The orderings of bicyclic graphs and connected graphs by algebraic connectivity*

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Abstract

The algebraic connectivity of a graph G is the second smallest eigenvalue of its Laplacian matrix. Let $\mathscr{B}_n$ be the set of all bicyclic graphs of order n. In this paper, we determine the last four bicyclic graphs (according to their smallest algebraic connectivities) among all graphs in $\mathscr{B}_n$ when $n \geqslant 13$. This result, together with our previous results on trees and unicyclic graphs, can be used to further determine the last sixteen graphs among all connected graphs of order n. This extends the results of Shao *et al.* [The ordering of trees and connected graphs by their algebraic connectivity, Linear Algebra Appl. 428 (2008) 1421-1438].

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1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the set of vertices which are adjacent to v in G and $d(v) = |N(v)|$ be the degree of v. For any $e \in E(G)$, we use $G - e$ to denote the graph obtained by deleting e from G. Readers are referred to [2] for undefined terms.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. It is easy to see that $L(G)$ is a symmetric positive semidefinite matrix having 0 as an eigenvalue. Thus, the eigenvalues $\mu_i(G)$'s of $L(G)$ (or the Laplacian eigenvalues of G) satisfy

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0,$$

repeated according to their multiplicities. Fiedler [6] showed that the second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ is 0 if and only if G is disconnected. Thus $\mu_{n-1}(G)$ is popularly known as the algebraic connectivity of G and is usually denoted by $\alpha(G)$. Recently, the algebraic connectivity has received much more attention, see [1] for survey. It has be found a lot of applications in theoretical chemistry, control theory, combinatorial optimization, *etc* (see [1, 4, 6]).

Let $\mathscr{T}_n$, $\mathscr{U}_n$ $\mathscr{B}_n$ and $\mathscr{G}_n$ be the sets of all trees, unicyclic graphs, bicyclic graphs and connected graphs of order n, respectively. Let $\mathscr{U}_n^g$ be the set of all unicyclic graphs of order n with girth g. Let $C_{n,g}$ be the graph obtained by appending a cycle C_g to a pendant vertex of P_{n-g}. Clearly, $C_{n,g} \in \mathscr{U}_n^g$.

Cvetković *et al.* [4] proposed some possible directions for further investigations on graph spectra. One of which is how to order graphs according to their (Laplacian) eigenvalues. Hence ordering graphs with various properties by their spectra, specially by their algebraic connectivity becomes an attractive topic. In particular, Shao *et al.* [12] determined the last four trees (according to their smallest algebraic connectivities) among all trees in $\mathscr{T}_n$. In [10], we further extend this result to the last eight trees. Those results can be combined into the following theorem.

Theorem 1.1 ([12, 10]) *Let $T \in \mathscr{T}_n \setminus \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ with $n \geqslant 13$. Then $\alpha(T) > \max\{\alpha(T_7), \alpha(T_8)\}$. Moreover, $\alpha(T_1) < \alpha(T_2) < \alpha(T_3) < \alpha(T_4) < \alpha(T_5) < \alpha(T_6) < \min\{\alpha(T_7), \alpha(T_8)\}$, where $T_1, \ldots, T_8$ are shown in Fig. 1.*

Guo [7, 8] proved the following theorem which was conjectured by Fallat and Kirkland [5].

Theorem 1.2 ([7, 8]) *Let G be a connected graph of order n with girth $g \geqslant 3$. Then*

(1) $\alpha(G) \geqslant \alpha(C_{n,g})$, and the equality holds if and only if $G \cong C_{n,g}$.

(2) $\alpha(C_{n,g+1}) > \alpha(C_{n,g})$.

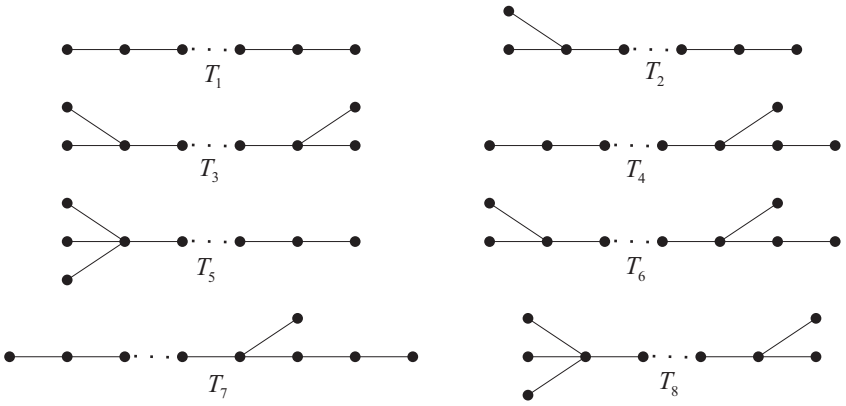

Figure 1: Trees T_i $(1 \leqslant i \leqslant 8)$.

Moreover, this result was used by Guo [8] to determine the graph with first smallest algebraic connectivity among all graphs in $\mathscr{U}_n$. Recently, Liu and Liu [11] further determined the graphs with the second and the third smallest algebraic connectivities among all graphs in $\mathscr{U}_n$, respectively. So, the last three unicyclic graphs (according to their smallest algebraic connectivities) are determined as U_1, U_2 and U_3 (shown in Fig. 2), respectively. In [9], we further determine the fourth to seventh unicyclic graphs, which are U_4, U_5, U_6 and U_7 (shown in Fig. 2), respectively. We combine these results into the following theorem.

Theorem 1.3 ([8, 11, 9]) *Let $U \in \mathscr{U}_n \setminus \{U_1, U_2, U_3, U_4, U_5, U_6, U_7\}$ with $n \geqslant 13$. Then $\alpha(U) > \alpha(U_7)$. Moreover, $\alpha(U_1) < \alpha(U_2) < \alpha(U_3) < \alpha(U_4) < \alpha(U_5) < \alpha(U_6) < \alpha(U_7)$.*

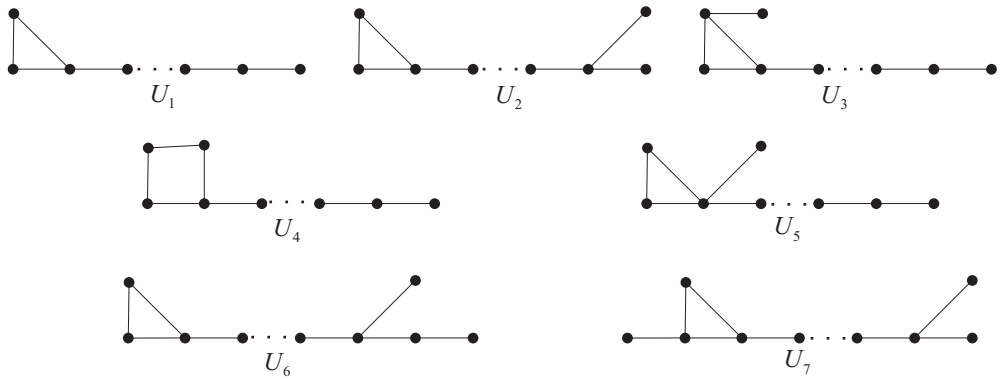

Figure 2: Unicyclic graphs U_i $(1 \leqslant i \leqslant 7)$.

Moreover, Shao *et al.* [12] determined the last six graphs (according to their smallest algebraic connectivities) among all connected graphs in $\mathscr{G}_n$ when $n \geqslant 9$. In this six graphs, only one graph B_1 (shown in Fig. 4) is a bicyclic graph. That is to say, they also determined the graph B_1 which has the minimum algebraic connectivity among all graphs

in $\mathscr{B}_n$. In this paper, we further extend their result to the last four bicyclic graphs. These together with the previous result on the trees and unicyclic graphs, we can extend the ordering of connected graphs by their smallest algebraic connectivities form the last six connected graphs to the last sixteen connected graphs.

2 Preliminaries

In this section, we present some lemmas which will be used in the subsequent sections.

Lemma 2.1 ([3]) *Let G be a graph of order n which does not isomorphic to the complete graph K_n and let $G' = G + e$ be the graph obtained from G by adding a new edge e. Then the Laplacian eigenvalues of G and G' interlace, that is*

$$\mu_{i+1}(G') \leqslant \mu_i(G) \leqslant \mu_i(G') \text{ for } 1 \leqslant i \leqslant n-1.$$

Lemma 2.2 ([12]) *Let G be a connected graph of order n. Suppose that $v_1, \ldots, v_s$ ($s \geqslant 2$) are non-adjacent vertices of G and $N(v_1) = \cdots = N(v_s)$. Let G_t be a graph obtained from G by adding any t ($0 \leqslant t \leqslant \frac{s(s-1)}{2}$) edges among $v_1, \ldots, v_s$. If $\alpha(G) \neq d(v_1)$, then $\alpha(G) = \alpha(G_t)$.*

In [9], we proved two useful results on the smallest algebraic connectivity of unicyclic graphs with girth 3 or 4.

Lemma 2.3 ([9]) *Let $U \in \mathscr{U}_n^3 \setminus \{U_1, U_2, U_3, U_5, U_6, U_7\}$ with $n \geqslant 13$, where U_i are shown in Fig. 2. Then $\alpha(U) > \alpha(U_7)$. Moreover $\alpha(U_1) < \alpha(U_2) < \alpha(U_3) < \alpha(U_5) < \alpha(U_6) < \alpha(U_7)$.*

Lemma 2.4 ([9]) *For each $U \in \mathscr{U}_n^4 \setminus \{C_{n,4} \cong U_4\}$ with $n \geqslant 8$, $\alpha(U) > \alpha(U_7)$.*

3 Bicyclic graphs

Firstly, we introduce some notations that are used in this section. Let ∞ (coalescence of two cycles C_3) be the graph shown in Fig. 3. Let $\mathscr{B}_n^{\infty}$ be the set of all bicyclic graphs of order n which consist of ∞ and five trees T_1, T_2, T_3, T_4 and T_5 attached at the vertices v_1, v_2, v_3, v_4 and v_5, respectively, where $v_i \in V(T_i)$ for $i = 1, 2, 3, 4, 5$. Let θ (shown in Fig. 3) be the graph obtained from a cycle C_4 ($= v_1v_2v_3v_4v_1$) by adding a new edge v_1v_3. Let $\mathscr{B}_n^{\theta}$ be the set of all bicyclic graphs of order n which consist of θ and four trees T_1, T_2, T_3 and T_4 attached at the vertices v_1, v_2, v_3 and v_4, respectively. Assume that $|V(T_i)| = n_i$ for $i = 1, 2, 3, 4$. Clearly, $n_1 + n_2 + n_3 + n_4 = n$. Then for each $B \in \mathscr{B}_n^{\theta}$, we write $B = \theta_4(T_1, T_2, T_3, T_4)$. We also write $B = \theta_4(i, j, k, l)$ instead of $\theta_4(P_{i+1}, P_{j+1}, P_{k+1}, P_{l+1})$, where $i, j, k, l \geqslant 0$. Clearly, $B_2 = \theta_4(0, n-4, 0, 0)$ and $B_3 = \theta_4(n-4, 0, 0, 0)$, where B_2 and B_3 are shown in Fig. 4.

Now, we give the first four bicyclic graphs of order $n \geqslant 13$ with smallest algebraic connectivity.

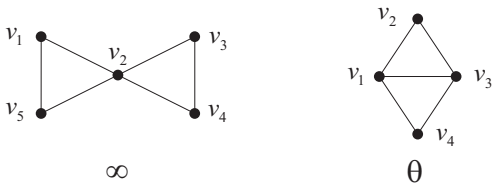

Figure 3: Bicyclic graphs ∞ and θ.

Theorem 3.1 *Let $B \in \mathscr{B}_n \setminus \{B_1, B_2, B_3, B_4\}$ with $n \geqslant 13$. Then $\alpha(B) > \alpha(B_4)$. Moreover, $\alpha(B_1) < \alpha(B_2) < \alpha(B_3) < \alpha(B_4)$, where B_1, B_2, B_3, B_4 are shown in Fig. 4.*

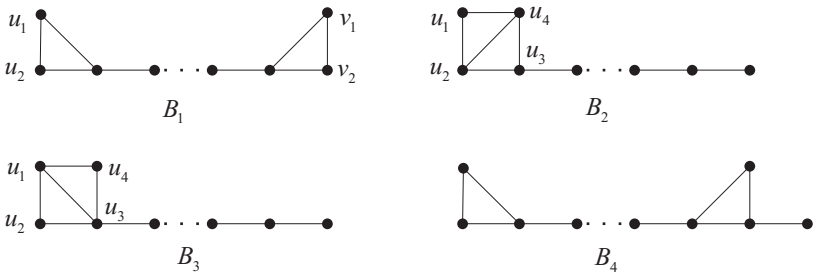

Figure 4: Bicyclic graphs B_i $(1 \leqslant i \leqslant 4)$.

Proof. Firstly, by Lemma 2.2, it is easy to see that $\alpha(B_1) = \alpha(U_2)$, $\alpha(B_2) = \alpha(U_4)$, $\alpha(B_3) = \alpha(U_5)$ and $\alpha(B_4) = \alpha(U_7)$. Thus, by Theorem 1.3, we have $\alpha(B_1) < \alpha(B_2) < \alpha(B_3) < \alpha(B_4)$.

For each $B \in \mathscr{B}_n$, let C_k and C_l be two independent cycles in B, where $3 \leqslant k \leqslant l$. If $l \geqslant 5$, then we may delete one of the edges in $E(C_k)$, say e, such that $B - e \in \mathscr{U}_n^l$. Thus, by Lemma 2.1 and Theorems 1.2 and 1.3, we have

$$\alpha(B) \geqslant \alpha(B-e) \geqslant \alpha(C_{n,l}) \geqslant \alpha(C_{n,5}) > \alpha(U_7) = \alpha(B_4).$$

In the following we suppose that $l \leqslant 4$. We consider the following two cases.

Case 1 $|V(C_k) \cap V(C_l)| \leqslant 1$.

(a) $l = 4$.
In this case, we always can choose some edge, say e, in C_k $(k = 3, 4)$ such that $B - e \in \mathscr{U}_n^4$ and $B - e$ does not isomorphic to $C_{n,4}$. Thus, Lemmas 2.1 and 2.4 imply that

$$\alpha(B) \geqslant \alpha(B-e) \ > \alpha(U_7) = \alpha(B_4).$$

(b) $k = l = 3$
If $|V(C_k) \cap V(C_l)| = 1$, then $B \in \mathscr{B}_n^\infty$. In this case, we always can choose

some edge, say e, in C_k or C_l such that $B-e \in \mathscr{U}_n^3$ and $B-e$ does not isomorphic to one of graphs $U_1, U_2, U_3, U_5, U_6, U_7$. Thus, Lemmas 2.1 and 2.3 imply that

$$\alpha(B) \geqslant \alpha(B-e) > \alpha(U_7) = \alpha(B_4).$$

If $|V(C_k) \cap V(C_l)| = 0$ and B does not isomorphic to B_1 or B_4, then B must be a bicyclic graph which consists of the graph H (here H is a bicyclic graph obtained by joining the vertex u of $C_3(= uu_1u_2u)$ and the vertex v of $C_3(= vv_1v_2v)$ with a path P_{uv}, shown in Fig. 5) and some trees which attached at some vertices of H, respectively. If there is a tree T_i with $|V(T_i)| \geqslant 2$

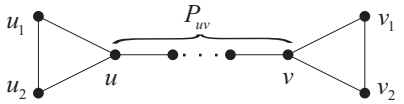

Figure 5: Bicyclic graph H, where $u \neq v$.

attached at some vertex belonging to P_{uv} (in H), then we have $B-u_1u_2 \in \mathscr{U}_n^3$ (or $B-v_1v_2 \in \mathscr{U}_n^3$) and $B-u_1u_2$ (or $B-v_1v_2$) does not isomorphic to one of graphs $U_1, U_2, U_3, U_5, U_6, U_7$. Thus, Lemmas 2.1 and 2.3 imply that

$$\alpha(B) \geqslant \alpha(B-u_1u_2)(\text{or } \alpha(B-v_1v_2)) > \alpha(U_7) = \alpha(B_4).$$

If there are four trees T_1, T_2, T_3 and T_4 attached at the vertices u_1, u_2, v_1 and v_2, respectively. Suppose that $|V(T_i)| = n_i \geqslant 1$ for $i = 1, 2, 3, 4$, where $n_1+n_2+n_3+n_4 = n-|V(P_{uv})|$. If one of n_1, n_2, n_3, n_4 is more than 3, then by the same reasoning, we may delete one of the edges u_1u_2 and v_1v_2 such that the resulting graph is in $\mathscr{U}_n^3$ and does not isomorphic to one of graphs $U_1, U_2, U_3, U_5, U_6, U_7$. Thus the result follows.

Similarly, if $n_1 = 2$ and $n_2 = 2$ (or $n_3 = 2$ and $n_4 = 2$), the result also follows. Now, recall that B does not isomorphic to B_1 or B_4, by symmetric, it suffices to consider $n_1 = 2, n_2 = 1, n_3 = 2$ and $n_4 = 1$, such a graph can be denoted by $B^\dagger$. Then by Lemmas 2.1 and 2.3, we have

$$\alpha(B^\dagger) \geqslant \alpha(B^\dagger-u_1u_2)(\text{or } \alpha(B^\dagger-v_1v_2)) > \alpha(U_7) = \alpha(B_4).$$

Case 2 $|V(C_k) \cap V(C_l)| > 1$.

(a) $l = 4$.

In this case $|V(C_k) \cap V(C_l)| \leqslant 3$. If $|V(C_k) \cap V(C_l)| = 3$, then $k = 4$. Therefore, B must be a bicyclic graph which consists of the graph H' (shown in Fig. 6) and five trees attached at each vertex of H', respectively. If B does not isomorphic to B^* (shown in Fig. 6), we may delete one of the common

edges, say e, in $E(C_k) \cap E(C_l)$ such that $B - e \in \mathscr{U}_n^4$ and $B - e$ does not isomorphic to $C_{n,4}$. Then Lemmas 2.1 and 2.4 imply that

$$\alpha(B) \geqslant \alpha(B - e) > \alpha(U_7) = \alpha(B_4);$$

if $B \cong B^*$, we may delete one of the edges in $E(C_k) \cup E(C_l)/E(C_k) \cap E(C_l)$, by the same reasoning, the result follows. If $|V(C_k) \cap V(C_l)| = 2$, then we can

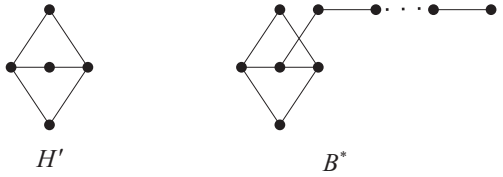

Figure 6: Bicyclic graphs H' and B^*.

delete the common edge, say e, of C_k and C_l such that $B - e \in \mathscr{U}_n^5$ or $\mathscr{U}_n^6$, the result follows from Theorem 1.2 and the fact $\alpha(C_{n,5}) > \alpha(B_4)$.

(b) $k = l = 3$.
Since $B \in \mathscr{B}_n^\theta$, B can be rewrote as $B = \theta_4(T_1, T_2, T_3, T_4)$, and $|V(T_i)| = n_i$ for $i = 1, 2, 3, 4$, where $n_1 + n_2 + n_3 + n_4 = n$.
If at least two of n_1, n_2, n_3, n_4 are great than 1, then $B - v_1v_3 \in \mathscr{U}_n^4$ and $B - v_1v_3$ does not not isomorphic to $C_{n,4}$. Then Lemmas 2.1 and 2.4 imply that

$$\alpha(B) \geqslant \alpha(B - e) > \alpha(U_7) = \alpha(B_4).$$

If only one of n_1, n_2, n_3, n_4 is more than 1, by symmetric, we may assume that $n_1 \geqslant 2$ or $n_2 \geqslant 2$. Therefore, $B \cong \theta_4(T_1, P_1, P_1, P_1)$ or $B \cong \theta_4(P_1, T_2, P_1, P_1)$. If $\theta_4(T_1, P_1, P_1, P_1)$ does not isomorphic to B_3 and $\theta_4(P_1, T_2, P_1, P_1)$ does not isomorphic to B_2. That is, $\theta_4(T_1, P_1, P_1, P_1) - v_1v_3$ and $\theta_4(P_1, T_2, P_1, P_1) - v_1v_3$ do not isomorphic to $C_{n,4}$, respectively. Then Lemma 2.1 and Theorem 2.4 imply that

$$\alpha(\theta_4(T_1, P_1, P_1, P_1)) \geqslant \alpha(\theta_4(T_1, P_1, P_1, P_1) - v_1v_3) > \alpha(U_7) = \alpha(B_4)$$

and

$$\alpha(\theta_4(P_1, T_2, P_1, P_1)) \geqslant \alpha(\theta_4(P_1, T_2, P_1, P_1) - v_1v_3) > \alpha(U_7) = \alpha(B_4).$$

This completes the proof. □

4 Connected graphs

In Section 3, we determined the last four bicyclic graphs according to their smallest algebraic connectivities among all graphs in $\mathscr{B}_n$ with $n \geqslant 13$. Combing with the results on

the orderings of the trees and unicyclic graphs, in this section, we extend the ordering of connected graphs from the last six connected graphs to the last sixteen connected graphs. Before giving the main result of this section, the following preliminary results are needed.

Lemma 4.1 *Let G be a connected graph of order $n \geqslant 13$ which contains exactly $n+2$ edges. If $\Delta(G) = 3$, then $\alpha(G) \geqslant \alpha(B_4)$.*

Proof. Let v be a vertex of degree 3 in G, e be an edge on some cycle C of G such that e is not incident with v, and $G' = G - e$. Then $G' \in \mathscr{B}_n$ with $\Delta(G') = 3$.

Case 1 G' does not isomorphic to B_1 or B_2.

Since $\Delta(G') = 3$, by Theorem 3.1, we have $\alpha(G') \geqslant \alpha(B_4)$. This together with Lemma 2.1 imply that $\alpha(G) \geqslant \alpha(G') \geqslant \alpha(B_4)$.

Case 2 $G' \cong B_1$ or $G' \cong B_2$.

If $G' \cong B_1$ (shown in Fig. 4), let $e_1 = u_1u_2$ and $e_2 = v_1v_2$ be the edges on the cycles C_1 and C_2 of G', respectively, such that the degrees of u_1, u_2, v_1 and v_2 in G' are all 2. Then in $G = G' + e$, at least one of u_1, u_2, v_1 and v_2, say u_1, has degree 2. Now, let $G'' = G - e_1$. Then $G'' \in \mathscr{B}_n$ with $\Delta(G'') = 3$. Clearly, G'' does not isomorphic to B_1 or B_2. Thus the result follows from Case 1.

Similarly, if $G' \cong B_2$ (shown in Fig. 4), then in $G = G' + e$, u_2 u_3 and u_4 have degrees 3, respectively. Let $G'' = G - u_1u_2$. Then $G'' \in \mathscr{B}_n$ with $\Delta(G'') = 3$. Clearly, G'' does not isomorphic to B_1 or B_2. Thus the result also follows from Case 1. This completes the proof. □

Lemma 4.2 *Let G be a connected graph of order $n \geqslant 13$ with maximum degree $\Delta(G) = 4$. If G does not isomorphic to one of $G_{11}, G_{12}, G_{13}, G_{14}$, then $\alpha(G) > \alpha(G_{15})$ (or $\alpha(G_{16})$), where $G_{11}, G_{12}, G_{13}, G_{14}, G_{15}$ and G_{16} are shown in Fig. 7.*

Proof. By Lemma 2.2, we have $\alpha(G_{15}) = \alpha(G_{16})$ and $\alpha(U_7) = \alpha(B_4)$. Thus Theorem 1.3 implies that $\alpha(B_4) > \alpha(G_{16})$. Let $m = |E(G)|$ be the edge number of G. Since $G \in \mathscr{G}_n$ with $n \geqslant 13$, we consider the following three cases.

Case 1 $m = n - 1, n, n + 1$

In this case, the results follow from Theorems 1.1, 1.3 and 3.1, respectively.

Case 2 $m = n + 2$

Let v be a vertex of degree 4 in G, e be an edge on some cycle C of G such that e is not incident with v. Let $G' = G - e$. Then $G' \in \mathscr{B}_n$ with $\Delta(G') = 4$.

If G' does not isomorphic to B_3 (also does not isomorphic to one of B_1, B_2, B_4), then from Theorem 3.1, we have $\alpha(G') > \alpha(B_4)$. This together with Lemma 2.1 and $\alpha(B_4) > \alpha(G_{16})$ lead to the result follows.

If $G' \cong B_3$ (shown in Fig. 4), since G does not isomorphic to G_{14}, then in $G = G' + e$, at least one of u_2 and u_4, say u_2, has degree 2. Let $G'' = G - u_1u_2$. Clearly, $G'' \in \mathscr{B}_n$ with $\Delta(G'') = 4$ and G'' does not isomorphic to B_3 since G'' contains a vertex u_2 with degree 1. Thus the result also follows from Lemma 2.1 and Theorem 3.1 with $\alpha(B_4) > \alpha(G_{16})$.

Case 3 $m \geqslant n + 3$

In this case, it suffices to prove that for any connected graph G of order $n \geqslant 13$ with exactly $n + 3$ edges, if $\Delta(G) = 4$, then $\alpha(G) > \alpha(G_{15})$ (or $\alpha(G_{16})$) (since if G with $m > n + 3$, we may delete $m - (n + 3)$ edges from G such that the resulting graph (with $n + 3$ edges) is connected). Let v be a vertex of degree 4 in G, e be an edge on some cycle C of G such that e is not incident with v, and $G' = G - e$. Then G' with exactly $n + 2$ edges and $\Delta(G') = 4$.

If G' does not isomorphic to G_{14}, then the result follows from Case 2 and Lemma 2.1.

If $G' \cong G_{14}$, let v_1, v_2 and v_3 (where v_1 is join to the vertex with degree 4) be the vertices of G' with degrees 3, respectively. In $G = G' + e$, we delete the edges v_1v_2 and v_1v_3, and the resulting graph is denoted by G''. Clearly, $G'' \in \mathscr{B}_n$ with $\Delta(G'') = 4$ and G'' does not isomorphic to B_3. Then by Lemma 2.1 and Theorem 3.1, we have $\alpha(G) \geqslant \alpha(G'') > \alpha(B_4) > \alpha(G_{15}) = \alpha(G_{16})$.

The proof is completed. □

Now, we give the main result of this section.

Theorem 4.3 *Let $G \in \mathscr{G}_n \setminus \{G_1, G_2, \ldots, G_{16}\}$ with $n \geqslant 13$. Then $\alpha(G) > \alpha(G_{16})$. Moreover, $\alpha(G_1) < \alpha(G_2) = \alpha(G_3) < \alpha(G_4) = \alpha(G_5) = \alpha(G_6) < \alpha(G_7) < \alpha(G_8) < \alpha(G_9) = \alpha(G_{10}) < \alpha(G_{11}) = \alpha(G_{12}) = \alpha(G_{13}) = \alpha(G_{14}) < \alpha(G_{15}) = \alpha(G_{16})$, where $G_1, \ldots, G_{16}$ are shown in Fig. 7 and $G_1 \cong T_1, G_2 \cong T_2, G_3 \cong U_1, G_4 \cong T_3, G_5 \cong U_2, G_6 \cong B_1, G_7 \cong T_4, G_8 \cong U_3, G_9 \cong U_4, G_{10} \cong B_2, G_{11} \cong T_5, G_{12} \cong U_5, G_{13} \cong B_3, G_{15} \cong T_6, G_{16} \cong U_6$.*

Proof. By Lemma 2.2, we have $\alpha(G_2) = \alpha(G_3)$, $\alpha(G_4) = \alpha(G_5) = \alpha(G_6)$, $\alpha(G_9) = \alpha(G_{10})$, $\alpha(G_{11}) = \alpha(G_{12}) = \alpha(G_{13}) = \alpha(G_{14})$, $\alpha(G_{15}) = \alpha(G_{16})$ and $\alpha(U_7) = \alpha(B_4)$. This together with Theorems 1.1, 1.3 and 3.1, we have $\alpha(G_1) < \alpha(G_2) = \alpha(G_3) < \alpha(G_4) = \alpha(G_5) = \alpha(G_6) < \alpha(G_7) < \alpha(G_8) < \alpha(G_9) = \alpha(G_{10}) < \alpha(G_{11}) = \alpha(G_{12}) = \alpha(G_{13}) = \alpha(G_{14}) < \alpha(G_{15}) = \alpha(G_{16})$ and $\alpha(B_4) > \alpha(G_{16})$.

Since $G \in \mathscr{G}_n$ with $n \geqslant 13$, we consider the following four cases.

Case 1 $\Delta(G) = 2$.

Then $G \cong C_n$ since G does not isomorphic to P_n (or G_1). From [3], we have $\alpha(C_n) = 4\sin^2\frac{\pi}{n}$ and $\alpha(P_n) = 4\sin^2\frac{\pi}{2(n-1)}$.

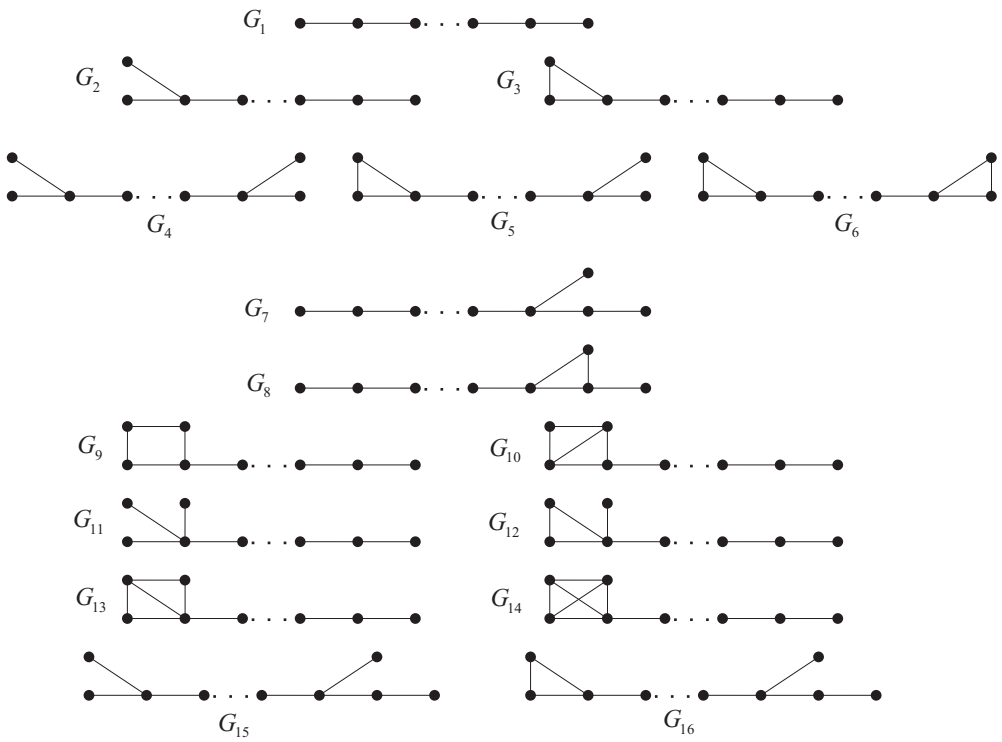

Figure 7: Connected graphs G_i $(1 \leqslant i \leqslant 16)$.

Moreover, P_{n-1} is a subtree of T_7. Combining with Lemma 2.1 and Theorem 1.1, we have

$\alpha(C_n) > \alpha(P_{n-1}) \geqslant \alpha(T_7) > \alpha(G_{15}) = \alpha(G_{16})$.

Case 2 $\Delta(G) = 3$.

Let $m = |E(G)|$. For $m = n-1, n, n+1$, the results follow from Theorems 1.1, 1.3 and 3.1, respectively. For $m = n + 2$, the result follows from Lemma 4.1 with $\alpha(B_4) > \alpha(G_{16})$. For $m > n + 2$, we can delete $m - (n + 2)$ edges from G such that the resulting graph is also a connected graph with $n + 2$ edges. Thus the result also follows from Lemmas 2.1 and 4.1.

Case 3 $\Delta(G) = 4$.

The result follows from Lemma 4.2.

Case 4 $\Delta(G) \geqslant 5$.

Then G contains a spanning tree T with $\Delta(T) = \Delta(G) \geqslant 5$. Clearly, T does not isomorphic to one of $T_1, \ldots, T_8$. Thus the result follows from Lemma 2.1 and Theorem 1.1.

The proof is completed. □

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