λ-factorials of n

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Abstract

Recently, by the Riordan identity related to tree enumerations,

$$\sum_{k=0}^{n}\binom{n}{k}(k+1)!(n+1)^{n-k}=(n+1)^{n+1},$$

Sun and Xu have derived another analogous one,

$$\sum_{k=0}^{n}\binom{n}{k}D_{k+1}(n+1)^{n-k}=n^{n+1},$$

where D_k is the number of permutations with no fixed points on $\{1,2,\ldots,k\}$. In the paper, we utilize the λ-factorials of n, defined by Eriksen, Freij and Wästlund, to give a unified generalization of these two identities. We provide for it a combinatorial proof by the functional digraph theory and two algebraic proofs. Using the umbral representation of our generalized identity and Abel's binomial formula, we deduce several properties for λ-factorials of n and establish interesting relations between the generating functions of general and exponential types for any sequence of numbers or polynomials.

Keywords: Derangement; λ-factorial of n; Charlier polynomial; Bell polynomial; Hermite polynomial.

1 Introduction

Let $\mathcal{S}_n$ denote the set of permutations of $[n]=\{1,2,\ldots,n\}$. A *fixed point* of a permutation $\pi\in\mathcal{S}_n$ is an element $i\in[n]$ such that $\pi(i)=i$. Denote by $fix(\pi)$ the number of fixed points of π. Recently, Eriksen, Freij and Wästlund [6] defined the polynomials, called the *λ-factorials of n*, by setting

$$f_n(\lambda) = \sum_{\pi \in \mathcal{S}_n} \lambda^{fix(\pi)}, \qquad f_0(\lambda) = 1.$$

They utilized the polynomials $f_n(\lambda)$ to give closed formulas for the number of derangements (permutations with no fixed points) with descents in prescribed positions and derived several nice properties for $f_n(\lambda)$ such as

$$f_n(\lambda + \mu) = \sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)\mu^{n-k}, \tag{1.1}$$

$$f_n(\lambda) = \sum_{k=0}^{n} \binom{n}{k} k!(\lambda - 1)^{n-k}, \tag{1.2}$$

$$f_n(\lambda) = n f_{n-1}(\lambda) + (\lambda - 1)^n, \tag{1.3}$$

$$\frac{d}{d\lambda} f_n(\lambda) = n f_{n-1}(\lambda). \tag{1.4}$$

Clearly, we have $f_n(0) = D_n$ [17, A000166] and $f_n(1) = n!$, where D_n is the number of derangements in $\mathcal{S}_n$. The relation (1.4) indicates that $f_n(\lambda)$ $(n = 0, 1, \dots)$ form a kind of special Appell polynomials [2]. According to the definition, $f_n(\lambda)$ also has another expression

$$f_n(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D_k \lambda^{n-k}. \tag{1.5}$$

It should be noted that $f_n(\lambda)$ has close relation to the (re-normalized) Charlier polynomials $C_n(\alpha, u)$ [7] defined by

$$C_n(\alpha, u) = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_k u^{n-k},$$

where $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$. Clearly, $f_n(\lambda) = C_n(1, \lambda - 1)$.

Using the Riordan identity [5, P173] and [13]

$$\sum_{k=0}^{n} \binom{n}{k} (k + 1)!(n + 1)^{n-k} = (n + 1)^{n+1},$$

Sun and Xu [20] deduced an analogous identity (also obtained by Riordan [12]),

$$\sum_{k=0}^{n} \binom{n}{k} D_{k+1}(n + 1)^{n-k} = n^{n+1}.$$

Motivated by these two remarkable identities, we give the following general one and provide with a combinatorial interpretation by the functional digraph theory.

Theorem 1.1. *For any integer $n \geq 0$ and any indeterminate λ, there holds*

$$\sum_{k=0}^{n} \binom{n}{k} f_{k+1}(\lambda)(n+1)^{n-k} = (n+\lambda)^{n+1}. \tag{1.6}$$

Using the umbral representation of (1.6) and Abel's binomial formula, we have the second main result.

Theorem 1.2. *For any sequence $(a_n)_{n\geq 0}$, let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. Then*

$$\sum_{n\geq 0} a_n f_n(\lambda) \frac{x^n}{n!} = \sum_{k\geq 0} (k+\lambda-1)^k \frac{x^k A^{(k)}(-kx)}{k!}, \tag{1.7}$$

where $A^{(k)}(-kx)$ denotes the k-th derivative of $A(x)$ taking value at $-kx$. In particular, the case $\lambda = 1$ generates

$$\sum_{n\geq 0} a_n x^n = \sum_{k\geq 0} \frac{(kx)^k A^{(k)}(-kx)}{k!}.$$

The organization of this paper is as follows. The next section is devoted to the proofs of (1.6). In the third section, we focus on the proof of (1.7) and give some applications. In the forth section, using the umbral representation of (1.6) we further investigate the properties of $f_n(\lambda)$ and present many identities. In the final section, we give some comments and provide several open problems.

2 Three Proofs for (1.6)

In this section, we give three different proofs for (1.6), one is a combinatorial proof by the functional digraph theory, one is a generating function proof, and the other is a proof by the umbral calculus method.

2.1 First proof of (1.6).

In order to give the combinatorial proof of (1.6), we need some notations. A *rooted labeled tree* on $[n]$ is an acyclic connected graph on the vertex set $[n]$ such that one vertex, called the *root*, is distinguished. A *labeled forest* is a graph such that every connected component is a rooted labeled tree. We denote by $\mathcal{F}_n$ the set of labeled forests on $[n+1]$ and by $\mathcal{F}_{n,k}$ the set of labeled forests $[n+1]$ with exactly $k+1$ components. It is well known that Cayley's formulas [3] state that $|\mathcal{F}_n| = (n+2)^n$ and $|\mathcal{F}_{n,k}| = \binom{n}{k}(n+1)^{n-k}$.

Let $\mathcal{M}_n$ denote the set of maps $\sigma : [n] \to [n]$. Clearly $|\mathcal{M}_n| = n^n$. For any $\sigma \in \mathcal{M}_n$, we represent σ as a directed graph G_σ by drawing arrows from i to $\sigma(i)$. For any component of G_σ, it contains equally many vertices and edges, and hence has exactly one directed cycle. Let $\mathcal{R}$ denote the set of all the vertices of these cycles of G_σ. Precisely, $\sigma|_{\mathcal{R}}$, the restriction of σ onto $\mathcal{R}$ is just a permutation on $\mathcal{R}$. If deleting all the directed edges in these cycles, the remainders (omitting the directions, since all edges are directed towards the roots) form a labeled forest F on $[n]$. Conversely, it is not difficult to recover the map σ from the pair (F, π), where π is a permutation of the set $\mathcal{R}_F$ of the roots of F. Hence there exists a bijection between the set $\mathcal{P}_n$ of the pairs (F, π) and $\mathcal{M}_{n+1}$, where $F \in \mathcal{F}_n$. See [1, 9] for more details.

Now we can give a combinatorial interpretation for (1.6).

It suffices to prove (1.6) for the cases when λ are nonnegative integers. Let $\mathcal{M}^*_{n+\lambda+1}$ be the set $\sigma \in \mathcal{M}_{n+\lambda+1}$ such that $\sigma^{-1}(n+1) = \emptyset$ and $\sigma(k) = k$ for $n+2 \leq k \leq n+\lambda+1$. Clearly, $|\mathcal{M}^*_{n+\lambda+1}| = (n+\lambda)^{n+1}$.

For any $\sigma \in \mathcal{M}^*_{n+\lambda+1}$, it can uniquely determine a map τ from $[n+1]$ to $[n+1]$ such that the fixed points of τ have λ colors, say, $c_1, c_2, \ldots, c_\lambda$. The map τ is defined as follows.

$$\tau(i) = \begin{cases} \sigma(i), & if\ \sigma(i) \neq i\ and\ \sigma(i) \in [n], \\ n+1, & if\ \sigma(i) = i\ and\ i \in [n], \\ i_{c_j}, & if\ \sigma(i) = n+j+1\ and\ i \in [n+1], j \in [\lambda], \end{cases}$$

where $\tau(i) = i_{c_j}$ means $\tau(i) = i$ and i has color c_j. Conversely, one can uniquely recover σ from τ by the following manner,

$$\sigma(i) = \begin{cases} \tau(i), & if\ \tau(i) \neq i\ and\ \tau(i) \in [n], \\ i, & if\ \tau(i) = n+1\ and\ i \in [n], \\ n+j+1, & if\ \tau(i) = i_{c_j}\ and\ i \in [n+1], j \in [\lambda]. \end{cases}$$

In other words, G_τ is obtained from G_σ by the three steps:

(i) Each directed cycle from i to itself for $i \in [n]$ is transferred to be a directed edge from i to $n+1$;

(ii) Each directed edge from i to $n+j+1$ for $j \in [\lambda]$ is transferred to be a directed cycle from i to itself for $i \in [n+1]$, and such i is assigned a color c_j;

(iii) Remove all the vertices $n+j+1$ for $j \in [\lambda]$.

It is clear that the procedure above is invertible and it is easy to recover G_σ from G_τ. So such maps τ are counted by $(n+\lambda)^{n+1}$. On the other hand, we have also known that τ is bijected to a pair $(F, \pi) \in \mathcal{P}_n$ such that the fixed points of π (also the fixed points of τ) have λ colors. If we restrict $F \in \mathcal{F}_{n,k}$, then π is a permutation on $\mathcal{R}_F$ with $k+1$ vertices such that the fixed points of π have λ colors. So such F are counted by $|\mathcal{F}_{n,k}| = \binom{n}{k}(n+1)^{n-k}$ and such π are counted by $f_{k+1}(\lambda)$. Summering all possible cases for $0 \leq k \leq n$, we get (1.6). ∎

2.2 Second proof of (1.6)

Let $y := y(x)$ denote the exponential generating function for the labeled rooted trees on $[n]$ which are counted by the sequence $(n^{n-1})_{n\geq 1}$, that is

$$y = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}.$$

This generating function satisfies the relation $y = xe^y$ [19]. By the Lagrange inversion formula, one can derive

$$\frac{y^k}{k!} = \sum_{n\geq k}\binom{n-1}{k-1}n^{n-k}\frac{x^n}{n!}, \tag{2.1}$$

$$\frac{e^{\lambda y}}{1-y} = \sum_{n\geq 0}(n+\lambda)^n\frac{x^n}{n!}. \tag{2.2}$$

By (1.2), the exponential generating function $f(\lambda, t)$ for $f_n(\lambda)$ can be easily deduced

$$f(\lambda, t) = \sum_{k\geq 0} f_k(\lambda)\frac{t^k}{k!} = \frac{e^{(\lambda-1)t}}{1-t}. \tag{2.3}$$

Setting $t := y$ in (2.3), by (2.1) and (2.2), extracting the coefficient of $\frac{x^n}{n!}$, we have

$$\sum_{k=1}^{n}\binom{n-1}{k-1}f_k(\lambda)n^{n-k} = (n+\lambda-1)^n, \tag{2.4}$$

which is equivalent to (1.6). ∎

2.3 Third proof of (1.6)

Let $\mathbf{D}$ denote the umbra, given by $\mathbf{D}^n = D_n$. See [7, 14, 15] for more information on umbral calculus. By (1.5), $f_n(\lambda)$ can be represented umbrally as

$$f_n(\lambda) = (\mathbf{D}+\lambda)^n. \tag{2.5}$$

Then, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)(n+1)^{n-k}\\
&= (\mathbf{D}+\lambda)(\mathbf{D}+\lambda+n+1)^n\\
&= (\mathbf{D}+\lambda+n+1)^{n+1} - (n+1)(\mathbf{D}+\lambda+n+1)^n\\
&= f_{n+1}(\lambda+n+1) - (n+1)f_n(\lambda+n+1)\\
&= (n+\lambda)^{n+1} \qquad\qquad (\text{by } (1.3)),
\end{aligned}$$

as desired. ∎

3 Proof of Theorem 1.2 and Its Applications

In this section, we first give a proof of Theorem 1.2, and then we provide several interesting examples.

3.1 Proof of Theorem 1.2

Recall that Abel's binomial theorem [5, P128] states

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a(a-kt)^{k-1}(b+kt)^{n-k}. \tag{3.1}$$

Let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ be the exponential generating function for any sequence $(a_n)_{n\geq 0}$, then (3.1) is equivalent to the form [5, P130]

$$A(x) = \sum_{k\geq 0} x(x-kt)^{k-1} \frac{A^{(k)}(kt)}{k!}, \tag{3.2}$$

where t is a new indeterminate and $A^{(k)}(kt)$ denotes the k-th derivative of $A(x)$ taking value at kt.

By (1.6) and (2.5), we have

$$(\mathbf{D}+\lambda)(\mathbf{D}+\lambda+n+1)^n = (n+\lambda)^{n+1}. \tag{3.3}$$

Setting $x := (\mathbf{D}+\lambda)x$ and $t = -x$ in (3.2), by (3.3), one can obtain (1.7).

3.2 Applications of Theorem 1.2

In this subsection, as applications of Theorem 1.2, we only consider three special cases when a_n are taken to be the Charlier, Bell and Hermite polynomials. Of course, one can also consider other interesting cases such as a_n are the Bessel, Chebyshev, Legendre, Jacobi, Laguerre, and ultraspherical polynomials and so on.

Example 3.1. Let $a_n = C_n(\alpha, u)$, the (re-normalized) Charlier polynomial, which has the exponential generating function $A(x) = \frac{e^{ux}}{(1-x)^\alpha}$. It is easy to derive $\frac{\partial}{\partial x}A(x) = \frac{\alpha+u(1-x)}{1-x}A(x)$ and the recurrence relation

$$C_{n+1}(\alpha, u) = \alpha C_n(\alpha+1, u) + uC_n(\alpha, u).$$

Using this recurrence and by induction on k, one can deduce

$$\frac{\partial^k}{\partial x^k}A(x) = \frac{C_k(\alpha, u(1-x))}{(1-x)^k}A(x) = \frac{C_k(\alpha, u(1-x))e^{ux}}{(1-x)^{\alpha+k}}.$$

Then, by Theorem 1.2, we have

$$\sum_{n\geq 0} C_n(\alpha,u)f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(k+\lambda-1)^k x^k C_k(\alpha,u(1+kx))e^{-ukx}}{k!(1+kx)^{\alpha+k}}. \tag{3.4}$$

More generally, if let $a_n = C_{m+n}(\alpha,u)$ or $A(x) = \frac{\partial^m}{\partial x^m}\frac{e^{ux}}{(1-x)^\alpha}$, then

$$\frac{\partial^k}{\partial x^k}A(x) = \frac{\partial^{m+k}}{\partial x^{m+k}}\frac{e^{ux}}{(1-x)^\alpha} = \frac{C_{m+k}(\alpha,u(1-x))e^{ux}}{(1-x)^{\alpha+m+k}}.$$

In this case Theorem 1.2 generates

$$\sum_{n\geq 0} C_{m+n}(\alpha,u)f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(k+\lambda-1)^k x^k C_{m+k}(\alpha,u(1+kx))e^{-ukx}}{k!(1+kx)^{\alpha+m+k}}. \tag{3.5}$$

The parameter specializations in (3.4) and (3.5) produce several consequences.

Case 1. When $\alpha = 1, u = \mu - 1$, $C_{m+n}(1,\mu-1) = f_{m+n}(\mu)$. Then, by (3.5), we have

$$\sum_{n\geq 0} f_{m+n}(\mu)f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(k+\lambda-1)^k x^k f_{m+k}(1+(\mu-1)(1+kx))e^{-(\mu-1)kx}}{k!(1+kx)^{m+k+1}}.$$

which, when $\mu = \lambda = 0$, by $f_n(0) = D_n$, yields

$$\sum_{n\geq 0} D_{m+n}D_n\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(k-1)^k x^k f_{m+k}(-kx)e^{kx}}{k!(1+kx)^{m+k+1}}, \tag{3.6}$$

and when $\lambda = 1$ leads to

$$\sum_{n\geq 0} f_{m+n}(\mu)x^n = \sum_{k\geq 0}\frac{(kx)^k f_{m+k}(1+(\mu-1)(1+kx))e^{-(\mu-1)kx}}{k!(1+kx)^{m+k+1}}. \tag{3.7}$$

The case $\mu = 0$ in (3.7) gives the ordinary generating function for D_{m+n},

$$\sum_{n\geq 0} D_{m+n}x^n = \sum_{k\geq 0}\frac{(kx)^k f_{m+k}(-kx)e^{kx}}{k!(1+kx)^{m+k+1}}.$$

Case 2. When $u = 0$, $C_{m+n}(\alpha,0) = (\alpha)_{m+n}$. Then, by (3.5), we have

$$\sum_{n\geq 0}(\alpha)_{m+n}f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(\alpha)_{m+k}(k+\lambda-1)^k x^k}{k!(1+kx)^{m+k+1}},$$

which, when $\alpha = 1, \lambda = 1$ and $\alpha = 1, m = 0$, leads respectively to the ordinary generating function for $(m+n)!$ and $f_n(\lambda)$,

$$\sum_{n\geq 0}(m+n)!x^n = \sum_{k\geq 0}\frac{(m+k)!}{k!}\frac{(kx)^k}{(1+kx)^{m+k+1}},$$

$$\sum_{n\geq 0} f_n(\lambda)x^n = \sum_{k\geq 0}\frac{(k+\lambda-1)^k x^k}{(1+kx)^{k+1}}. \tag{3.8}$$

Remark 3.2. Gessel [7] utilized the umbral calculus method to derive the bilinear generating function for Charlier polynomials

$$\sum_{n\geq 0} C_n(\alpha,u)C_n(\beta,v)\frac{x^n}{n!} = e^{uvx}\sum_{k\geq 0}\frac{(\alpha)_k}{(1-vx)^{\alpha+k}}\frac{(\beta)_k}{(1-ux)^{\beta+k}}\frac{x^k}{k!},$$

which, when $\alpha=\beta=1, u=v=-1$, gives us another more interesting but considerably more recondite formula analogous to the case $m=0$ in (3.6),

$$\sum_{n\geq 0} D_n D_n \frac{x^n}{n!} = e^x \sum_{k\geq 0}\frac{k!x^k}{(1+x)^{2k+2}}.$$

Remark 3.3. Clarke, Han and Zeng [4] utilized the Laplace transformation to deduce another ordinary generating function for $f_n(\mu)$ analogous to (3.8) or the case $m=0$ in (3.7),

$$\sum_{n\geq 0} f_n(\mu)x^n = \sum_{k\geq 0}\frac{k!x^k}{(1-(\mu-1)x)^{k+1}}.$$

Example 3.4. Let $a_n = B_n(u)$, the nth Bell polynomial, which has the exponential generating function $A(x)=\exp(u(e^x-1))$. It is easy to derive $\frac{\partial}{\partial x}A(x) = ue^x A(x)$ and the recurrence relation

$$B_{n+1}(u) = uB_n(u) + u\frac{d}{du}B_n(u).$$

Using this recurrence and by induction on k, one can deduce

$$\frac{\partial^k}{\partial x^k}A(x) = B_k(ue^x)A(x) = B_k(ue^x)\exp(u(e^x-1)).$$

Then, by Theorem 1.2, we have

$$\sum_{n\geq 0} B_n(u)f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(k+\lambda-1)^k x^k B_k(ue^{-kx})\exp\left(u(e^{-kx}-1)\right)}{k!}.$$

More generally, if let $a_n = B_{m+n}(u)$ or $A(x) = \frac{\partial^m}{\partial x^m}\exp(u(e^x-1))$, then

$$\frac{\partial^k}{\partial x^k}A(x) = \frac{\partial^{m+k}}{\partial x^{m+k}}\exp(u(e^x-1)) = B_{m+k}(ue^x)\exp(u(e^x-1)).$$

In this case Theorem 1.2 generates

$$\sum_{n\geq 0} B_{m+n}(u)f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0}\frac{(k+\lambda-1)^k x^k B_{m+k}(ue^{-kx})\exp\left(u(e^{-kx}-1)\right)}{k!},$$

which, when $u=\lambda=1$, leads to the ordinary generating function for the Bell numbers $B_{m+n}(1) = B_{m+n}$ [17, A000110],

$$\sum_{n\geq 0} B_{m+n}x^n = \sum_{k\geq 0}\frac{(kx)^k B_{m+k}(e^{-kx})\exp\left(e^{-kx}-1\right)}{k!}. \tag{3.9}$$

Remark 3.5. Another classical ordinary generating function for the Bell numbers B_n is

$$\sum_{n\geq 0} B_n x^n = \sum_{k\geq 0} \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Klazar [10] in depth investigated this generating function and proved that it satisfies no algebraic differential equation over the complex field.

Example 3.6. Let $a_n = H_n(u)$, the nth (re-normalized) Hermite polynomial, whose exponential generating function is $A(x) = \exp(ux + \frac{x^2}{2})$. The polynomial $H_n(u)$ also counts involutions on $[n]$ such that the fixed points have u colors. It is easy to derive $\frac{\partial}{\partial x}A(x) = (u+x)A(x)$ and the recurrence relation

$$H_{n+1}(u) = uH_n(u) + \frac{d}{du}H_n(u).$$

Using this recurrence and by induction on k, one can deduce

$$\frac{\partial^k}{\partial x^k}A(x) = H_k(u+x)A(x) = H_k(u+x)\exp(ux + \frac{x^2}{2}).$$

Then, by Theorem 1.2, we have

$$\sum_{n\geq 0} H_n(u) f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0} \frac{(k+\lambda-1)^k x^k H_k(u-kx)\exp\left(-ukx + \frac{(kx)^2}{2}\right)}{k!}.$$

More generally, if let $a_n = H_{m+n}(u)$ or $A(x) = \frac{\partial^m}{\partial x^m}\exp(ux + \frac{x^2}{2})$, then

$$\frac{\partial^k}{\partial x^k}A(x) = \frac{\partial^{m+k}}{\partial x^{m+k}}\exp(ux + \frac{x^2}{2}) = H_{m+k}(u+x)\exp(ux + \frac{x^2}{2}).$$

In this case Theorem 1.2 generates

$$\sum_{n\geq 0} H_{m+n}(u) f_n(\lambda)\frac{x^n}{n!} = \sum_{k\geq 0} \frac{(k+\lambda-1)^k x^k H_{m+k}(u-kx)\exp\left(-ukx + \frac{(kx)^2}{2}\right)}{k!}. \quad (3.10)$$

The cases when $\lambda = 1$ and $u = 1$ or $u = 0$ in (3.10), lead respectively to the ordinary generating functions for the involution numbers $I_{m+n} = H_{m+n}(1)$ [17, A000085] and the matching numbers $M_{m+n} = H_{m+n}(0)$ [17, A001147],

$$\sum_{n\geq 0} I_{m+n}x^n = \sum_{k\geq 0} \frac{(kx)^k H_{m+k}(1-kx)\exp\left(-kx + \frac{(kx)^2}{2}\right)}{k!},$$

$$\sum_{n\geq 0} M_{m+n}x^n = \sum_{k\geq 0} \frac{(kx)^k H_{m+k}(-kx)\exp\left(\frac{(kx)^2}{2}\right)}{k!}.$$

4 Further Properties of $f_n(\lambda)$

Theorem 4.1. *For any integer $n \geq 0$ and any indeterminates λ, μ, there hold*

$$\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)(\mu + k - n)\mu^{n-k-1} = (\lambda + \mu - 1)^n, \tag{4.1}$$

$$\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda) f_{n-k}(\mu + 1) = (\lambda + \mu - 1)^{n+1} + (n - \lambda - \mu + 2) f_n(\lambda + \mu). \tag{4.2}$$

Proof. For (4.1), we have

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)(\mu + k - n)\mu^{n-k-1} \\
&= f_n(\lambda + \mu) - \frac{\partial}{\partial \mu} f_n(\lambda + \mu) \qquad \text{(by (1.1))} \\
&= f_n(\lambda + \mu) - n f_{n-1}(\lambda + \mu) \qquad \text{(by (1.4))} \\
&= (\lambda + \mu - 1)^n \qquad \text{(by (1.3))}.
\end{aligned}$$

For (4.2), we have

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)(\mu + k - n)\mu^{n-k}\Big|_{\mu := \mathbf{D} + \mu + n + 1} \\
&= \sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)(\mathbf{D} + \mu + n + 1 + k - n)(\mathbf{D} + \mu + n + 1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)\big(f_{n-k+1}(\mu + n + 1) - (n - k) f_{n-k}(\mu + n + 1)\big) \qquad \text{(by (2.5))} \\
&= \sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)\big(f_{n-k}(\mu + n + 1) + (\mu + n)^{n-k+1}\big) \qquad \text{(by (1.3))} \\
&= \sum_{k=0}^{n} \binom{n}{k} f_k(\lambda) f_{n-k}(\mu + n + 1) + (\mu + n) f_n(\lambda + \mu + n) \qquad \text{(by (1.1))}.
\end{aligned} \tag{4.3}$$

On the other hand, we have

$$\begin{aligned}
&\mu(\lambda + \mu - 1)^n\Big|_{\mu := \mathbf{D} + \mu + n + 1} \\
&= (\mathbf{D} + \mu + n + 1)(\mathbf{D} + \lambda + \mu + n)^n \\
&= (\mathbf{D} + \lambda + \mu - 1)(\mathbf{D} + \lambda + \mu + n)^n + (n - \lambda + 2)(\mathbf{D} + \lambda + \mu + n)^n \\
&= (\lambda + \mu + n - 1)^{n+1} + (n - \lambda + 2) f_n(\lambda + \mu + n) \qquad \text{(by (1.3))}.
\end{aligned} \tag{4.4}$$

Then, by (4.1), (4.2) can be deduced by setting $\mu := \mu - n$ in (4.3) and (4.4). ∎

Remark 4.2. It should be noted that (4.1) in the case $\mu = 1$ and (1.5) form a new inverse relation. In general, for any two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$,

$$b_n = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} a_k \quad \Leftrightarrow \quad a_n = \sum_{k=0}^{n} \binom{n}{k} (1+k-n) b_k.$$

Remark 4.3. The case $\mu = n$ and $n := n+1$ in (4.1) reduces to (1.6).

When $\mu = \lambda + n + 1$ in (4.1), we have

Corollary 4.4. *For any integer $n \geq 0$ and any indeterminate λ, there holds*

$$\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda)(\lambda+k+1)(\lambda+n+1)^{n-k-1} = (n+2\lambda)^n. \tag{4.5}$$

When $\lambda + \mu = n + 2$, (4.2) reduces to the surprising result.

Corollary 4.5. *For any integer $n \geq 0$ and any indeterminate λ, there holds*

$$\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda) f_{n-k}(n-\lambda+3) = (n+1)^{n+1}.$$

Theorem 4.6. *For any integer $n \geq 0$ and any indeterminates λ, μ, there holds*

$$f_n(\lambda+\mu) = \sum_{k=0}^{n} \binom{n}{k} (\lambda+k)^k (\mu-k-1)^{n-k}. \tag{4.6}$$

Proof. Setting $t = -1, a = \mathbf{D} + \lambda + 1, b = \mu - 1$ in (3.1), we have

$$(\mathbf{D}+\lambda+\mu)^n = \sum_{k=0}^{n} \binom{n}{k} (\mathbf{D}+\lambda+1)(\mathbf{D}+\lambda+1+k)^{k-1}(\mu-k-1)^{n-k},$$

which, by (2.5), is equivalent to (4.6). ∎

Remark 4.7. Note that (4.6) in the case $\lambda := 2\lambda, \mu = -\lambda$ and (4.5) form another inverse relation. In general, for any two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$,

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (\lambda+k)(\lambda+n)^{n-k-1} a_k \quad \Leftrightarrow \quad a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\lambda+k)^{n-k} b_k, \tag{4.7}$$

which is an Abel inverse pair [11, P95], also a special case of Gould-Hsu inversions [8]. When $\lambda = 0$, (4.7) reduces to another known inverse relation [5, P164], [11, P96],

$$b_n = \sum_{k=0}^{n} \binom{n-1}{k-1} n^{n-k} a_k \quad \Leftrightarrow \quad a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^{n-k} b_k.$$

Using the inverse relation (4.7), by (4.6) in the case $\mu := -\mu + 1$, we have

$$\sum_{k=0}^{n} \binom{n}{k} f_k(\lambda-\mu+1)(\mu+k)(\mu+n)^{n-k-1} = (\lambda+n)^n.$$

When $\mu = 1-\lambda$ in (4.6), by $f_n(1) = n!$, we have the well-known difference identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\lambda+k)^n = n!.$$

By the inverse relation (4.7) in the case $a_k = k!, b_k = (\lambda+k)^k$, we have

$$\sum_{k=0}^{n}\binom{n}{k}k!(\lambda+k)(\lambda+n)^{n-k-1} = (\lambda+n)^n, \tag{4.8}$$

which, when $\lambda = 1$ or $\lambda = 0$ and $n := n+1$, reduces to the Riordan identity [13]. Note that (4.8) is also a special case when $\lambda = 1, \mu = \lambda + n$ in (4.1).

When $\mu = -\lambda$ in (4.6), by $f_n(0) = D_n$, we have the following result.

Corollary 4.8. *For any integer $n \geq 0$ and any indeterminate λ, there holds*

$$D_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\lambda+k)^k(\lambda+k+1)^{n-k}. \tag{4.9}$$

Remark 4.9. The case $\lambda = -1$ in (4.9) was obtained by Ryser [16] using the permanent theory, and also appeared in [5, P201]. In fact, D_n is also the permanent of the matrix $\mathbf{J} - \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ unit matrix and $\mathbf{J}$ is the $n \times n$ matrix with all entries being equal to 1.

Remark 4.10. By the inverse relation (4.7) in the case $a_k = D_k, b_k = (\lambda+k-1)^k$ and by (4.9) in the case $\lambda := \lambda - 1$, we have

$$\sum_{k=0}^{n}\binom{n}{k}D_k(\lambda+k)(\lambda+n)^{n-k-1} = (\lambda+n-1)^n. \tag{4.10}$$

The inverse pair of (4.9) and (4.10) in the case $\lambda = 0$ have appeared in [8]. The case $\lambda = 0$ in (4.10) reduces to the identity obtained by Riordan [12], Sun and Xu [20]. Note that (4.10) is also a special case when $\lambda = 0, \mu = \lambda + n$ in (4.1).

Theorem 4.11. *For any integer $n \geq 0$ and any indeterminates λ, μ, there hold*

$$\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)\mu^{n-k} = (\mu-(n+1))\sum_{k=0}^{n}\binom{n}{k}(\lambda+k)^{k+1}(\mu-k-1)^{n-k-1}, \tag{4.11}$$

$$\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)f_{n-k}(\mu+1) = \sum_{k=0}^{n}\binom{n}{k}(\lambda+k)^{k+1}(\mu-k-1)^{n-k}. \tag{4.12}$$

Proof. For (4.11), setting $t = -1, a = \mu - n - 1, b = \mathbf{D} + \lambda + n + 1$ in (3.1), we have

$$\begin{aligned}(\mathbf{D}+\lambda+\mu)^n &= \sum_{k=0}^{n}\binom{n}{k}(\mathbf{D}+\lambda+n-k+1)^{n-k}(\mu-n-1)(\mu-n+k-1)^{k-1}\\ &= \sum_{k=0}^{n}\binom{n}{k}(\mathbf{D}+\lambda+k+1)^{k}(\mu-n-1)(\mu-k-1)^{n-k-1}.\end{aligned}$$

Then, by (2.5), we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)\mu^{n-k}=(\mathbf{D}+\lambda)(\mathbf{D}+\lambda+\mu)^n\\
&=\sum_{k=0}^{n}\binom{n}{k}(\mathbf{D}+\lambda)(\mathbf{D}+\lambda+k+1)^k(\mu-n-1)(\mu-k-1)^{n-k-1}\\
&=(\mu-(n+1))\sum_{k=0}^{n}\binom{n}{k}(\lambda+k)^{k+1}(\mu-k-1)^{n-k-1}.
\end{aligned}$$

For (4.12), setting $\mu:=\mathbf{D}+\mu+n+1$ in (4.11), by (2.5), we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)f_{n-k}(\mu+n+1)\\
&=\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)(\mathbf{D}+\mu+n+1)^{n-k}\\
&=\sum_{k=0}^{n}\binom{n}{k}(\lambda+k)^{k+1}(\mathbf{D}+\mu)(\mathbf{D}+\mu+n-k)^{n-k-1}\\
&=\sum_{k=0}^{n}\binom{n}{k}(\lambda+k)^{k+1}(\mu+n-k-1)^{n-k},
\end{aligned}$$

which, by setting $\mu:=\mu-n$, generates (4.12). ∎

Remark 4.12. The case $\mu=n+1$ in (4.11) produces (1.6).

Setting $\mu=1-\lambda$ in (4.11) and (4.12), using the general difference identity [18]

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\lambda+k)^m=\sum_{k=n}^{m}(-1)^kS(m,k)(-k)_n(-\lambda)_{k-n},\tag{4.13}$$

where $S(m,k)$ is the Stirling number of the second kind [17, A008277], and by $S(n+1,n)=\binom{n+1}{2}$, we have

Corollary 4.13. *For any integer $n\geq 0$ and any indeterminate λ, there hold*

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)(1-\lambda)^{n-k}&=n!(\lambda+n),\\
\sum_{k=0}^{n}\binom{n}{k}f_{k+1}(\lambda)f_{n-k}(2-\lambda)&=(n+1)!(\lambda+\frac{n}{2}).
\end{aligned}$$

Remark 4.14. The special case $m = n + 1$ in (4.13) produces

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\lambda+k)^{n+1} = (n+1)!(\lambda+\frac{n}{2}),$$

which, by the inverse relation (4.7), generates

$$\sum_{k=0}^{n}\binom{n}{k}(k+1)!(\lambda+\frac{k}{2})(\lambda+k)(\lambda+n)^{n-k-1} = (\lambda+n)^{n+1}.$$

5 Comments and Open Questions

In general, we can consider the generalization of (1.1) and (4.11), that is

$$Q_{n,m}(\lambda,\mu) = \sum_{k=0}^{n}\binom{n}{k}f_{k+m}(\lambda)\mu^{n-k}.$$

By (1.3), one can deduce the first recurrence relation for $Q_{n,m}(\lambda,\mu)$,

$$Q_{n,m}(\lambda,\mu) = nQ_{n-1,m}(\lambda,\mu) + mQ_{n,m-1}(\lambda,\mu) + (\lambda-1)^m(\lambda+\mu-1)^n \tag{5.1}$$

with the initial conditions $Q_{0,0}(\lambda,\mu) = 1, Q_{n,0}(\lambda,\mu) = Q_{0,m}(\lambda,\mu) = 0$ whenever $n, m < 0$. Clearly, (5.1) reduces to (1.3) when $n = 0$ and $m := n$ or $m = 0$ and $\mu = 0$.

Let $Q(\lambda,\mu;x,t)$ denote the exponential generating function for $Q_{n,m}(\lambda,\mu)$, i.e.,

$$Q(\lambda,\mu;t,x) = \sum_{n,m\geq 0} Q_{n,m}(\lambda,\mu)\frac{t^n}{n!}\frac{x^m}{m!}.$$

From (5.1), we can derive the explicit formula for $Q(\lambda,\mu;t,x)$,

$$Q(\lambda,\mu;t,x) = \frac{e^{(\lambda+\mu-1)t}e^{(\lambda-1)x}}{1-t-x}. \tag{5.2}$$

By (5.2), one has

$$\frac{\partial Q(\lambda,\mu;t,x)}{\partial t} = \frac{\partial Q(\lambda,\mu;t,x)}{\partial x} + \mu Q(\lambda,\mu;t,x),$$

which implies that there holds another recurrence relation for $Q_{n,m}(\lambda,\mu)$,

$$Q_{n+1,m}(\lambda,\mu) = Q_{n,m+1}(\lambda,\mu) + \mu Q_{n,m}(\lambda,\mu).$$

Note that the type of the exponential generating function $Q(\lambda,\mu;t,x)$ brings it into the general framework considered in [20], which signifies that $Q_{n,m}(\lambda,\mu)$ has many other interesting properties. For examples, setting $t := tx$ in (5.2) and comparing the coefficients of $\frac{x^N}{N!}$, we get

$$\sum_{n=0}^{N} \binom{N}{n} Q_{N-n,n}(\lambda,\mu) t^{N-n} = (t+1)^N f_N(\lambda + \frac{\mu t}{t+1}).$$

Using the series expansion, we have

$$\begin{aligned}
Q(\lambda,\mu;t,x) &= \frac{e^{(\lambda+\mu-1)t}e^{(\lambda-1)x}}{1-t-x} = \frac{e^{(\lambda+\mu-1)t}}{1-t}\frac{e^{(\lambda-1)x}}{1-\frac{x}{1-t}} \\
&= \sum_{m\geq 0}\frac{x^m}{m!}\sum_{j=0}^{m}\binom{m}{j} j!(\lambda-1)^{m-j}\frac{e^{(\lambda+\mu-1)t}}{(1-t)^{j+1}} \\
&= \sum_{m\geq 0}\frac{x^m}{m!}\sum_{j=0}^{m}\binom{m}{j} j!(\lambda-1)^{m-j}\sum_{n\geq 0}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(j+1)_k(\lambda+\mu-1)^{n-k} \\
&= \sum_{n\geq 0}\sum_{m\geq 0}\frac{t^n}{n!}\frac{x^m}{m!}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}(k+j)!(\lambda+\mu-1)^{n-k}(\lambda-1)^{m-j}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n x^m}{n!m!}$, we get an explicit formula for $Q_{n,m}(\lambda,\mu)$,

$$Q_{n,m}(\lambda,\mu) = \sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}(k+j)!(\lambda+\mu-1)^{n-k}(\lambda-1)^{m-j}.$$

But here we have more interest in the type of formulas for $Q_{n,m}(\lambda,\mu)$ similar to (4.6) and (4.11).

Lemma 5.1. *For any integers $n, m \geq 0$ and any indeterminates λ, μ, there holds*

$$Q_{n,m}(\lambda,\mu) = mQ_{n,m-1}(\lambda, \mathbf{D}+\mu+1) + (\lambda-1)^m f_n(\lambda+\mu), \tag{5.3}$$

or equivalently

$$Q_{n,m}(\lambda,\mu) = m\sum_{k=0}^{n}\binom{n}{k} f_{k+m-1}(\lambda) f_{n-k}(\mu+1) + (\lambda-1)^m f_n(\lambda+\mu). \tag{5.4}$$

Proof. Note that

$$\begin{aligned}
\sum_{n\geq 0} Q_{n,m}(\lambda,\mu)\frac{t^n}{n!} &= \frac{\partial^m}{\partial x^m}Q(\lambda,\mu;t,x)\Big|_{x=0} \\
&= \frac{\partial^m}{\partial x^m}\frac{e^{\mu t}e^{(\lambda-1)(t+x)}}{1-(t+x)}\Big|_{x=0} = e^{\mu t}\frac{\partial^m}{\partial t^m}\frac{e^{(\lambda-1)t}}{1-t} \\
&= f_m((\lambda-1)(1-t)+1)\frac{e^{(\lambda+\mu-1)t}}{(1-t)^{m+1}} \qquad \text{(by Example 3.1)} \\
&= mf_{m-1}((\lambda-1)(1-t)+1)\frac{e^{(\lambda+\mu-1)t}}{(1-t)^{m+1}} + (\lambda-1)^m\frac{e^{(\lambda+\mu-1)t}}{1-t} \qquad \text{(by (1.3))} \\
&= m\frac{e^{\mu t}}{1-t}\frac{\partial^{m-1}}{\partial t^{m-1}}\frac{e^{(\lambda-1)t}}{1-t} + (\lambda-1)^m\frac{e^{(\lambda+\mu-1)t}}{1-t}.
\end{aligned}$$

By (2.3), comparing the coefficient of $\frac{t^n}{n!}$, we get (5.4). By (2.5), $f_{n-k}(\mu+1)$ can be represented umbrally as $(\mathbf{D}+\mu+1)^{n-k}$, which means that (5.4) is equivalent to (5.3) by the definition of $Q_{n,m}(\lambda,\mu)$. ∎

Setting $m=2$ in (5.4), by (4.6) and (4.12), we obtain

Theorem 5.2. *For any integer $n \geq 0$ and any indeterminates λ, μ, there holds*

$$\sum_{k=0}^{n} \binom{n}{k} f_{k+2}(\lambda)\mu^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (\lambda^2+2k+1)(\lambda+k)^k(\mu-k-1)^{n-k}.$$

In general, it seems to be not easy to derive the explicit formula similar to (4.6) and (4.11) for $Q_{n,m}(\lambda,\mu)$, we leave it as an open problem to the interested readers. One can also be asked to give combinatorial proofs for Corollary 4.4, 4.5 and 4.8.

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