, γ and δ are not π_0-equivalent. Let π be an equivalence of Π such that $\alpha \sim_\pi \beta$ (and thus $\gamma \sim_\pi \delta$). If all such equivalence relations satisfy that $\beta \sim_\pi \gamma$, then $\alpha \sim_\pi \beta$ implies $\beta \sim_\pi \gamma$ for each $\pi \in \Pi$. Set $\alpha' = \alpha$, $\beta' = \delta' = \beta$, $\gamma' = \gamma$ and apply the argument for the case when $\beta = \delta$. If this is not the case, there exists $\pi \in \Pi$ such that $\alpha \sim_\pi \beta$ but $\beta \not\sim_\pi \gamma$. Fix such a relation π.

Consider i, j and j' such that $\xi_{ij} = \xi_{ij'}$. By the construction of H and the existence of π_0 and π, there is an edge E of H such that $\alpha = v_{ij1}$, $\beta = v_{ij'1}$, $\gamma = v_{i'j''1}$ and $\delta = v_{i'j'''1}$ for any $i' \neq i$, $j'' \neq j, j'$ and $j''' \neq j, j', j''$. But then $\xi_{i'j''} = \xi_{i'j'''}$. Hence, each row except possibly for the i-th row is monochromatic with a possible exception for the elements of the j-th and j'-th column. Using a similar arguments as in the previous, we may conclude that all the rows are actually completely monochromatic. Hence, either all the rows are completely monochromatic or there is a completely polychromatic row. Again, the feasible set of H is broken. $\square$

We show in the last lemma of the series that if Π is not up-group-closed, then Lemma 14 or Lemma 15 can be applied:

Lemma 16. *If Π is an edge type that is not up-group-closed, then there exist a projection Π_0 of Π and (not necessarily distinct) elements α, β, γ and δ that have Property $P\!\sim$ or Property $P\!\not\sim$ for Π_0.*

Proof. If $\rho(\Pi) \notin \Pi$, then the statement follows from Lemma 13. Hence, we can assume that $\rho(\Pi) \in \Pi$. Since Π is not up-group-closed, there exists an equivalence relation $\pi \in \Pi$ such that a refinement π' of π with respect to $\rho(\Pi)$ is not contained in Π. Note that $\pi' \neq \rho(\Pi)$ since we have assumed that $\rho(\Pi) \in \Pi$. Hence, there exists α and β such that $\alpha \not\sim_{\rho(\Pi)} \beta$ and $\alpha \sim_{\pi'} \beta$.

Consider the $\mathcal{C}$-projection Π' of Π with respect to the pair α and β. Note that $\pi \in \Pi'$. We first consider the case when there are two elements γ and δ such that $\gamma \sim_{\rho(\Pi')} \delta$ and

$\gamma \not\sim_{\pi'} \delta$, i.e., the relation $\rho(\Pi')$ is not finer than π'. Note that since π' is coarser than $\rho(\Pi)$, it holds that $\gamma \not\sim_{\rho(\Pi)} \delta$. If $\alpha \sim_{\rho(\Pi)} \gamma$ and $\beta \sim_{\rho(\Pi)} \gamma$, then $\alpha \sim_{\rho(\Pi)} \beta$, which is impossible by the choice of α and β. Similarly, it does not hold that $\alpha \sim_{\rho(\Pi)} \delta$ and $\beta \sim_{\rho(\Pi)} \delta$, or $\alpha \sim_{\rho(\Pi)} \gamma$ and $\alpha \sim_{\rho(\Pi)} \delta$, or $\beta \sim_{\rho(\Pi)} \gamma$ and $\beta \sim_{\rho(\Pi)} \delta$. If $\alpha \sim_{\rho(\Pi)} \gamma$ and $\beta \sim_{\rho(\Pi)} \delta$, then $\alpha \sim_{\pi'} \beta$ implies $\gamma \sim_{\pi'} \delta$ which is impossible by the choice of γ and δ. An analogous argument yields that it cannot simultaneously hold that $\alpha \sim_{\rho(\Pi)} \delta$ and $\beta \sim_{\rho(\Pi)} \gamma$. Hence, it can be assumed that $\alpha \not\sim_{\rho(\Pi)} \gamma$, $\alpha \not\sim_{\rho(\Pi)} \delta$ and $\beta \not\sim_{\rho(\Pi)} \gamma$. Note that if $\alpha \sim_{\pi''} \beta$ for an equivalence relation $\pi'' \in \Pi$, then $\gamma \sim_{\pi''} \delta$ because $\gamma \sim_{\rho(\Pi')} \delta$ and Π' is the $\mathcal{C}$-projection of Π with respect to α and β. Since $\alpha \not\sim_{\rho(\Pi)} \beta$ and $\gamma \not\sim_{\rho(\Pi)} \delta$ by our previous discussion, the elements α, β, γ and δ satisfy the conditions of the statement of the lemma with $\Pi_0 = \Pi$.

The other case is that the relation $\rho(\Pi')$ is finer than π'. Since $\pi \in \Pi'$, $\pi' \notin \Pi'$ and $\rho(\Pi')$ is finer than π', Π' is not up-group-closed. Hence, we can apply the same procedure for a new pair of elements α' and β'. Since each time the number of equivalence relations contained in the edge type decreases (note that we have chosen α and β such that $\alpha \not\sim_{\rho(\Pi)} \beta$), we eventually end with the quadruple α, β, γ and δ and a suitable projection Π_0 of Π. □

We are now ready to prove our main results:

Theorem 17. *Let Π be an edge type. There exists a Π-hypergraph with a broken feasible set if and only if Π is neither simply-closed, down-closed, up-closed nor up-group-closed.*

Proof. We distinguish the following four cases based on whether the trivial and the universal equivalence relation is contained in Π:

- Π **contains both the trivial and the universal equivalence relation.**

  Note that if Π is down-closed, up-closed or up-group-closed, then it is also simply-closed. If Π is simply closed, then any Π-hypergraph has an unbroken feasible set by Lemma 3. If Π is not simply-closed, then there is a Π-hypergraph with a broken feasible set by Lemma 7.

- Π **contains the trivial equivalence relation but it does not contain the universal equivalence relation.**

  Π is neither simply-closed nor down-closed. Note that if Π is up-group-closed, then it is also up-closed. If Π is up-closed, then the feasible set of any Π-hypergraph is unbroken by Lemma 5. On the other hand, if Π is not up-closed, then there is a Π-hypergraph with a broken feasible set by Lemma 8.

- Π **contains the universal equivalence relation but it does not contain the trivial equivalence relation.**

  The edge-type Π is neither simply-closed nor up-closed. Note that if Π is up-group-closed, then it is also down-closed. If Π is down-closed, then any Π-hypergraph has unbroken feasible set by Lemma 4, and if Π is not down-closed, then there is a Π-hypergraph with a broken feasible set by Lemma 9.

- Π **contains neither the trivial nor the universal equivalence relation.**

  Π is not simply-closed, down-closed or up-closed. If Π is up-group-closed, then the feasible set of every Π-hypergraph is unbroken by Lemma 6. If Π is not up-group-closed, then there exists a $(\Pi, \mathcal{C}_2, \mathcal{D}_2)$-hypergraph with a broken feasible set by Lemma 14 or Lemma 15 applied to the quadruple α, β, γ and δ and the projection Π_0 of Π from Lemma 16. Consequently, there is a Π-hypergraph with a broken feasible set by Lemma 12.

□

We now state a version of Theorem 17 for pattern hypergraphs with more edge types:

Theorem 18. *Let $\Pi_1, \Pi_2, \ldots, \Pi_\lambda$ be fixed edge types for $i = 1, 2, \ldots, \lambda$. There exists a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set if and only if there exist (not necessarily distinct) integers i_1, i_2, i_3 and i_4, $1 \leqslant i_1, i_2, i_3, i_4 \leqslant \lambda$, such that:*

- *the edge type Π_{i_1} is not simply-closed, and*
- *the edge type Π_{i_2} is not down-closed, and*
- *the edge type Π_{i_3} is not up-closed, and*
- *the edge type Π_{i_4} is not up-group-closed.*

Proof. If all the edge types are simply-closed, then the feasible set of every $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph is unbroken by Lemma 3. Similarly, Lemmas 4, 5 and 6 imply that the feasible set of every $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph is unbroken if all the edge types are down-closed, up-closed or up-group-closed. In the rest of the proof, we show that otherwise there exists a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set.

Let l_i be the size of the edge type Π_i, $1 \leqslant i \leqslant \lambda$. We construct a new edge type Π of size $l_1 + \ldots + l_\lambda$. The support set A of Π consists of elements x_{ij} with $1 \leqslant i \leqslant \lambda$ and $1 \leqslant j \leqslant l_i$. For every i, $1 \leqslant i \leqslant \lambda$, the elements x_{ij} are identified with the elements of the support set of Π_i. An equivalence relation π is contained in Π if there exist equivalence relations $\pi_i \in \Pi_i$, $1 \leqslant i \leqslant \lambda$, such that the following holds for every i, j and j':

- $x_{ij} \sim_\pi x_{ij'}$ if and only if $x_{ij} \sim_{\pi_i} x_{ij'}$.

In other words, the equivalence relation π is comprised of some equivalence relations contained in $\Pi_1, \ldots, \Pi_\lambda$. Note that Π contains relations π with $x_{ij} \sim_\pi x_{i'j'}$ as well as with $x_{ij} \not\sim_\pi x_{i'j'}$.

Since Π_{i_1} is not simply-closed, the edge type Π is also not simply closed. Similarly, Π is not down-closed, up-closed or up-group-closed. By Theorem 17, there exists a Π-hypergraph H with a broken feasible set. We turn H into a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph H': each Π-edge E of H is decomposed into λ parts of sizes $l_1, \ldots, l_\lambda$ corresponding to the support sets of the original edge types and the parts form Π_i-edges of H', $1 \leqslant i \leqslant \lambda$. It is clear that every proper coloring of H is also a proper coloring of H' and vice versa. Hence, H' is a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set. □

5 Conclusion

There are several modifications of the concept of pattern hypergraphs that can look more general than pattern hypergraphs at the first sight, but they are all covered by Theorem 18:

- The edges are formed by unordered tuples. In this case, we may just assume that each edge type Π contains together with an equivalence relation π all the equivalence relations obtained from π by a permutation of the elements.
- Cycle systems considered in [9]. In this case, we just assume that each edge type Π contains together with an equivalence π all its rotations.
- The edges are ordered but a vertex may be contained in the same edge several times. There is a simple strategy how to cope with this setting: for each edge type Π, add all the edge types Π' of smaller sizes obtained by identifying some elements of the support set of Π and then consider the pattern hypergraphs with edges of all the types obtained in this way.

It is straightforward to state Theorem 18 in any of the above mentioned setting. As examples, we restate Theorem 18 in the first and the third case. Let us start with the latter one:

Theorem 19. *Let Π_i be edge types for $1 \leqslant i \leqslant \lambda$. Assume that a single vertex is allowed to be contained several times in an edge of a hypergraph. There exists a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set if and only if there exist (not necessarily distinct) integers i_1 and i_2, $1 \leqslant i_1, i_2 \leqslant \lambda$, such that:*

- *the edge type Π_{i_1} is not down-closed, and*
- *the edge type Π_{i_2} is not up-group-closed.*

In the first case, we need definitions. The unoriented edge type Π is a set of multisets of positive integers such that the integers in each multiset sum to a fixed integer ℓ which represents the edge size. The coloring of an edge with ℓ vertices of type Π is proper if the numbers of vertices of each color form one of the multisets contained in Π. An unoriented pattern hypergraph is a hypergraph with edges of unoriented edge types. The sought characterization theorem reads as follows.

Theorem 20. *Let Π_i be unoriented edge types for $1 \leqslant i \leqslant \lambda$. There exists a $(\Pi_1, \ldots, \Pi_\lambda)$-hypergraph with a broken feasible set if and only if there exist (not necessarily distinct) integers i_1, i_2 and i_3, $1 \leqslant i_1, i_2, i_3 \leqslant \lambda$, such that:*

- *there exists k such that Π_{i_1} does not contain the multiset $\{1, \ldots, 1, k\}$ tough the integers in this multiset sum to the integer ℓ corresponding to Π_{i_1},*
- *there exist $\{a_1, \ldots, a_k\} \in \Pi_{i_2}$ and $k \geqslant 2$ such that $\{a_1, \ldots, a_{k-1} + a_k\} \notin \Pi_{i_2}$, and*
- *there exist $\{a_1, \ldots, a_k\} \in \Pi_{i_3}$ with $a_k \geqslant 2$ such that $\{a_1, \ldots, a_{k-1}, a_k - 1, 1\} \notin \Pi_{i_3}$.*

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