Regular factors of regular graphs from eigenvalues*

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Abstract

Let r and m be two integers such that $r \geqslant m$. Let H be a graph with order $|H|$, size e and maximum degree r such that $2e \geqslant |H|r - m$. We find a best lower bound on spectral radius of graph H in terms of m and r. Let G be a connected r-regular graph of order $|G|$ and $k < r$ be an integer. Using the previous results, we find some best upper bounds (in terms of r and k) on the third largest eigenvalue that is sufficient to guarantee that G has a k-factor when $k|G|$ is even. Moreover, we find a best bound on the second largest eigenvalue that is sufficient to guarantee that G is k-critical when $k|G|$ is odd. Our results extend the work of Cioabă, Gregory and Haemers [*J. Combin. Theory Ser. B*, 1999] who obtained such results for 1-factors.

1 Introduction

Throughout this paper, G denotes a simple graph of *order* n (the number of vertices) and *size* e (the number of edges). For two subsets $S, T \subseteq V(G)$, let $e_G(S,T)$ denote the number of edges of G joining S to T. The eigenvalues of G are the *eigenvalues* λ_i of its adjacency matrix A, indexed so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The largest eigenvalue is often called *spectral radius*. If G is k-regular, then it is easy to see that $\lambda_1 = k$ and also, $\lambda_2 < k$ if and only if G is connected. A *matching* of a graph G is a set of mutually disjoint edges. A matching is *perfect* if every vertex of G is incident with an edge of the matching. Let a be a nonnegative integer and we denote a matching of size a by M_a. Let $\overline{G}$ denote the complement of a graph G. The join $G + H$ denotes the graph with vertex $V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G) \text{ and } y \in V(H)\}.$$

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For a general graph G and an integer k, a spanning subgraph F of G such that

$$d_F(x) = k \text{ for all } x \in V(G)$$

is called a *k-factor*. Given a subgraph H of G, we define the *deficiency* of H with respect to k-factor as

$$def_H(G) = \sum_{v \in V} |k - d_H(v)|.$$

The *total deficiency* of a graph G is defined as

$$def(G) = \min_{H \subseteq G} def_H(G).$$

F is called a *k-optimal subgraph* of G if $def_F(G) = def(G)$. Clearly, G has a k-factor if and only if $def(G) = 0$. We call a graph G *k-critical*, if G contains no k-factors, but for any fixed vertex x of $V(G)$, there exists a subgraph H of G such that $d_H(x) = k \pm 1$ and $d_H(y) = k$ for any vertex y ($y \neq x$). Tutte [13] obtained the well-known k-Factor Theorem in 1952.

Theorem 1.1 (Tutte [13]) *Let $k \geqslant 1$ be an integer and G be a general graph. Then G has a k-factor if and only if for all disjoint subsets S and T of $V(G)$,*

$$\begin{aligned}\delta_G(S,T) &= k|T| + e_G(S,T) + \tau_G(S,T) - k|S| - \sum_{x \in T} d_G(x)\\ &= k|T| + \tau_G(S,T) - k|S| - \sum_{x \in T} d_{G-S}(x) \leqslant 0,\end{aligned}$$

where $\tau_G(S,T)$ denotes the number of components C, called k-odd components of $G-(S \cup T)$ such that $e_G(V(C),T) + k|C| \equiv 1 \pmod 2$. Moreover, $\delta(S,T) \equiv k|V(G)| \pmod 2$.

Furthermore, Lovász proved the well-known k-defficiency Theorem in 1970.

Theorem 1.2 (Lovász [10]) *Let G be a graph and k a positive integer. Then*

$$\begin{aligned}def(G) &= \max \delta_G(S,T)\\ &= \max\{k|T| + \tau_G(S,T) - k|S| - \sum_{x \in T} d_{G-S}(x) \mid S, T \subseteq V(G), \text{ and } S \cap T = \emptyset\}\end{aligned}$$

where $\tau_G(S,T)$ is the number of components C of $G-(S \cup T)$ such that $e(V(C),T)+k|C| \equiv 1 \pmod 2$. Moreover, $\delta_G(S,T) \equiv k|V(G)| \pmod 2$. Furthermore, G is not k-critical if and only if there exist two disjoint subsets S and T with $S \cup T \neq \emptyset$ such that $\delta_G(S,T) > 0$.

In [2], Brouwer and Haemers gave sufficient conditions for a graph to have a 1-factor in terms of its Laplacian eigenvalues and, for a regular graph, gave an improvement in terms of the third largest adjacency eigenvalue λ_3. Cioabă and Gregory [4] also studied relations between 1-factors and eigenvalues. Later, Cioabă, Gregory and Haemers [5] found a best upper bound on λ_3 that is sufficient to guarantee that a regular graph G of order v has a 1-factor when v is even, and a matching of order $v-1$ when v is odd. In [11], the author studied the relation of eigenvalues and regular factors of regular graphs.

We are now able to state our main theorems and prove them in Section 2. Recently, Suil O and Cioabă [12] also independently proved Theorems 1.3 and 1.4 with different method and applied their results to matching problems.

Theorem 1.3 *Let $r \geqslant 4$ be an integer and m an even integer, where $2 \leqslant m \leqslant r+1$. Let $\mathcal{H}(r,m)$ denote the class of all connected irregular graphs with order $n \neq r \pmod 2$, maximum degree r, and size e with $2e \geqslant rn - m$. Let*

$$\rho_1(r,m) = \frac{1}{2}(r - 2 + \sqrt{(r+2)^2 - 4m}). \tag{1}$$

Then $\lambda_1(H) \geqslant \rho_1(r,m)$ for each $H \in \mathcal{H}(r,m)$ with equality if H is the join of K_{r+1-m} and $\overline{M_{m/2}}$.

Theorem 1.4 *Let r and m be two integers such that $m \equiv r \pmod 2$ and $1 \leqslant m \leqslant r$. Let $\mathcal{H}(r,m)$ denote the class of all connected irregular graphs with order $n \equiv r \pmod 2$, maximum degree r, and size e with $2e \geqslant rn - m$.*

(*i*) *If $m \geqslant 3$, let*

$$\rho_2(r,m) = \frac{1}{2}(r - 3 + \sqrt{(r+3)^2 - 4m}), \tag{2}$$

then $\lambda_1(H) \geqslant \rho_2(r,m)$ for each $H \in \mathcal{H}(r,m)$ with equality if H is the join of $\overline{M_{(r+2-m)/2}}$ and $\overline{C}$, where C with order m consists of disjoint cycles;

(*ii*) *if $m = 1$, let $\rho_2(r,m)$ is the greatest root of $P(x)$, where $P(x) = x^3 - (r-2)x^2 - 2rx + (r-1)$, then $\lambda_1(H) \geqslant \rho_2(r,m)$ for each $H \in \mathcal{H}(r,m)$ with equality if H is the join of $\overline{K_{1,2}}$ and $\overline{M_{(r-1)/2}}$;*

(*iii*) *if $m = 2$, let $\rho_2(r,m)$ is the greatest root of $f_1(x)$, where $f_1(x) = x^3 - (r-2)x^2 - (2r-1)x + r$, then $\lambda_1(H) \geqslant \rho_2(r,m)$ for each $H \in \mathcal{H}(r,m)$ with equality if H is the join of $\overline{P_4}$ and $\overline{M_{(r-2)/2}}$, where P_4 denote the path of length three.*

Theorems 1.3 and 1.4 improve the recent results from [11]. The proofs of these theorems are contained in Section 2.

Theorem 1.5 *Suppose that r is even, k is odd. Let G be a connected r-regular graph with order n. Let $m \geqslant 3$ be an integer and $m_0 \in \{m, m-1\}$ be an odd integer. Suppose that $\frac{r}{m} \leqslant k \leqslant r(1 - \frac{1}{m})$.*

(*i*) *If n is odd and $\lambda_2(G) < \rho_1(r, m_0 - 1)$, then G is k-critical;*

(*ii*) *if n is even and $\lambda_3(G) < \rho_1(r, m_0 - 1)$, then G has a k-factor.*

Theorem 1.6 *Let r and k be two integers. Let m be an integer such that $m^* \in \{m, m+1\}$ and $m^* \equiv 1 \pmod 2$. Let G be a connected r-regular graph with order n. Suppose that*

$$\lambda_3(G) < \begin{cases} \rho_1(r, m-1) & \text{if } m \text{ is odd,} \\ \rho_2(r, m-1) & \text{if } m \text{ is even.} \end{cases}$$

If one of the following conditions holds, then G has a k-factor.

(*i*) *r is odd, k is even and $k \leqslant r(1 - \frac{1}{m^*})$;*

(*ii*) *both r and k are odd and $\frac{r}{m^*} \leqslant k$.*

The main tool in our arguments is eigenvalue interlacing (see [9]).

Theorem 1.7 (Interlacing Theorem) *If A is a real symmetric $n \times n$ matrix and B is a principal submatrix of A with order $m \times m$, then for $1 \leqslant i \leqslant m$, $\lambda_i(A) \geqslant \lambda_i(B) \geqslant \lambda_{n-m+i}(A)$.*

2 The proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Let H be a graph in $\mathcal{H}(r, m)$ with $\lambda_1(H) \leqslant \rho_1(r, m)$. Firstly, we prove the following claim.

Claim 1. H has order n and size e, where $n = r + 1$ and $2e = rn - m$.

Suppose that $2e > rn - m$. Then, since $rn - m$ is even, so $2e \geqslant rn - m + 2$. Because the spectral radius of a graph is at least the average degree, $\lambda_1(H) \geqslant \frac{2e}{n} \geqslant r - \frac{m-2}{r+1}$. Since

$$\begin{aligned} \rho_1(r, m) &= \frac{1}{2}(r - 2 + \sqrt{(r+2)^2 - 4m}) \\ &= \frac{1}{2}(r-2) + \frac{1}{2}(r+2)\sqrt{1 - \frac{4m}{(r+2)^2}} \\ &< \frac{1}{2}(r-2) + \frac{1}{2}(r+2)(1 - \frac{2m}{(r+2)^2}) \\ &= r - \frac{m}{r+2} \\ &< r - \frac{m-2}{r+1}, \end{aligned}$$

so $\lambda_1(H) > \rho_1(r, m)$. Thus $2e = rn - m$. Because H has order n with maximum degree r, we have $n \geqslant r + 1$. If $n > r + 1$, since $n + r$ is odd, so $n \geqslant r + 3$, it is straightforward to check that

$$\lambda_1(H) > \frac{2e}{n} \geqslant r - \frac{m}{r+3} > \rho_1(r, m),$$

a contradiction. This completes the claim.

Then by Claim 1, H has order $n = r+1$ and at least $r+1-m$ vertices of degree r. Let G_1 be the subgraph of H induced by $n_1 = n+1-m$ vertices of all the vertices of degree r and G_2 be the subgraph induced by the remaining $n_2 = m$ vertices. Also, let G_{12} be the bipartite subgraph induced by the partition and let e_{12} be the size of G_{12}. A theorem of Haemers [7] shows that eigenvalues of the quotient matrix of the partition interlace the eigenvalues of the adjacency matrix of G. Because each vertex in G_1 is adjacent to all other vertices in H, the quotient matrix Q is the following

$$Q = \begin{pmatrix} \frac{2e_1}{n_1} & \frac{e_{12}}{n_1} \\ \frac{e_{12}}{n_2} & \frac{2e_2}{n_2} \end{pmatrix} = \begin{pmatrix} r-m & m \\ r+1-m & m-2 \end{pmatrix}.$$

Applying eigenvalue interlacing to the greatest eigenvalue of G, we get

$$\lambda_1(H) \geqslant \lambda_1(Q) = \frac{1}{2}(r-2+\sqrt{(r+2)^2-4m}), \tag{3}$$

with the equality if the partition is equitable [[9], p.202]; equivalently, if G_1 and G_2 are regular, and G_{12} is semiregular; or equivalently, if $G_2 = \overline{M_{m/2}}$, $G_1 = K_{r+1-m}$ and $G_{12} = K_{r+1-m,m}$. Hence $\lambda_1(R) \geqslant \rho_1(r,m)$ for each $R \in \mathcal{H}(r,m)$ and the equality holds if $R = K_{r+1-m} + \overline{M_{m/2}}$. This completes the proof. □

Proof of Theorem 1.4. Let H be a graph in $\mathcal{H}(r,m)$ with $\lambda_1(H) \leqslant \rho_2(r,m)$. With similar proof of Claim 1 in Theorem 1.3, we obtain the following claim.

Claim 1. H has order n and size e, where $n = r+2$ and $2e = rn - m$.

By Claim 1, H has order $n = r+2$ and at least $r+2-m$ vertices of degree r. Let G_1 be the subgraph of H induced by the $n_1 = n+2-m$ vertices of degree r and G_2 be the subgraph induced by the remaining $n_2 = m$ vertices. Also, let G_{12} be the bipartite subgraph induced by the partition and let e_{12} be the size of G_{12}. The quotient matrix Q is the following

$$Q = \begin{pmatrix} \frac{2e_1}{n_1} & \frac{e_{12}}{n_1} \\ \frac{e_{12}}{n_2} & \frac{2e_2}{n_2} \end{pmatrix}.$$

Suppose that $e_{12} = t$. Then $2e_1 = (r+2-m)r - t$ and $2e_2 = rm - m - t$. Applying eigenvalue interlacing to greatest eigenvalue

$$\begin{aligned} \lambda_1(G) \geqslant \lambda_1(Q) &= \frac{2e_1}{n_1} + \frac{2e_2}{n_2} + \sqrt{(\frac{2e_1}{n_1} - \frac{2e_2}{n_2})^2 + \frac{e_{12}^2}{n_1 n_2}} \\ &= \frac{2r-1}{2} - \frac{(r+2)t}{2m(r+2-m)} + \sqrt{(\frac{1}{2} + \frac{t(r+2-2m)}{2m(r+2-m)})^2 + \frac{t^2}{m(r+2-m)}}. \end{aligned}$$

Let $s = \frac{t}{m(r+2-m)}$, where $0 < s \leqslant 1$, then we have

$$2\lambda_1(Q) = f(s) = (2r-1) - s(r+2) + \sqrt{1 + 2s(r+2-2m) + s^2(r+2)^2}.$$

For $s > 0$, since

$$f'(s) = -(r+2) + \frac{(r+2-2m) + s(r+2)^2}{\sqrt{1+2s(r+2-2m)+s^2(r+2)^2}} < 0.$$

Then $0 < t \leqslant m(r+2-m)$, so we have

$$\begin{aligned} 2\lambda_1(Q) \geqslant f(1) &= (r-3) + \sqrt{1+2(r+2-2m)+(r+2)^2} \\ &= (r-3) + \sqrt{(r+3)^2 - 4m}. \end{aligned}$$

Hence

$$\lambda_1(H) \geqslant \lambda_1(Q) \geqslant \frac{1}{2}(r-3) + \frac{1}{2}\sqrt{(r+3)^2-4m}, \tag{4}$$

with equality if $t = m(r+2-m)$, both G_1 and G_2 are regular and G_{12} is semiregular; equivalently, if $\overline{G_1}$ is a perfect matching with order $r+2-m$ and $\overline{G_2}$ is a 2-regular graph with order m. Hence $\lambda_1(R) \geqslant \rho_2(r,m)$ for each $R \in \mathcal{H}(r,m)$ and the equality holds if $R = \overline{M_{(r+2-m)/2}} + \overline{C}$, where C is a 2-regular graph with order m.

Now we consider $m = 1$. Then r is odd and $n = r+2$. So H contains one vertex of degree $r-1$, say v and the rest vertices have degree r. Hence $\overline{H} = K_{1,2} \cup M_{(r-1)/2}$. Partition the vertex of $V(\overline{H})$ into three parts: the two endpoints of $K_{1,2}$; the internal vertex of $K_{1,2}$; the $(r-1)$ vertices of $M_{(r-1)/2}$. This is an equitable partition of H with quotient matrix

$$Q = \begin{pmatrix} 0 & 0 & r-1 \\ 0 & 1 & r-1 \\ 1 & 2 & r-3 \end{pmatrix}.$$

The characteristic polynomial of the quotient matrix is

$$P(x) = x^3 - (r-2)x^2 - 2rx + (r-1).$$

Since the partition is equitable, so $\lambda_1(H) = \lambda_1(Q)$ and $\lambda_1(H)$ is a root of $P(x)$.

Finally, we consider $m = 2$. Then r is even. Let $G \in \mathcal{H}(r,m)$ be the graph with order $r+2$ and size $e = (r(r+2)-2)/2$.

We discuss three cases.

Case 3.1. G has two nonadjacent vertices of degree $r-1$.

Then $G = \overline{P_4} + \overline{M_{(r-2)/2}}$ and $\overline{G} = P_4 \cup M_{(r-2)/2}$. Partition the vertex of $V(\overline{G})$ into three parts: the two endpoints of P_4; the two internal vertices of P_4; the $(r-2)$ vertices of $M_{(r-1)/2}$. This is an equitable partition of G with quotient matrix

$$Q_1 = \begin{pmatrix} 1 & 1 & r-2 \\ 0 & 1 & r-2 \\ 2 & 2 & r-4 \end{pmatrix}.$$

The characteristic polynomial of the quotient matrix is

$$f_1(x) = x^3 - (r-2)x^2 - (2r-1)x + r.$$

Case 3.2. G has two adjacent vertices of degree $r-1$.

Then $\overline{G} = 2P_3 \cup M_{(r-4)/2}$. Partition the vertex of $V(\overline{G})$ into three parts: the four endpoints of two P_3; the two internal vertices of two P_3; the $(r-4)$ vertices of $M_{(r-4)/2}$. This is an equitable partition of G with quotient matrix

$$Q_3 = \begin{pmatrix} 3 & 1 & r-4 \\ 2 & 1 & r-4 \\ 4 & 2 & r-6 \end{pmatrix}.$$

The characteristic polynomial of the quotient matrix is

$$f_2(x) = x^3 - (r-2)x^2 - (2r-1)x + r - 2.$$

Case 3.3. G has one vertex of degree $r-2$.

Then $\overline{G} = K_{1,3} \cup M_{(r-2)/2}$. Partition the vertex set of $\overline{G}$ into three parts: the center vertex of $K_{1,3}$; the three endpoints of $K_{1,3}$; the $(r-2)$ vertices of $M_{(r-2)/2}$. This is an equitable partition of G with quotient matrix

$$Q_2 = \begin{pmatrix} 0 & 0 & r-2 \\ 0 & 2 & r-2 \\ 1 & 3 & r-4 \end{pmatrix}.$$

The characteristic polynomial of the quotient matrix is

$$f_3(x) = x^3 - (r-2)x^2 - 2rx + 2(r-2).$$

Note that $\lambda_1(Q_1) < \lambda_1(Q_2) < \lambda_1(Q_3)$. We have $\rho_2(r,m) = \lambda_1(Q_1)$. So $\overline{H} = P_4 \cup M_{(r-2)/2}$. Hence $\lambda_1(H)$ is a root of $f_1(x) = 0$. This completes the proof. □

3 The proof of Theorems 1.5 and 1.6

We will need the following technical lemma whose proof is an easy modification of the proof of Theorem 2.2 from [11]. We provide the proof here for completeness.

Lemma 3.1 *Let r and k be integers such that $1 \leqslant k < r$. Let G be a connected r-regular graph with n vertices. Let m be an integer and $m^* \in \{m, m+1\}$ be an odd integer. Suppose that one of the following conditions holds*

(i) r is even, k is odd, and $\frac{r}{m} \leqslant k \leqslant r(1-\frac{1}{m})$;

(ii) r is odd, k is even and $k \leqslant r(1-\frac{1}{m^*})$;

(iii) both r and k are odd and $\frac{r}{m^*} \leqslant k$.

If G contains no a k-factor and is not k-critical, then G contains $def(G)+1$ vertex disjoint induced subgraph $H_1, H_2, \ldots, H_{def(G)+1}$ such that $2e(H_i) \geqslant r|V(H_i)| - (m-1)$ for $i = 1, 2, \ldots, def(G)+1$.

Proof. Suppose that the result does not hold. Let $\theta = k/r$. Since G is not k-critical and contains no k-factors, so by Theorem 1.2, there exist two disjoint subsets S and T of $V(G)$ such that $S \cup T \neq \emptyset$ and $\delta(S,T) = def(G) \geqslant 1$. Let $C_1, \ldots, C_\tau$ be the k-odd components of $G - (S \cup T)$. We have

$$def(G) = \delta(S,T) = k|T| + e_G(S,T) + \tau - k|S| - \sum_{x \in T} d_G(x). \tag{5}$$

Claim 1. $\tau \geqslant def(G) + 1$.

Otherwise, let $\tau \leqslant def(G)$. Then we have

$$0 \geqslant k|S| + \sum_{x \in T} d_{G-S}(x) - k|T|. \tag{6}$$

So we have $|S| \leqslant |T|$, and equality holds only if $\sum_{x \in T} d_{G-S}(x) = 0$. Since G is r-regular, so we have

$$r|S| \geqslant e_G(S,T) = r|T| - \sum_{x \in T} d_{G-S}(x). \tag{7}$$

By (6) and (7), we have

$$(r-k)(|T| - |S|) \leqslant 0.$$

Hence $|T| = |S|$ and $\sum_{x \in T} d_{G-S}(x) = 0$. So we have $\tau = def(G) > 0$. Since G is connected, then $e_G(C_i, S \cup T) > 0$ and so $e_G(C_1, S) > 0$. Note that G is r-regular, then we have $r|S| \geqslant r|T| - \sum_{x \in T} d_{G-S}(x) + e(C_i, S)$, a contradiction. We complete the claim.

By the hypothesis, without loss of generality, we can say $e(S \cup T, C_i) \geqslant m$ for $i = 1, \ldots, \tau - def(G)$. Then $0 < \theta < 1$, and we have

$$\begin{aligned}
&- def(G) \\
=& - \delta(S,T) = k|S| + \sum_{x \in T} d_G(x) - k|T| - e_G(S,T) - \tau \\
=& k|S| + (r-k)|T| - e_G(S,T) - \tau \\
=& \theta r|S| + (1-\theta) r|T| - e_G(S,T) - \tau \\
=& \theta \sum_{x \in S} d_G(x) + (1-\theta) \sum_{x \in T} d_G(x) - e_G(S,T) - \tau \\
\geqslant& \theta \Big(e_G(S,T) + \sum_{i=1}^{\tau} e_G(S, C_i)\Big) + (1-\theta)\Big(e_G(S,T) + \sum_{i=1}^{\tau} e_G(T, C_i)\Big) - e_G(S,T) - \tau \\
=& \sum_{i=1}^{\tau} \big(\theta e_G(S, C_i) + (1-\theta) e_G(T, C_i) - 1\big).
\end{aligned}$$

Since G is connected, so we have $\theta e_G(S, C_i) + (1-\theta)e_G(T, C_i) > 0$ for $1 \leqslant i \leqslant \tau$. Hence it suffices to show that for every $C = C_i$, $1 \leqslant i \leqslant \tau - def(G)$,

$$\theta e_G(S, C_i) + (1-\theta)e_G(T, C_i) \geqslant 1. \tag{8}$$

Since C is a k-odd component of $G - (S \cup T)$, we have

$$k|C| + e_G(T, C) \equiv 1 \pmod 2. \tag{9}$$

Moreover, since $r|C| = e_G(S \cup T, C) + 2|E(C)|$, then we have

$$r|C| \equiv e_G(S \cup T, C) \pmod 2. \tag{10}$$

It is obvious that the two inequalities $e_G(S, C) \geqslant 1$ and $e_G(T, C) \geqslant 1$ implies

$$\theta e_G(S, C) + (1-\theta)e_G(T, C) \geqslant \theta + (1-\theta) = 1.$$

Hence we may assume $e_G(S, C) = 0$ or $e_G(T, C) = 0$. We consider two cases.

First we consider (i). If $e_G(S, C) = 0$, since $1 \leqslant k \leqslant r(1 - \frac{1}{m})$, then $\theta \leqslant 1 - \frac{1}{m}$ and so $1 \leqslant (1-\theta)m$. Note that $e(T, C) \geqslant m$, so we have

$$(1-\theta)e_G(T, C) \geqslant (1-\theta)m \geqslant 1.$$

If $e_G(T, C) = 0$, since $k \geqslant r/m$, so $m\theta \geqslant 1$. Hence we obtain

$$\theta e_G(S, C) \geqslant m\theta \geqslant 1.$$

In order to prove that (ii) implies the claim, it suffices to show that (8) holds under the assumption that $e_G(S, C)$ or $e_G(T, C) = 0$. If $e_G(S, C) = 0$, then by (9), we have $e_G(T, C) \equiv 1 \pmod 2$. Hence $e_G(T, C) \geqslant m^*$, and thus

$$(1-\theta)e_G(T, C) \geqslant (1-\theta)m^* \geqslant 1.$$

If $e_G(T, C) = 0$, then by (10), we have $k|C| \equiv 1 \pmod 2$, which contradicts the assumption that k is even.

We next consider (iii), i.e., we assume that both r and k are odd and $\frac{r}{m^*} \leqslant k$. If $e_G(S, C) = 0$, then by (9) and (10), we have

$$|C| + e_G(T, C) \equiv 1 \pmod 2 \text{ and } |C| \equiv e_G(T, C) \pmod 2.$$

This is a contradiction. If $e_G(T, C) = 0$, then by (9) and (10), we have

$$|C| \equiv 1 \pmod 2 \text{ and } |C| \equiv e_G(S, C) \pmod 2,$$

which implies $e_G(S, C) \geqslant m^*$. Thus

$$\theta e_G(S, C) \geqslant \theta m^* \geqslant 1.$$

So we have

$$-def(G) \geqslant \delta(S,T) > -def(G),$$

a contradiction. This completes the proof. □

Proof of Theorem 1.5. Firstly, we prove (i). Suppose that G is not k-critical. By Lemma 3.1, G contains two vertex disjoint induced subgraphs H_1 and H_2 such that $2e(H_i) \geqslant rn_i - (m-1)$, where $n_i = |V(H_i)|$ for $i = 1, 2$. Hence we have $2e(H_i) \geqslant rn_i - (m_0 - 1)$. So by Interlacing Theorem, we have

$$\begin{aligned}\lambda_2(G) &\geqslant \min\{\lambda_1(H_1), \lambda_1(H_2)\}\\ &\geqslant \min\{\rho_1(r, m_0-1), \rho_2(r, m_0-1)\} = \rho_1(r, m_0-1).\end{aligned}$$

So we have $\lambda_2(G) \geqslant \rho_1(r, m_0 - 1)$, a contradiction.

Now we prove (ii). Suppose that G contains no a k-factor. Then we have $def(G) \geqslant 2$. So by Lemma 3.1, G contains three vertex disjoint induced subgraphs H_1, H_2 and H_3 such that $2e(H_i) \geqslant rn_i - (m-1)$, where $n_i = |V(H_i)|$ for $i = 1, 2, 3$. Since r is even , so $2e(H_i) \geqslant rn_i - (m_0 - 1)$ for $i = 1, 2, 3$. So by Interlacing Theorem, we have

$$\begin{aligned}\lambda_3(G) &\geqslant \min\{\lambda_1(H_1), \lambda_1(H_2), \lambda_1(H_3)\}\\ &\geqslant \min\{\rho_1(r, m_0-1), \rho_2(r, m_0-1)\} = \rho_1(r, m_0-1),\end{aligned}$$

a contradiction. We complete the proof. □

Remark. Now we show that the upper bound in Theorems 1.5 (ii) is the best possible function of r and m when $2m^2 < r$. Let r be even and m be odd. Let k be an odd integer such that $r/(m-1) > k \geqslant r/m$. Let $m_0 = m - 1$ and $H(r, m_0) = K_{r+1-m_0} + \overline{M_{m_0/2}}$. Let $G(r, m_0)$ be the r-regular graph obtained by matching the m_0 vertices of degree $r-1$ in each r copies of $H(r, m_0)$ to a set S of $|S| = m_0$ independent vertices. Then $G(r, m_0) - S$ has $r > km_0$ copies of odd order graph $H(r, m_0)$ as its components and so, by Theorem 1.1, $G(r, m_0)$ has no k-factors. Moreover,

$$\lambda_2(G(r, m_0)) = \lambda_3(G(r, m_0)) = \rho_1(r, m_0).$$

(For the proof, we refer the reader to [5], where the statement is proved for 1-factors.) For (i), let k be even such that $(r-1)/(m-1) > k \geqslant r/m$. Let $G'(r, m_0)$ be the r-regular graph obtained by matching the m_0 vertices of degree $r-1$ in each $r-1$ copies of $H(r, m_0)$ to a set S of $M_{m_0/2}$. Then $G'(r, m_0)$ has $n = m - 1 + (r-1)(r+1)$ vertices. Since $(r-1)/(m-1) > k \geqslant r/m$ and $\delta_{G'(r,m_0)}(S, \emptyset) = (r-1) - k(m-1) > 0$, so by Theorem 1.2, $G'(r, m_0)$ is not k-critical. Similarly, we have

$$\lambda_2(G'(r, m_0)) = \lambda_3(G'(r, m_0)) = \rho_1(r, m_0).$$

Proof of Theorem 1.6. Suppose that G contains no a k-factor. By Lemma 3.1, G contains three vertex disjoint induced subgraph H_1, H_2, H_3 such that $2e(H_i) \geqslant r|V(H_i)| - (m-1)$ for $i = 1, 2, 3$. Firstly, let m be odd. By Interlacing Theorem we have

$$\lambda_3(G) \geqslant \min_{1 \leqslant i \leqslant 3} \lambda_1(H_i) \geqslant \min\{\rho_1(r, m-1), \rho_2(r, m-2)\} = \rho_1(r, m-1).$$

So we have $\lambda_3(G) \geqslant \rho_2(r, m-1)$, a contradiction.

Next, let m be even. By Interlacing Theorem we have

$$\lambda_3(G) \geqslant \min_{1 \leqslant i \leqslant 3} \lambda_1(H_i) \geqslant \min\{\rho_1(r, m-2), \rho_2(r, m-1)\} = \rho_2(r, m-1),$$

a contradiction. We complete the proof. □

Remark. The upper bound in Theorems 1.6 is best possible when m is even and $m^2 < r$. Let r and k be two odd integers. Let G be an r-regular graph. Note that G contains a k-factor if and only if G contains an $(r-k)$-factor. So we only need to show that the upper bound in Theorems 1.6 (ii) is best possible. Let m be an even integer and $m^* = m+1$ such that $r/m^* \leqslant k < r/(m-1)$. Let $H(r, m-1)$ denote the extremal graph in Theorem 1.4. Let $G(r, m-1)$ be the r-regular graph obtained by matching the $m-1$ vertices of degree $r-1$ in each r copies of $H(r, m-1)$ to a set S of $|S| = m-1$ independent vertices. Similarly, we have

$$\lambda_3(G(r, m-1)) = \rho_2(r, m-1).$$

But $G(r, m-1)$ contains no k-factors.

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