Climbing elements in finite Coxeter groups

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Abstract

We define the notion of a climbing element in a finite real reflection group relative to a total order on the reflection set and we characterise these elements in the case where the total order arises from a bipartite Coxeter element.

1 Introduction

Suppose (W, S) is a finite Coxeter system. Each reduced expression for an element w of W determines a total order on the inversion set of w. The inversion set of the longest element w_0 of W is equal to the set, T, of all the reflections and a particular reduced expression for w_0 gives a total order, $\leqslant_T$, on T. For some elements w of W, the restriction of $\leqslant_T$ to the inversion set of w coincides with the order determined by one of its reduced expressions. We will call such an element w a *climbing* element of W. Geometrically, this means that there is a gallery from the fundamental domain C to $w(C)$ which crosses hyperplanes in increasing order.

In this paper, we characterise the climbing elements in the case where the reduced expression for w_0 is obtained by iterating a bipartite factorisation of a Coxeter element.

This characterisation is obtained using the construction from [6] of a copy of the type-W generalised associahedron, whose cone is a coarsening of the fan determined by the W reflection hyperplanes. This coarsening determines an equivalence relation on W whose equivalence classes we prove directly to be intervals in the left weak order. The least elements of these intervals are precisely the climbing elements. It follows that the number of climbing elements is equal to the W-Catalan number. The maximal elements in these intervals are translates of the falling elements of W, a notion that is analagous to that of climbing elements but which is defined using the reverse of the order $\leqslant_T$.

For any minimal factorisation of a Coxeter element, the interval property of the equivalence classes can be deduced from [9] and [10], where the minimal elements are the corresponding Coxeter-sortable elements. Thus we provide a different characterisation of Coxeter-sortable elements in the case of a bipartite factorisation of the Coxeter element. Indeed, the notion of climbing element arose from our attempts to show that the equivalence classes had the interval property without explicitly using Coxeter-sortable elements.

The paper is organised as follows. In §2 we collect some facts about inversion sets, extend a theorem of Papi and recall some results from [5] and [6] about orderings of roots and the geometry of the generalised associahedron. In §3 we define climbing elements and we show that each facet of the generalised associahedron determines such an element. We characterise climbing elements in §4 while in §5 we introduce and characterise falling elements.

2 Preliminaries

2.1 Inversion sets

For background on reflection groups, root systems and inversion sets we refer to [3] and [4]. Throughout this paper, (W, S) is a Coxeter system with W finite, acting effectively on $\mathbb{R}^n$ and with standard generating set $S = \{s_1, \ldots s_n\}$. Denote by T the reflection set of W, that is, the set of congugates of elements of S. Let C be the fundamental chamber with inward unit normals given by the simple roots $\{\alpha_1, \ldots, \alpha_n\}$, where s_i is the reflection in the hyperplane normal to α_i. Let $\{\beta_1, \ldots, \beta_n\}$ be the dual basis so that $\alpha_i \cdot \beta_j = \delta_{ij}$.

For each $w \in W$ we define $\text{Inv}_\rho(w)$ to be the set of positive roots λ such that $w^{-1}(\lambda)$ is a negative root. Thus $\text{Inv}_\rho(w)$ is the set of positive roots whose orthogonal hyperplanes separate the fundamental chamber C from its image $w(C)$. The corresponding set of reflections is denoted by $\text{Inv}(w)$, that is, $\text{Inv}(w) = \{R(\lambda) \mid \lambda \in \text{Inv}_\rho(w)\}$, where $R(\lambda)$ is the reflection in the hyperplane orthogonal to λ. We refer to $\text{Inv}(w)$ as the *inversion set* of w. If $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ is a reduced word, then $\text{Inv}(w) = \{t_1, \ldots, t_k\}$ where

$$t_1 = s_{i_1} \ < \ t_2 = s_{i_1} s_{i_2} s_{i_1} \ < \ t_3 = s_{i_1} s_{i_2} s_{i_3} s_{i_2} s_{i_1} \ < \ \ldots \tag{1}$$

as in section 1.3 of [3]. This defines a linear order on $\text{Inv}(w)$ and the corresponding linear order on $\text{Inv}_\rho(w)$ is given by

$$\alpha_{i_1} \ < \ s_{i_1}(\alpha_{i_2}) \ < \ s_{i_1} s_{i_2}(\alpha_{i_3}) \ < \ \ldots$$

In [8], Papi characterises ordered inversion sets among ordered subsets of T. His proof is given for crystallographic groups although he notes that it can be generalised to apply to all Coxeter groups. The following theorem modifies Papi's characterisation and is valid in the general finite case. For completeness the proof is included in an appendix.

Theorem 2.1 *An ordered subset Σ of positive roots for W is derived from a reduced expression for some element of W if and only if Σ satisfies both of the following conditions on triples $\{\sigma, \tau, \rho\}$ of positive roots satisfying $\rho = a\sigma + b\tau$ for some numbers $a > 0$ and $b > 0$.*
(i) Whenever σ and τ are elements of Σ with $\sigma < \tau$ then $\rho \in \Sigma$ and $\sigma < \rho < \tau$.
(ii) Whenever ρ is an element of Σ, then either (a) $\sigma \in \Sigma$ and $\sigma < \rho$ or (b) $\tau \in \Sigma$ and $\tau < \rho$.

2.2   Geometry of the generalised associahedron

Let Π^+ be the set of all positive roots. We recall from [5] the special features of the linear order on Π^+ determined by iterating a so-called bipartite Coxeter element. First assume that the elements of the simple system are ordered so that $\{\alpha_1, \dots, \alpha_s\}$ and $\{\alpha_{s+1}, \dots, \alpha_n\}$ are orthonormal sets. Let $c = R(\alpha_1)R(\alpha_2) \dots R(\alpha_n)$ be the corresponding Coxeter element. Because of this partitioning, such a c is called bipartite. If h denotes the order of c then W contains $nh/2$ reflections. Denoting by w_0 the longest element of W it follows from the proof of Corollary 4.5 of [11] that w_0 has the reduced expression

$$w_0 = \begin{cases} c^{h/2} & \text{if } h \text{ is even} \\ c^{(h-1)/2}R(\alpha_1)\dots R(\alpha_s) & \text{if } h \text{ is odd.} \end{cases}$$

It follows that the ordered set $\mathrm{Inv}_\rho(w_0)$ is equal to $\{\rho_1, \rho_2, \dots, \rho_{nh/2}\}$ where

$$\rho_i = R(\alpha_1)R(\alpha_2) \dots R(\alpha_{i-1})\alpha_i,$$

and we define $\alpha_i = \alpha_{i-n}$ for $i > n$. In fact, $\mathrm{Inv}_\rho(w_0) = \Pi^+$ and we denote this order on Π^+ by $\leqslant_\rho$ and by $\leqslant_T$ the corresponding order on the reflection set T.

Furthermore, in [5], we define the vectors

$$\mu_i = R(\alpha_1)R(\alpha_2) \dots R(\alpha_{i-1})\beta_i, \quad i = 1, 2, \dots, nh$$

where we similarly define $\beta_i = \beta_{i-n}$ for $i > n$. It is immediate from the definitions of ρ_i and μ_j that $\rho_{i+n} = c(\rho_i)$, $\mu_{j+n} = c(\mu_j)$ and $\rho_i \cdot \mu_i = 1$. We recall that $\mu_i = \mu(\rho_i)$ where μ is the linear map defined by $\mu = 2(I - c)^{-1}$. In particular, $\rho_i = (1/2)(I - c)\mu_i$. Furthermore, we have

Proposition 2.2 (Proposition 4.6 of [5])
(a) $\mu_i \cdot \rho_j = -\mu_{j+n} \cdot \rho_i$ for all i and j.
(b) $\mu_i \cdot \rho_j \geqslant 0$, for $1 \leqslant i \leqslant j \leqslant nh/2$.
(c) $\mu_{i+t} \cdot \rho_i = 0$, for $1 \leqslant t \leqslant n - 1$ and for all i.
(d) $\mu_j \cdot \rho_i \leqslant 0$ for $1 \leqslant i < j \leqslant nh/2$.

We recall from [6] that a copy, denoted $\mu AX(c)$, of the type-W associahedron has a facet with vertex set $\{\mu(\tau_1), \ldots, \mu(\tau_n)\}$ whenever both

$$\rho_1 \leqslant \tau_1 < \tau_2 < \ldots < \tau_n \leqslant \rho_{nh/2+n} \quad \text{and} \quad c = R(\tau_n) \ldots R(\tau_1).$$

We also recall that $\mu AX(c)$ determines a particular coarsening of the Coxeter fan, that is, of the fan defined by the W reflection hyperplanes. This coarsening has rays in the directions $\mu_1, \ldots, \mu_{nh/2+n}$ and each maximal cone is of the form $\mathrm{cone}(F)$, where F is a facet $\mu AX(c)$ and $\mathrm{cone}(F)$ denotes the positive cone on F. We define an equivalence relation on W by $w \sim w'$ if and only if $w(C)$ and $w'(C)$ are contained in the same maximal cone.

Finally, we will use the filtration of $\mu AX(c)$ inherited from the filtration of $X(c)$ used in [5]. For each root ρ we define the subsets ρ^+, ρ^- and $\rho^\perp$ by

$$\begin{aligned}
\rho^+ &= \{x \in \mathbb{R} \mid x \cdot \rho \geqslant 0\} \\
\rho^- &= \{x \in \mathbb{R} \mid x \cdot \rho \leqslant 0\} \\
\rho^\perp &= \{x \in \mathbb{R} \mid x \cdot \rho = 0\}.
\end{aligned}$$

For $n \leqslant i \leqslant nh/2 + n$, we define $V_i = \{\mu_1, \ldots, \mu_i\}$, μX_i to be the subcomplex of $\mu AX(c)$ consisting of those simplices with vertices in V_i and

$$\mu Z_i = \rho^+_{i-n+1} \cap \rho^+_{i-n+2} \cap \ldots \cap \rho^+_{nh/2}.$$

It follows that the closure of $\mu Z_i \setminus \mu Z_{i-1}$ is equal to

$$\rho^-_{i-n} \cap \rho^+_{i-n+1} \cap \ldots \cap \rho^+_{nh/2}.$$

We note that μZ_n and $\mu Z_{nh/2+n}$ coincide with the fundamental chamber C and with $\mathbb{R}^n$ respectively. We also note that Proposition 7.6 of [5] (in the case $\alpha = c$) can be extended to show that μZ_i coincides with both the positive cone on μX_i and the positive span of V_i.

3   Climbing elements

In this section we define climbing elements and show that each subset of the vertex set of $\mu AX(c)$ determines a climbing element. In the case of the vertex set of a facet we will show that this climbing element is the minimum in the corresponding equivalence class of $(W, \sim)$.

Definition 3.1 *An element w of W is climbing (with respect to the reflection order $\leqslant_T$) if the order on $\mathrm{Inv}(w)$ given by $\leqslant_T$ coincides with the order determined by one of the reduced expressions for w.*

Definition 3.2 *For each subset A of $V_{nh/2+n}$ we define the set $N(A)$ of positive roots by*

$$N(A) = \{\rho_i \mid 1 \leqslant i \leqslant nh/2 \text{ and } \rho_i \cdot \mu \leqslant 0 \text{ for all } \mu \in A\}.$$

Thus a positive root ρ belongs to N(A) if and only if $A \subseteq \rho^-$.

Example 3.3 *If $A = \{\mu(\rho_i)\}$ then Proposition 2.2 implies that*

$$N(A) = \{\rho_j : j < i \text{ or } \rho_j \cdot \mu(\rho_i) = 0\}.$$

For a larger set B, $N(B)$ is the intersection of sets of this form.

Proposition 3.4 *For each subset A of $V_{nh/2+n}$ there exists a element $w \in W$ such that the ordered set $(N(A), \leqslant_\rho)$ coincides with the ordered set $\mathrm{Inv}_\rho(w)$ for some reduced expression of w. In particular, w is climbing.*

Proof: We show that $N(A)$ satisfies the criteria (i) and (ii) of Theorem 2.1. First suppose $\rho_i, \rho_j \in N(A)$ with $i < j$ and that $a, b > 0$ are such that $\rho_k = a\rho_i + b\rho_j$ is a positive root. For each $\mu \in A$ we have

$$\rho_k \cdot \mu = (a\rho_i + b\rho_j) \cdot \mu = a(\rho_i \cdot \mu) + b(\rho_j \cdot \mu) \leqslant 0$$

since $\rho_i, \rho_j \in N(A)$. Thus, $\rho_k \in N(A)$. As the order $\leqslant_\rho$ on Π^+ is derived from a particular reduced expression for the longest element w_0, the 'only if' part of Theorem 2.1 yields $\rho_i \leqslant_\rho \rho_k \leqslant_\rho \rho_j$ and criterion (i) follows.

Next, suppose that ρ_i and ρ_j are positive roots with $i < j$ and that $a, b > 0$ are such that $\rho_k = a\rho_i + b\rho_j \in N(A)$. As in the previous paragraph, Theorem 2.1 yields $\rho_i \leqslant_\rho \rho_k \leqslant_\rho \rho_j$. It remains to show that $\rho_i \in N(A)$.

If $\rho_i \notin N(A)$ then $\rho_i \cdot \mu > 0$ for some $\mu \in A$. By definition of $\mu AX(c)$, $\mu = \mu(\rho_q)$ for some root ρ_q with $1 \leqslant q \leqslant nh/2 + n$. In fact, $1 \leqslant q \leqslant nh/2$ since $\{\mu_{nh/2+1}, \ldots, \mu_{nh/2+n}\}$ are the rays of the cone $w_0(C)$, the opposite chamber to C. Now part (d) of Proposition 2.2 gives $q \leqslant i$. Therefore $q < j$ and, hence, part (b) of Proposition 2.2 implies that $\rho_j \cdot \mu = \rho_j \cdot \mu(\rho_q) \geqslant 0$. Thus

$$\rho_k \cdot \mu = (a\rho_i + b\rho_j) \cdot \mu = a(\rho_i \cdot \mu) + b(\rho_j \cdot \mu) \geqslant a(\rho_i \cdot \mu) > 0,$$

contradicting the assumption that $\rho_k \in N(A)$.

If F is a facet of $\mu AX(c)$, we denote its set of vertices by V_F. That is $V_F = F \cap V_{nh/2+n}$. Such vertex sets will be particularly important in the sequel.

Proposition 3.5 *If F is a facet of $\mu AX(c)$ and $x_F \in W$ is the climbing element with $\mathrm{Inv}_\rho(x_F) = N(V_F)$, then $x_F(C) \subseteq cone(F)$.*

Proof: The set $cone(F)$ can be characterised as an intersection of halfspaces determined by the roots ρ_i. We show that $x_F(C)$ is contained in the same intersection. If ρ_i is a positive root with F contained in ρ_i^-, then $\rho_i \in N(V_F)$. Since $N(V_F) = \mathrm{Inv}_\rho(x_F)$, it follows that $x_F(C)$ must also be contained in ρ_i^-. On the other hand, if ρ_j is a positive root with F contained in ρ_j^+ then $\rho_j \notin N(V_F)$ since F has nonempty interior and hence cannot be contained in $\rho_j^\perp$. Thus $x_F(C)$ must also be contained in ρ_j^+.

Corollary 3.6 *Each equivalence class of $(W, \sim)$ contains a minimum in the left weak order on W.*

Proof: Let F be a facet of $\mu AX(c)$ with vertex set V_F and let x_F be the element of W whose inversion set is $N(V_F)$ (Proposition 3.4). By Proposition 3.5, $x_F(C)$ is contained in cone(F). If $w \sim x_F$ then $w(C) \subseteq$ cone(F), by definition, and it follows that $w(C) \subset \rho_i^-$ for each $\rho_i \in N(V_F)$. Thus $N(V_F) \subseteq \text{Inv}_\rho(w)$ and Proposition 3.1.3 of [3] now implies that x_F precedes w in the left weak order on W.

4 Characterising climbing elements

The proof of Corollary 3.6 shows that the number of facets of $\mu AX(c)$ does not exceed the number of climbing elements. In fact the theorem below implies that these two numbers are equal. The number of facets of $\mu AX(c)$ is one of the quantities counted by the W-Catalan number. For a description of these numbers and their properties see Chapter 1 of [1].

Lemma 4.1 *If $\mu(\rho_i)$ is the last vertex of a facet F of $\mu AX(c)$ and if w is a climbing element for which $w(C) \subset$ cone(F), then $R(\rho_{i-n})w$ is also a climbing element.*

Proof: Assume that $\mu(\rho_{i_1})$, $\mu(\rho_{i_2})$, ..., $\mu(\rho_{i_{n-1}})$, $\mu(\rho_i)$ are the vertices of F where $1 \leqslant i_1 < \ldots < i_{n-1} < i \leqslant nh/2 + n$ and $c = R(\rho_i)R(\rho_{i_{n-1}}) \ldots R(\rho_{i_1})$. Since $1 \leqslant i - n \leqslant nh/2$ and

$$c = R(\rho_i)R(\rho_{i_{n-1}}) \ldots R(\rho_{i_1}) = R(\rho_{i_{n-1}}) \ldots R(\rho_{i_1})R(\rho_{i-n}),$$

Lemma 2.2 of [2] implies that $\rho_{i-n} \cdot \mu(\rho_{i_k}) = 0$ for $k = 1, 2, \ldots, n-1$. Thus the face of F opposite to the vertex $\mu(\rho_i)$ is contained in the hyperplane $\rho_{i-n}^\perp$. It follows that

$$F \subseteq \overline{\mu Z_i \setminus \mu Z_{i-1}} = \rho_{i-n}^- \cap \rho_{i-n+1}^+ \cap \ldots \cap \rho_{nh/2}^+$$

and, hence, the last wall crossed by any increasing gallery for w is $\rho_{i-n}^\perp$. If we delete the last chamber from such an increasing gallery, we obtain an increasing gallery for $R(\rho_{i-n})w$. Therefore $R(\rho_{i-n})w$ is a climbing element, as required.

Theorem 4.2 *Each equivalence class of $(W, \sim)$ contains exactly one climbing element. In particular the number of climbing elements is equal to the W-Catalan number.*

Proof: Fix an associahedron facet F whose vertices are $\mu(\rho_{i_1})$, $\mu(\rho_{i_2})$, ..., $\mu(\rho_{i_{n-1}})$, $\mu(\rho_i)$ where $1 \leqslant i_1 < \ldots < i_{n-1} < i \leqslant nh/2 + n$ and $c = R(\rho_i)R(\rho_{i_{n-1}}) \ldots R(\rho_{i_1})$. We need to show that there is only one climbing element $w \in W$ for which $w(C) \subset$ cone(F). Our proof is by induction on i.

First note that $i \geqslant n$ and if $i = n$ then cone(F) must coincide with the fundamental domain C. In this case the identity element of W is the only element for which $w(C) \subset$ cone(F).

Assume now that $i > n$ and that for each associahedron facet $F' \subseteq \mu Z_{i-1}$ there is a unique climbing element w' for which $w'(C) \subseteq \text{cone}(F')$. Let G be the only other associahedron facet which contains the face $F \cap \rho_{i-n}^\perp$. Since $\mu(\rho_i) \cdot \rho_{i-n} < 0$, G is contained in μZ_{i-1}. Then $[R(\rho_{i-n})w](C)$ also lies in $\text{cone}(G)$ since G shares the face $\rho_{i-n}^\perp \cap F$ with F. As $R(\rho_{i-n})w$ is climbing (by Lemma 4.1), the induction hypothesis implies that $R(\rho_{i-n})w = w'$, the unique climbing element for which $w'(C) \subseteq \text{cone}(G)$. Hence $w = R(\rho_{i-n})w'$ is uniqely determined.

Corollary 4.3 *The set of climbing elements in W coincides with the set of Coxeter-sortable elements of W.*

Proof: By Theorem 1.1 of [9] the Coxeter-sortable elements of W are precisely the minima of the equivalence classes of $(W, \sim)$. By Theorem 4.2 and the proof of Corollary 3.6, the climbing elements are also the minima of these equivalence classes.

5 Falling elements

In this section we show that each equivalence class of $(W, \sim)$ contains a maximum in the left weak order on W. Just as a climbing element is reached from the fundamental chamber C via a gallery which crosses hyperplanes in increasing order, each of these maxima is reached from the opposite chamber $w_0(C)$ via a gallery which crosses hyperplanes in decreasing order. In order to use the results of sections 3 and 4 our strategy is to rebuild the fan determined by $\mu AX(c)$ with $w_0(C)$ taking the place of C and c^{-1} taking the place of c. This will give an ordering on T which is the reverse of the order $\leqslant_T$ and we will refer to the corresponding notion of climbing element as a falling element. The required maxima will then have the form fw_0 where f is falling.

Since the inward pointing normals for $w_0(C)$ are just the negatives of the inward pointing normals for C, the new simple system will be $\{-\alpha_1, \ldots, -\alpha_n\}$. We will order this simple system by using the corresponding order on the dual basis. Sometimes this order is different than the order $-\alpha_n, \ldots, -\alpha_1$ but we will see that it gives the reverse of the order $\leqslant_T$ on T.

Definition 5.1 *For $1 \leqslant j \leqslant n$ we define $\beta_i' = \mu_{nh/2+n-i+1}$ and we define $\{\alpha_1', \ldots, \alpha_n'\}$ to be the dual basis to $\{\beta_1', \ldots, \beta_n'\}$.*

Proposition 5.2 *The set $\{\beta_1', \ldots, \beta_{n-s}'\}$ is a permutation of $\{-\beta_{s+1}, \ldots, -\beta_n\}$ and the set $\{\beta_{n-s+1}', \ldots, \beta_n'\}$ is a permutation of $\{-\beta_1, \ldots, -\beta_s\}$.*

Proof: This follows from Steinberg's proof of Theorem 4.2 of [11], where the vectors he denotes by σ and τ lie in the non-negative linear spans of our $\beta_1, \ldots, \beta_s$ and $\beta_{s+1}, \ldots, \beta_n$, respectively.

Corollary 5.3 *The set $\{\alpha_1', \ldots, \alpha_{n-s}'\}$ is a permutation of $\{-\alpha_{s+1}, \ldots, -\alpha_n\}$ while the set $\{\alpha_{n-s+1}', \ldots, \alpha_n'\}$ is a permutation of $\{-\alpha_1, \ldots, -\alpha_s\}$. In particular, the product $c^{-1} = R(\alpha_1')R(\alpha_2') \ldots R(\alpha_n')$ is a bipartite factorisation.*

Definition 5.4 *With the convention that* $\alpha'_{i+n} = \alpha'_i$ *and* $\beta'_{i+n} = \beta'_i$, *we define* $\mu'_i = R(\alpha'_1)R(\alpha'_2)\ldots R(\alpha'_{i-1})\beta'_i$ *and* $\rho'_i = R(\alpha'_1)R(\alpha'_2)\ldots R(\alpha'_{i-1})\alpha'_i$.

Note that $\mu'_j = \beta'_j$ for $1 \leqslant j \leqslant n$ and $\mu'_{i+n} = c^{-1}\mu'_i$.

Proposition 5.5 *The vectors* μ'_j *and* ρ'_i *are related to* μ_j *and* ρ_i *by* $\mu'_j = \mu_{nh/2+n-j+1}$ *and* $\rho'_i = -\rho_{nh/2-i+1}$, *for* $1 \leqslant j \leqslant nh/2 + n$ *and* $1 \leqslant i \leqslant nh/2$ *respectively.*

Proof: For the first identity, write $j = mn + k$ with $0 \leqslant k < n$. Then

$$
\begin{aligned}
\mu'_j &= R(\alpha'_1)\ldots R(\alpha'_{j-1})\beta'_j \\
&= [c^{-1}]^m R(\alpha'_1)\ldots R(\alpha'_{k-1})\beta'_k \\
&= c^{-m}\beta'_k \quad \text{since } \beta'_k \perp \alpha'_1,\ldots,\alpha'_{k-1} \\
&= c^{-m}\mu_{nh/2+n-k+1} \\
&= \mu_{nh/2+n-mn-k+1} \\
&= \mu_{nh/2+n-j+1}.
\end{aligned}
$$

For the second identity, we use the first identity and the relationships

$$
c\mu_i = \mu_i - 2\rho_i, \quad c^{-1}\mu'_i = \mu'_i - 2\rho'_i
$$

to get

$$
\begin{aligned}
2\rho'_i &= (I - c^{-1})\mu'_i \\
&= (I - c^{-1})\mu_{nh/2+n-i+1} \\
&= (I - c^{-1})c\mu_{nh/2-i+1} \\
&= -(I - c)\mu_{nh/2-i+1} \\
&= -2\rho_{nh/2-i+1},
\end{aligned}
$$

for $1 \leqslant i \leqslant nh/2$.

We now construct a copy of the type-W associahedron using c^{-1} instead of c and $\{\mu'_1,\ldots,\mu'_{nh/2+n}\}$ instead of $\{\mu_1,\ldots,\mu_{nh/2+n}\}$. We find that the geometric complex is exactly the same since the vertex sets coincide by Proposition 5.5 and there is a facet on a set

$$
\{\mu'_{i_1},\ldots,\mu'_{i_n}\}
$$

if and only if there is facet on the corresponding set

$$
\{\mu_{nh/2+n-i_1+1},\ldots,\mu_{nh/2+n-i_n+1}\}.
$$

The reflection ordering $\rho'_1, \rho'_2, \rho'_3, \ldots$ is the reverse of $\leqslant_T$ and determines a different notion of climbing element which we will now call falling.

Definition 5.6 *An element w of W is falling if the order on $Inv(w)$ given by the reverse of the total order $\leqslant_T$ coincides with the order determined by one of the reduced expressions for w.*

The results of sections 3 and 4 apply to give

Theorem 5.7 *Each equivalence class of $(W, \sim)$ determines a unique falling element f. The element fw_0 is the maximal element in the corresponding equivalence class in the left weak order.*

Corollary 5.8 *Each equivalence class of $(W, \sim)$ is an interval in the left weak order on W.*

6 Appendix

Before proving Theorem 2.1, we prove some elementary facts.

Lemma 6.1 *If the positive root ρ is not simple then we can write $\rho = a\sigma + b\tau$ for some real numbers $a, b > 0$ and some positive roots σ, τ.*

Proof: First, if ρ is any positive root then we can write

$$\rho = a_1 \alpha_1 + \ldots + a_n \alpha_n \quad \text{with} \quad a_i \geqslant 0 \ \text{ for } \ 1 \leqslant i \leqslant n$$

and it follows that

$$0 < \rho \cdot \rho = \rho \cdot \left(\sum_i a_i \alpha_i \right) = \sum_i a_i (\rho \cdot \alpha_i).$$

yielding $\rho \cdot \alpha_i > 0$ for some simple root α_i.

Now suppose that ρ is a non-simple, positive root and that α_i is a simple root with $\rho \cdot \alpha_i > 0$ as above. Since ρ is not a simple root, it follows that $s_i(\rho) = \sigma$ is a positive root. However, $\sigma = \rho - b\alpha_i$, and hence $\rho = \sigma + b\alpha_i$ where $b = 2(\rho \cdot \alpha_i) > 0$, as required.

Define the vector v_0 by $v_0 = \beta_1 + \ldots + \beta_n$ and note that v_0 lies in the interior of the fundamental chamber C since $v_0 \cdot \alpha_i = 1$, for each i. Note also that for each $w \in W$, the set $Inv_\rho(w)$ is equal to the set of positive roots λ such that $\lambda \cdot w(v_0) < 0$.

Lemma 6.2 *If $w \in W$ and $w(\sigma) \in Inv_\rho(w)$ then σ is a negative root.*

Proof: Directly from the definition of $Inv_\rho(w)$ we have

$$\sigma \cdot v_0 = w(\sigma) \cdot w(v_0) < 0.$$

Proof of Theorem 2.1: First, assume that the ordered set Σ is derived from a reduced expression $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ for some element $w \in W$. Extend this to a reduced expression

$$s_{i_1} s_{i_2} \ldots s_{i_{nh/2}}$$

for the longest element of W, as in Section 1.8 of [7]. For each $1 \leqslant j \leqslant nh/2$, let $w_j = s_{i_1} s_{i_2} \ldots s_{i_j}$ be the jth prefix of this expression and note that $w = w_k$.

For condition (i), assume that $\sigma < \tau$ are elements of Σ and that $\rho = a\sigma + b\tau$ is a positive root for some $a, b > 0$. Then $R(\sigma) = t_{r_1}$, $R(\rho) = t_{r_2}$ and $R(\tau) = t_{r_3}$ for some $1 \leqslant r_1 < r_3 \leqslant k$ (by our assumption on Σ) and some $1 \leqslant r_2 \leqslant nh/2$, and where the t_j are given by equation (1). We show that $r_1 < r_2 < r_3$ by eliminating the other possibilities. It then follows that $\rho \in \Sigma$. First, if $r_2 < r_1 < r_3$ then

$$w_{r_2}(v_0) \cdot \sigma > 0 \quad \text{and} \quad w_{r_2}(v_0) \cdot \tau > 0 \quad \text{while} \quad w_{r_2}(v_0) \cdot \rho < 0.$$

This is impossible since ρ is a positive linear combination of σ and τ. Similarly, if $r_1 < r_3 < r_2$ then

$$w_{r_3}(v_0) \cdot \sigma < 0 \quad \text{and} \quad w_{r_3}(v_0) \cdot \tau < 0 \quad \text{while} \quad w_{r_3}(v_0) \cdot \rho > 0$$

which is also impossible since ρ is a positive linear combination of σ and τ.

For condition (ii), assume that σ and τ are positive roots and that $a, b > 0$ are such that $\rho = a\sigma + b\tau$ is an element of Σ. Thus $R(\rho) = t_r$, for some $1 \leqslant r \leqslant k$, and hence $w_r(v_0) \cdot \rho < 0$. Since ρ is a positive linear combination of σ and τ, at least one of $w_r(v_0) \cdot \sigma$ and $w_r(v_0) \cdot \tau$ must be strictly negative. Thus, either $R(\sigma) \in \text{Inv}(w_r)$ and hence $\sigma \leqslant \rho$ or $R(\tau) \in \text{Inv}(w_r)$ and hence $\tau \leqslant \rho$. As $a, b > 0$, we can exclude the possibilities of $\sigma = \rho$ or $\tau = \rho$.

For the converse, assume that Σ is a set of positive roots which satisfies conditions (i) and (ii). As in [8], we proceed by induction on the cardinality of Σ. To start the induction we assume that $\Sigma = \{\rho\}$. It suffices to show that ρ is a simple root, for then $(R(\rho)$ is the required group element. If ρ is not a simple root, then Lemma 6.1 implies that $\rho = a\sigma + b\tau$ for some other positive roots σ and τ and some $a, b > 0$. By condition (ii), either σ or τ is also in Σ, contradicting the assumption that Σ has cardinality one.

For the inductive step, assume that $k > 1$ and that the result is true for sets of cardinality less than k. Assume that $\Sigma = \{\rho_1, \rho_2, \ldots, \rho_k\}$ satisfies conditions (i) and (ii). Then the ordered set $\Sigma' = \{\rho_1, \rho_2, \ldots, \rho_{k-1}\}$ also satisfies these two conditions and hence there is a reduced expression $u = s_{i_1} s_{i_2} \ldots s_{i_{k-1}}$, of some element $u \in W$, such that

$$\rho_1 = \alpha_{i_1}, \quad \rho_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \rho_{k-1} = s_{i_1} s_{i_2} \ldots s_{i_{k-2}}(\alpha_{i_{k-1}}).$$

If $u^{-1}(\rho_k)$ is a simple root, α_{i_k} say, then the positivity of α_{i_k} implies that $l(s_{i_1} s_{i_2} \ldots s_{i_k}) = l(u) + 1$ and, hence, $w = u s_{i_k} = s_{i_1} s_{i_2} \ldots s_{i_k}$ is the required minimal expression. Thus it remains to show that $u^{-1}(\rho_k)$ must be simple.

Assume that $u^{-1}(\rho_k)$ is not simple. As $\rho_k \notin \text{Inv}_\rho(u)$, it follows that $u^{-1}(\rho_k)$ is a positive root. Then $u^{-1}(\rho_k) = a\sigma + b\tau$ for some positive roots σ and τ and some real numbers $a, b > 0$, by Lemma 6.1. Thus

$$\rho_k = au(\sigma) + bu(\tau). \tag{2}$$

In order to apply condition (ii) to this equation, we need to show that neither $u(\sigma)$ nor $u(\tau)$ can be a negative root. For example, if $u(\sigma)$ is negative, then

$$-u(\sigma) \cdot v_0 > 0 \quad \text{and} \quad -u(\sigma) \cdot u(v_0) = -\sigma \cdot v_0 < 0$$

putting $-u(\sigma)$ in $\mathrm{Inv}_\rho(u)$. Thus $-u(\sigma) = \rho_i$ for some $i < k$. Condition (i) applied to the expression $u(\tau) = (1/b)\rho_k + (a/b)\rho_i$ now implies that $u(\tau) \in \Sigma$ and $u(\tau) < \rho_k$, so that, in fact, $u(\tau) \in \Sigma' = \mathrm{Inv}_\rho(u)$. This gives a contradiction by Lemma 6.2.

Thus both $u(\sigma)$ and $u(\tau)$ must be positive and by Lemma 6.2 again neither belong to $\mathrm{Inv}_\rho(u)$. This gives a contradiction since condition (ii) applied to equation (2) implies that one of $u(\sigma)$ and $u(\tau)$ is in Σ and precedes ρ_k, putting one of $u(\sigma)$ and $u(\tau)$ in $\Sigma' = \mathrm{Inv}_\rho(u)$.

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