Graph Powers and Graph Homomorphisms*

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Abstract

In this paper, we investigate some basic properties of fractional powers. In this regard, we show that for any non-bipartite graph G and positive rational numbers $\frac{2r+1}{2s+1} < \frac{2p+1}{2q+1}$, we have $G^{\frac{2r+1}{2s+1}} < G^{\frac{2p+1}{2q+1}}$. Next, we study the power thickness of G, that is, the supremum of rational numbers $\frac{2r+1}{2s+1}$ such that G and $G^{\frac{2r+1}{2s+1}}$ have the same chromatic number. We prove that the power thickness of any non-complete circular complete graph is greater than one. This provides a sufficient condition for the equality of the chromatic number and the circular chromatic number of graphs. Finally, we introduce an equivalent definition for the circular chromatic number of graphs in terms of fractional powers. Also, we show that for any non-bipartite graph G if $0 < \frac{2r+1}{2s+1} \leqslant \frac{\chi(G)}{3(\chi(G)-2)}$, then $\chi(G^{\frac{2r+1}{2s+1}}) = 3$. Moreover, $\chi(G) \neq \chi_c(G)$ if and only if there exists a rational number $\frac{2r+1}{2s+1} > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^{\frac{2r+1}{2s+1}}) = 3$.

1 Introduction

Throughout this paper we only consider finite simple graphs, unless otherwise stated. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Denote two isomorphic graphs G and H by the symbol $G \cong H$. Also, a *homomorphism* from G to H is a map $f : V(G) \longrightarrow V(H)$ such that adjacent vertices in G are mapped into adjacent vertices in H, i.e., $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. For simplicity, the existence of a homomorphism is indicated by the symbol $G \longrightarrow H$. Two graphs G and H are homomorphically equivalent, denoted by $G \longleftrightarrow H$, if $G \longrightarrow H$ and $H \longrightarrow G$. Also, $G < H$ means that $G \longrightarrow H$ and there is no homomorphism from H to G. The symbol $\mathrm{Hom}(G, H)$ is used to denote the set of all homomorphisms from G to H. In this

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terminology, we say that H is an upper bound for a class $\mathcal{C}$ of graphs, if $G \longrightarrow H$ for all $G \in \mathcal{C}$. The problem of the existence of an upper bound for a class of graphs with some special properties has been a subject of study in the theory of graph homomorphism.

Suppose that H is a subgraph of G. We say that G *retracts* to H, if there exists a homomorphism $r : G \longrightarrow H$, called a *retraction*, such that $r(u) = u$ for any vertex u of H. A *core* is a graph which does not retract to a proper subgraph. Any graph is homomorphically equivalent to a unique core (for more on graph homomorphisms see [4, 5, 10, 13, 14]).

If n and d are positive integers with $n \geqslant 2d$, then the *circular complete graph* $K_{\frac{n}{d}}$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in which v_i is connected to v_j if and only if $d \leqslant |i - j| \leqslant n - d$. A graph G is said to be (n, d)-colorable if G admits a homomorphism to $K_{\frac{n}{d}}$. The *circular chromatic number* (also known as the *star chromatic number* [27]) $\chi_c(G)$ of a graph G is the minimum of those ratios $\frac{n}{d}$ such that G admits a homomorphism to $K_{\frac{n}{d}}$. It is also known that one may equivalently define $\chi_c(G)$ in a similar way, by a restriction to onto-vertex homomorphisms [28]. It is known [27, 28] that for any graph G, $\chi(G) - 1 < \chi_c(G) \leqslant \chi(G)$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$. The reader may consult [28] as an excellent survey on this subject.

A rational number p is called an *odd rational number* if numerator and denominator are both odd integers. As usual, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The *Kneser graph* $\mathrm{KG}(m, n)$ is the graph on the vertex set $\binom{[m]}{n}$, in which A is connected to B if and only if $A \cap B = \varnothing$. It was conjectured by Kneser [16] in 1955, and proved by Lovász [18] in 1978, that $\chi(\mathrm{KG}(m, n)) = m - 2n + 2$. The *Schrijver graph* $SG(m, n)$ is the subgraph of $\mathrm{KG}(m, n)$ induced by all 2-stable n-subsets of $[m]$. It was proved by Schrijver [21] that $\chi(SG(m, n)) = \chi(\mathrm{KG}(m, n))$ and that every proper subgraph of $SG(m, n)$ has a chromatic number smaller than that of $SG(m, n)$. Also, for a given graph G, the notation $og(G)$ stands for the odd girth of G.

For a graph G, let G^k be the kth power of G, which is obtained on the vertex set $V(G)$, by connecting any two vertices u and v for which there exists a walk of length k between u and v in G. Note that the kth power of a simple graph is not necessarily a simple graph itself. For instance, the kth power may have loop edges on its vertices if k is an even integer. The chromatic number of graph powers has been studied in the literature (see [3, 7, 8, 11, 22, 26]).

Remark 1. It should be noted that throughout the literature one may encounter another definition of the kth power of a graph for which two vertices are joined by an edge if the length of the shortest path between them is at most k (e.g. [1, 2]). Although, in this paper, the edge set of kth power of a graph G consists of all pairs u and v for which there exists a walk of length k between u and v in G. In fact, we stick to this definition of power, since it inherits some properties from power in numbers. Furthermore, the adjacency matrix of G^k is obtained from the kth power of the adjacency matrix of G, by replacing any non-zero entries with one.

The following simple and useful lemma was used in several papers (e.g. [7, 20, 26]).

Lemma A. *Let G and H be two simple graphs such that* $\mathrm{Hom}(G,H) \neq \varnothing$. *Then for any positive integer k,* $\mathrm{Hom}(G^{k}, H^{k}) \neq \varnothing$.

Note that if H contains a closed walk of length k, then H^{k} contains a loop edge. In this case, Lemma A trivially holds. Now, we recall a definition from [11].

Definition 1. Let m, n, and k be positive integers with $m \geqslant 2n$. Set $H(m,n,k)$ to be the *helical graph* whose vertex set consists of all k-tuples $(A_1, \ldots, A_k)$ such that for any $1 \leqslant r \leqslant k$, $A_r \subseteq [m], |A_1| = n, |A_r| \geqslant n$ and for any $s \leqslant k-1$ and $t \leqslant k-2$, $A_s \cap A_{s+1} = \varnothing, A_t \subseteq A_{t+2}$. Also, two vertices $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ of $H(m,n,k)$ are adjacent if for any $1 \leqslant i, j+1 \leqslant k$, $A_i \cap B_i = \varnothing, A_j \subseteq B_{j+1}$, and $B_j \subseteq A_{j+1}$. ♠

Roughly speaking, the vertices of $H(m,n,k)$ encode the set of colors that can be found in certain walks in an n-tuple coloring. Note that $H(m,1,1)$ is the complete graph K_m and $H(m,n,1)$ is the Kneser graph $\mathrm{KG}(m,n)$. Also, it is easy to verify that if $m > 2n$, then the odd girth of $H(m,n,k)$ is greater than or equal to $2k+1$.

Theorem A. [11] *Let m, n, and k be positive integers with $m \geqslant 2n$ and G be a non-empty graph with odd girth at least $2k+1$. Then we have* $\mathrm{Hom}(G^{2k-1}, \mathrm{KG}(m,n)) \neq \varnothing$ *if and only if* $\mathrm{Hom}(G, H(m,n,k)) \neq \varnothing$. *Moreover, the chromatic number of the helical graph $H(m,n,k)$ is equal to $m-2n+2$.*

A graph H is said to be a subdivision of a graph G if H is obtained from G by subdividing some of the edges. The graph $G^{\frac{1}{s}}$ is said to be the s-subdivision of a graph G if $G^{\frac{1}{s}}$ is obtained from G by replacing each edge with a path with exactly $s-1$ inner vertices. In this terminology, $G^{\frac{1}{1}}$ is isomorphic to G.

Hereafter, for a given graph G, we use the following notation for convenience. Set

$$G^{\frac{r}{s}} \stackrel{\mathrm{def}}{=} (G^{\frac{1}{s}})^{r},$$

where r and s are positive integers. Note that when s is an even integer, then $G^{\frac{r}{s}}$ is a bipartite graph. Furthermore, if r is an even integer and G is a non-empty graph, then the graph $G^{\frac{r}{s}}$ contains loop edges. On the other hand, for bipartite graphs or graphs with loop edges, one can easily recognize the existence of a graph homomorphism. Hence, hereafter we consider just odd rational numbers as power of graphs. The symbol C_n stands for the cycle on n vertices.

Theorem B. [11] *Let G be a graph with odd girth at least $2k+1$. Then* $\chi(G^{\frac{2k+1}{3}}) \leqslant 3$ *if and only if* $\mathrm{Hom}(G, C_{2k+1}) \neq \varnothing$.

For given graphs G and H with $v \in V(G)$, set

$$N_i(v) \stackrel{\mathrm{def}}{=} \{u | \text{there is a walk of length } i \text{ joining } u \text{ and } v\}.$$

Also, for a graph homomorphism $f : G \longrightarrow H$, define

$$f(N_i(v)) \stackrel{\text{def}}{=} \bigcup_{u \in N_i(v)} f(u).$$

For two subsets A and B of the vertex set of a graph G, we write $A \bowtie B$ if every vertex of A is joined to every vertex of B. Also, for any non-negative integer s, define the graph $G^{-\frac{1}{2s+1}}$ as follows.

$$V(G^{-\frac{1}{2s+1}}) \stackrel{\text{def}}{=} \{(A_1, \ldots, A_{s+1})|\ A_i \subseteq V(G), |A_1| = 1, \varnothing \neq A_i \subseteq N_{i-1}(A_1)\ \ , i \leqslant s+1\}.$$

Two vertices $(A_1, \ldots, A_{s+1})$ and $(B_1, \ldots, B_{s+1})$ are adjacent in $G^{-\frac{1}{2s+1}}$ if for any $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant s+1$, $A_i \subseteq B_{i+1}$, $B_i \subseteq A_{i+1}$, and $A_j \bowtie B_j$. Also, for any graph G define the graph $G^{-\frac{2r+1}{2s+1}}$ as follows.

$$G^{-\frac{2r+1}{2s+1}} \stackrel{\text{def}}{=} (G^{-\frac{1}{2s+1}})^{2r+1}.$$

The graph $G^{-\frac{1}{3}}$ was first defined by C. Tardif, with different notation ($P_3^{-1}(G)$), to study multiplicative graphs, see [26]. Also, the graph $K_n^{-\frac{1}{2k+1}}$ was defined in a completely different way in [3, 8, 9, 22]. It is readily seen that $K_n^{-\frac{1}{2k+1}} \cong H(n, 1, k+1)$ and $K_3^{-\frac{1}{2k+1}} \cong C_{6k+3}$.

Theorem C. [11] *Let G and H be two graphs and $2r+1 < og(G)$. We have $G^{2r+1} \longrightarrow H$ if and only if $G \longrightarrow H^{-\frac{1}{2r+1}}$.*

In what follows we are concerned with fractional powers. The paper is organized as follows. In the second section, we introduce some basic properties of fractional powers. In this regard, we introduce some properties of graph powers similar to power in numbers. For instance, we show that when q is an odd integer, then $G^{\frac{rq}{sq}}$ and $G^{\frac{r}{s}}$ are homomorphically equivalent. Also, we present reduction results for the graph homomorphism problem. Next, we introduce some density-type results. Indeed, we show that $G^{\frac{2r+1}{2s+1}} < G^{\frac{2p+1}{2q+1}}$ provided that G is a non-bipartite graph and $0 < \frac{2r+1}{2s+1} < \frac{2p+1}{2q+1} < og(G)$. In the third section, we investigate some properties of power thickness, that is, the supremum of rational numbers $\frac{2r+1}{2s+1}$ such that G and $G^{\frac{2r+1}{2s+1}}$ have the same chromatic number. In this section, we determine the power thickness of helical graphs and uniquely colorable graphs. In the fourth section, we introduce an equivalent definition for the circular chromatic number of graphs in terms of fractional powers. Also, we introduce some necessary and sufficient conditions for the equality of the chromatic number and the circular chromatic number of graphs. In this regard, we prove that the power thickness of any non-complete circular complete graph is greater than one. This provides a sufficient condition for the equality of the chromatic number and the circular chromatic number of graphs. Also, we show that for any non-bipartite graph G if $0 < \frac{2r+1}{2s+1} \leqslant \frac{\chi(G)}{3(\chi(G)-2)}$, then $\chi(G^{\frac{2r+1}{2s+1}}) = 3$. Moreover, $\chi(G) \neq \chi_c(G)$ if and only if there exists a rational number $\frac{2r+1}{2s+1} > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^{\frac{2r+1}{2s+1}}) = 3$. Finally, in the fifth section, we make some concluding remarks about open problems and natural directions of generalization.

2 Fractional Powers

2.1 Basic Properties

In this subsection, we investigate some basic properties of graph powers. First, we introduce some notation used for the remainder of the paper. Let G be a graph which does not contain isolated vertices. Set the vertex set of $G^{\frac{1}{2s+1}}$ as follows. By abuse of notation, for any edge $uv \in E(G)$, define $(uv)_0 \stackrel{\text{def}}{=} u$ and $(vu)_0 \stackrel{\text{def}}{=} v$. Note that a vertex may have several representations. Moreover, $(2s+1)$th subdivision of the edge uv is a path of length $2s+1$, say P_{uv}, set the vertices and the edges of this path, respectively, as follows.

$$V(P_{uv}) \stackrel{\text{def}}{=} \{(uv)_0, (uv)_1, \ldots, (uv)_s, (vu)_0, (vu)_1, \ldots, (vu)_s\}$$

and

$$E(P_{uv}) \stackrel{\text{def}}{=} \{(uv)_i(vu)_{s-i},\ (vu)_{s-j+1}(uv)_j |\ 0 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant s\}.$$

Also, note that the graph $G^{\frac{2r+1}{2s+1}}$ is $(2r+1)$th power of $G^{\frac{1}{2s+1}}$. Hence, we follow the aforementioned notation for the vertex set of $G^{\frac{2r+1}{2s+1}}$.

If $\frac{2r+1}{2s+1} \leqslant 1$, then $\mathrm{Hom}(G^{\frac{2r+1}{2s+1}}, G) \neq \varnothing$. To see this, for any vertex $(uv)_i \in G^{\frac{2r+1}{2s+1}}$ $(0 \leqslant i \leqslant s)$, set $f((uv)_i) \stackrel{\text{def}}{=} u$. One can check that $f \in \mathrm{Hom}(G^{\frac{2r+1}{2s+1}}, G)$. Similarly, the following simple lemma can easily be proved by constructing graph homomorphisms and its proof is omitted for the sake of brevity.

Lemma 1. *Let G be a graph.*

a) If q, r and s are non-negative integers, then $G^{\frac{(2r+1)(2q+1)}{(2s+1)(2q+1)}} \longleftrightarrow G^{\frac{2r+1}{2s+1}}$.

b) If s is a non-negative integer where $2s+1 < og(G)$, then $(G^{2s+1})^{\frac{1}{2s+1}} \longrightarrow G$.

The next lemma will be useful throughout the paper. We should mention that an extended version of this lemma has been appeared in [14] as Lemma 5.5.

Lemma 2. *Let G and H be two graphs where $2s+1 < og(H)$. Then $G^{\frac{1}{2s+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{2s+1}$.*

Proof. Let $G^{\frac{1}{2s+1}} \longrightarrow H$, then $(G^{\frac{1}{2s+1}})^{2s+1} \longrightarrow H^{2s+1}$. In view of Lemma 1(a), we have $G \longrightarrow (G^{\frac{1}{2s+1}})^{2s+1} \longrightarrow H^{2s+1}$. Conversely, assume that $G \longrightarrow H^{2s+1}$. Hence, $G^{\frac{1}{2s+1}} \longrightarrow (H^{2s+1})^{\frac{1}{2s+1}}$. On the other hand, Lemma 1(b) shows that $(H^{2s+1})^{\frac{1}{2s+1}} \longrightarrow H$, as desired. ■

It is easy to verify that if r is a non-negative integer and H is a non-bipartite graph, then the odd girth of $H^{-\frac{1}{2r+1}}$ is greater than or equal to $2r+3$. To see this, note that the statement is true for $r = 0$, hence, assume that $r \geqslant 1$. Indirectly, assume that C_{2l+1} is an odd cycle of $H^{-\frac{1}{2r+1}}$ where $1 \leqslant l \leqslant r$. Suppose that $u = (A_1, \ldots, A_{r+1}) \in V(C_{2l+1})$.

Consider two adjacent vertices $v = (B_1, \ldots, B_{r+1})$ and $w = (B'_1, \ldots, B'_{r+1})$ of C_{2l+1} at distance exactly l from u. In view of the definition of $H^{-\frac{1}{2r+1}}$, we should have $A_1 \subseteq B_{r+1}$ and $A_1 \subseteq B'_{r+1}$. On the other hand, v and w are adjacent, consequently, $B_{r+1} \cap B'_{r+1} = \varnothing$ which is a contradiction.

Lemma 3. *Let H be a non-bipartite graph and r be a non-negative integer. Then*

$$(2r+1)(og(H)-2) < og(H^{-\frac{1}{2r+1}}) \leqslant (2r+1)og(H).$$

Proof. First, assume that $og(H^{-\frac{1}{2r+1}}) = 2l+1 \geqslant 2r+3$. Hence, $C_{2l+1} \longrightarrow H^{-\frac{1}{2r+1}}$. Subsequently, in view of Theorem C, $C_{2l+1}^{2r+1} \longrightarrow H$, which implies that $og(C_{2l+1}^{2r+1}) \geqslant og(H)$. Also, it is easy to check that $og(C_{2l+1}^{2r+1})$ is the smallest positive odd integer greater than or equal to $\frac{2l+1}{2r+1}$. Thus, $og(H^{-\frac{1}{2r+1}}) > (2r+1)(og(H)-2)$. Next, in view of Theorem C, we have $H^{\frac{1}{2r+1}} \longrightarrow H^{-\frac{1}{2r+1}}$. Consequently, $og(H^{-\frac{1}{2r+1}}) \leqslant (2r+1)og(H)$. ■

The following theorem is a generalization of Theorem C and Lemma 3(ii) of [26].

Theorem 1. *Let G and H be two graphs. Also, assume that $\frac{2r+1}{2s+1} < og(G)$ and $2s+1 < og(H^{-\frac{1}{2r+1}})$. We have $G^{\frac{2r+1}{2s+1}} \longrightarrow H$ if and only if $G \longrightarrow H^{-\frac{2s+1}{2r+1}}$.*

Proof. Assume that $G^{\frac{2r+1}{2s+1}} \longrightarrow H$. In view of Theorem C, one has $G^{\frac{1}{2s+1}} \longrightarrow H^{-\frac{1}{2r+1}}$, consequently, $G \longrightarrow (G^{\frac{1}{2s+1}})^{2s+1} \longrightarrow (H^{-\frac{1}{2r+1}})^{2s+1}$. Conversely, suppose that $G \longrightarrow H^{-\frac{2s+1}{2r+1}}$. Considering Lemma 2, we have $G^{\frac{1}{2s+1}} \longrightarrow H^{-\frac{1}{2r+1}}$. Now, in view of Theorem C, one can conclude that $G^{\frac{2r+1}{2s+1}} \longrightarrow H$. ■

Although, we do not know the exact value of $og(H^{-\frac{1}{2r+1}})$, we specify the odd girth of $K_{\frac{n}{d}}^{-\frac{1}{2r+1}}$ in Corollary 1. Moreover, we introduce another lower bound for $og(H^{-\frac{1}{2r+1}})$ in Corollary 2 in terms of the circular chromatic number.

Lemma 4. *Let G be a non-bipartite graph. For any non-negative integer r we have*

$$G^{-\frac{2r+1}{2r+1}} \longleftrightarrow G.$$

Proof. First, note that $G^{-\frac{1}{2r+1}} \longrightarrow G^{-\frac{1}{2r+1}}$. Hence, in view of Theorem 1, we have $(G^{-\frac{1}{2r+1}})^{2r+1} \longrightarrow G$. Next, $G^{\frac{2r+1}{2r+1}} \longrightarrow G$. Considering Theorem 1, we have $G^{\frac{1}{2r+1}} \longrightarrow G^{-\frac{1}{2r+1}}$. Thus, $G \longrightarrow (G^{\frac{1}{2r+1}})^{2r+1} \longrightarrow (G^{-\frac{1}{2r+1}})^{2r+1}$, as required. ■

The graphs $(G^{2r+1})^{-\frac{1}{2r+1}}$ and G are not homomorphically equivalent in general. For instance, $(C_5^3)^{-\frac{1}{3}} = K_5^{-\frac{1}{3}}$ is not homomorphically equivalent to C_5. In fact, $\chi(K_5^{-\frac{1}{3}}) = \chi(H(5,1,2)) = 5$, while $\chi(C_5) = 3$. Also, in view of Lemma 4, Theorem A, and Theorem C, one can see that for given positive integers k, m, and n where $m > 2n$, the helical graph $H(m,n,k)$ and the graph $\mathrm{KG}(m,n)^{-\frac{1}{2k-1}}$ are homomorphically equivalent. Although, if $k \geqslant 2$ and $n \geqslant 2$, then the number of vertices of $H(m,n,k)$ is less than that of $\mathrm{KG}(m,n)^{-\frac{1}{2k-1}}$.

Lemma 5. *Let G be a non-bipartite graph. Also, assume that p and q are positive odd rational numbers and t is a non-negative integer.*

a) If $p(2t+1) < og(G)$, then $G^{p(2t+1)} \longleftrightarrow (G^p)^{2t+1}$.

b) If $p < og(G)$ and $pq < og(G)$, then $(G^p)^q \longrightarrow G^{pq}$.

Proof. Part (a) follows by a simple argument. Assume that $q = \frac{2r+1}{2s+1}$. To prove Part (b), note that

$$\begin{array}{lll} & G^p \longrightarrow G^{\frac{p(2s+1)}{2s+1}} & \text{(by Lemma 1(a))} \\ \Rightarrow & G^p \longrightarrow (G^{\frac{p}{2s+1}})^{2s+1} & \text{(by Lemma 5(a))} \\ \Rightarrow & (G^p)^{\frac{1}{2s+1}} \longrightarrow G^{\frac{p}{2s+1}} & \text{(by Lemma 2)} \\ \Rightarrow & ((G^p)^{\frac{1}{2s+1}})^{2r+1} \longrightarrow (G^{\frac{p}{2s+1}})^{2r+1} & \text{(by Lemma A)} \\ \Rightarrow & (G^p)^{\frac{2r+1}{2s+1}} \longrightarrow G^{\frac{p(2r+1)}{2s+1}} & \text{(by Lemma 5(a))} \\ \Rightarrow & (G^p)^q \longrightarrow G^{pq}. & \end{array}$$

■

An important observation is that the circular complete graph $K_{\frac{2n+1}{n-t}}$ is isomorphic to C_{2n+1}^{2t+1}. This allows us to investigate some coloring properties of circular complete graph powers. The next lemma follows by a simple discussion.

Lemma 6. *Let n and t be non-negative integers where $n > t$. Then $C_{2n+1}^{2t+1} \cong K_{\frac{2n+1}{n-t}}$.*

Now, we are ready to specify the odd girth of $K_{\frac{n}{d}}^{-\frac{1}{2r+1}}$.

Corollary 1. *Let n, d, and r be positive integers where $n > 2d$. The odd girth of $K_{\frac{n}{d}}^{-\frac{1}{2r+1}}$ is equal to $2r+1+2\lceil\frac{2r+1}{\frac{n}{d}-2}\rceil$.*

Proof. Assume that $og(K_{\frac{n}{d}}^{-\frac{1}{2r+1}}) = 2l+1 \geqslant 2r+3$. Then

$$\begin{array}{llll} C_{2l+1} \longrightarrow K_{\frac{n}{d}}^{-\frac{1}{2r+1}} & \Longleftrightarrow & C_{2l+1}^{2r+1} \longrightarrow K_{\frac{n}{d}} & \text{(by Theorem 1)} \\ & \Longleftrightarrow & K_{\frac{2l+1}{l-r}} \longrightarrow K_{\frac{n}{d}} & \text{(by Lemma 6)} \\ & \Longleftrightarrow & \frac{2l+1}{l-r} \leqslant \frac{n}{d} & \\ & \Longleftrightarrow & 2l+1 \geqslant 2r+1+2\lceil\frac{2r+1}{\frac{n}{d}-2}\rceil. & \end{array}$$

Thus,

$$og(K_{\frac{n}{d}}^{-\frac{1}{2r+1}}) = 2r+1+2\lceil\frac{2r+1}{\frac{n}{d}-2}\rceil.$$

■

The following corollary is a consequence of Lemma 4, Theorem 1, and the aforementioned corollary.

Corollary 2. *Let G be a non-bipartite graph and r be a positive integer. Then we have $og(G^{-\frac{1}{2r+1}}) \geqslant 2r+1+2\lceil\frac{2r+1}{\chi_c(G)-2}\rceil$.*

2.2 Density-Type Results

An important property of the family of circular complete graphs is that $K_{\frac{r}{s}} < K_{\frac{p}{q}}$ if and only if $\frac{r}{s} < \frac{p}{q}$. Fortunately, for a given non-bipartite graph G, we have a similar property for the family of fractional powers of G.

Theorem 2. *Let G be a non-bipartite graph and $0 < \frac{2r+1}{2s+1} < \frac{2p+1}{2q+1} < og(G)$. Then*

$$G^{\frac{2r+1}{2s+1}} < G^{\frac{2p+1}{2q+1}}.$$

Proof. We first show that if $1 < \frac{2r+1}{2s+1} < og(G)$, then $G < G^{\frac{2r+1}{2s+1}}$. We know that $G \longrightarrow G^{\frac{2r+1}{2s+1}}$. Hence, it is sufficient to show that there is no homomorphism from $G^{\frac{2r+1}{2s+1}}$ to G. First, we prove that if G is a core, then the statement is true. On the contrary, suppose that there is a homomorphism from $G^{\frac{2r+1}{2s+1}}$ to G. Since, G is a core and a subgraph of $G^{\frac{2r+1}{2s+1}}$, this homomorphism provides an isomorphism between two copies of G. For any edge $e = uv \in E(G)$, the vertex $(uv)_1 \in V(G^{\frac{2r+1}{2s+1}})$ (resp. $(vu)_1 \in V(G^{\frac{2r+1}{2s+1}})$) is adjacent to all the neighborhoods of the vertex u (resp. v) in $G \subseteq G^{\frac{2r+1}{2s+1}}$ (as a subgraph of $G^{\frac{2r+1}{2s+1}}$). The graph G is a core, therefore, the image of $(uv)_1$ (resp. $(vu)_1$) should be the same as u (resp. v). By induction, one can show that the image of $(uv)_k$ (resp. $(vu)_k$) should be the same as u (resp. v) whenever $1 \leqslant k \leqslant s$. Note that, since G is a non-bipartite graph, it contains a triangle or an induced path of length three. Assume that G contains a triangle with vertex set $\{u, v, w\}$. Consider two vertices $(uv)_s$ and $(uw)_s$. It was shown that images of $(uv)_s$ and $(uw)_s$ should be u. Also, $1 < \frac{2r+1}{2s+1}$, consequently, $(uv)_s$ and $(uw)_s$ are adjacent which is a contradiction. Similarly, if G contains an induced path of length three, we get a contradiction.

Now, suppose that G is an arbitrary non-bipartite graph. It is well known that G contains a core, say H, as an induced subgraph. On the contrary, suppose that $G^{\frac{2r+1}{2s+1}} \longrightarrow G$. Then we have $H^{\frac{2r+1}{2s+1}} \longrightarrow G^{\frac{2r+1}{2s+1}} \longrightarrow G \longrightarrow H$, which is a contradiction. Consequently, if $1 < \frac{2r+1}{2s+1} < og(G)$, then $G < G^{\frac{2r+1}{2s+1}}$.

Now, it is easy to verify that

$$G^{\frac{2r+1}{2s+1}} \longleftrightarrow G^{\frac{(2r+1)(2q+1)}{(2s+1)(2q+1)}} \quad \text{and} \quad G^{\frac{2p+1}{2q+1}} \longleftrightarrow G^{\frac{(2p+1)(2s+1)}{(2q+1)(2s+1)}}.$$

On the other hand, we have $\frac{2r+1}{2s+1} < \frac{2p+1}{2q+1}$, hence, $G^{\frac{(2r+1)(2q+1)}{(2s+1)(2q+1)}} \longrightarrow G^{\frac{(2p+1)(2s+1)}{(2q+1)(2s+1)}}$. It remains to show that the inequality is strict. On the contrary, assume that $G^{\frac{2p+1}{2q+1}} \longrightarrow G^{\frac{2r+1}{2s+1}}$. Then, in view of Lemma 5(b), we have

$$(G^{\frac{2p+1}{2q+1}})^{\frac{(2s+1)(2p+1)}{(2r+1)(2q+1)}} \longrightarrow (G^{\frac{2r+1}{2s+1}})^{\frac{(2s+1)(2p+1)}{(2r+1)(2q+1)}} \longrightarrow G^{\frac{2p+1}{2q+1}}.$$

Note that $\frac{(2s+1)(2p+1)}{(2r+1)(2q+1)} > 1$ which is a contradiction. ■

A similar result can be obtained for negative powers as follows.

Theorem 3. *Let G be a non-bipartite graph, $0 < \frac{2r+1}{2s+1} < \frac{2p+1}{2q+1}$, and $2p+1 < og(G^{-\frac{1}{2q+1}})$. Then*

$$G^{-\frac{2r+1}{2s+1}} < G^{-\frac{2p+1}{2q+1}}.$$

Proof. In view of Theorem 1, one can conclude that $(G^{-\frac{1}{2s+1}})^{-\frac{1}{2q+1}}$ is homomorphically equivalent to $G^{-\frac{1}{(2s+1)(2q+1)}}$. Subsequently,

$$\begin{array}{rcll} G^{-\frac{2r+1}{2s+1}} & \longleftrightarrow & (((G^{-\frac{1}{2s+1}})^{-\frac{1}{2q+1}})^{2q+1})^{2r+1} & \text{(by Lemma 4)} \\ & \longleftrightarrow & ((G^{-\frac{1}{(2s+1)(2q+1)}})^{2q+1})^{2r+1} & \\ & \longleftrightarrow & G^{-\frac{(2r+1)(2q+1)}{(2s+1)(2q+1)}} & \text{(by Lemma 5(a))} \\ & < & G^{-\frac{(2p+1)(2s+1)}{(2q+1)(2s+1)}} & \text{(by Theorem 2)} \\ & \longleftrightarrow & (((G^{-\frac{1}{2q+1}})^{-\frac{1}{2s+1}})^{2s+1})^{2p+1} & \\ & \longleftrightarrow & G^{-\frac{2p+1}{2q+1}}. & \text{(by Lemma 4)} \end{array}$$

■

3 Power Thickness

Theorem B shows that the chromatic number of graph powers can be used to investigate the existence of graph homomorphisms into odd cycles, and this is our motivation for the following definition.

Definition 2. Assume that G is a non-bipartite graph. Also, let $i \geqslant -\chi(G)+3$ be an integer. The *ith power thickness* of G is defined as follows.

$$\theta_i(G) \stackrel{\text{def}}{=} \sup\{\frac{2r+1}{2s+1} | \chi(G^{\frac{2r+1}{2s+1}}) \leqslant \chi(G)+i, \frac{2r+1}{2s+1} < og(G)\}.$$

For simplicity, when i = 0, the parameter is called the power thickness of G and is denoted by $\theta(G)$. ♠

Note that, in view of Theorem 2, if G is a non-bipartite graph and $0 < \frac{2r+1}{2s+1} < \frac{2p+1}{2q+1} < og(G)$, then $\chi(G^{\frac{2r+1}{2s+1}}) \leqslant \chi(G^{\frac{2p+1}{2q+1}})$. Consequently, $\theta_i(G) > \frac{2r+1}{2s+1}$ implies that $\chi(G^{\frac{2r+1}{2s+1}}) \leqslant \chi(G)+i$.

As an example, one can see that $\theta(C_{2n+1}) = \frac{2n+1}{3}$. To see this, note that $C_{2n+1}^{\frac{2r+1}{2s+1}}$ and $C_{(2n+1)(2s+1)}^{2r+1}$ are isomorphic. Now, by considering Lemma 6 we have $\theta(C_{2n+1}) = \frac{2n+1}{3}$.

Lemma 7. *Let G and H be two non-bipartite graphs with $\chi(G) = \chi(H) - j$, $j \geqslant 0$. If $G \longrightarrow H$ and $i+j \geqslant -\chi(G)+3$, then*

$$\theta_{i+j}(G) \geqslant \theta_i(H).$$

Proof. Consider a rational number $\frac{2r+1}{2s+1} < og(H)$ for which $\chi(H^{\frac{2r+1}{2s+1}}) \leqslant \chi(H) + i$. We know that $og(G) \geqslant og(H)$ since $G \longrightarrow H$. Hence, $\frac{2r+1}{2s+1} < og(G)$ and $G^{\frac{2r+1}{2s+1}} \longrightarrow H^{\frac{2r+1}{2s+1}}$ which implies that $\chi(G^{\frac{2r+1}{2s+1}}) \leqslant \chi(H) + i = \chi(G) + i + j$. ■

Hereafter, we introduce some results to compute the power thickness of some graphs. Now, we compute the power thickness of some helical graphs.

Theorem 4. *Let k, l, and m be positive integers where $m \geqslant 3$ and $\frac{2l-1}{2k-1} \leqslant 1$. Then*

$$\theta(H(m,1,k)^{2l-1}) = \frac{2k-1}{2l-1}.$$

Proof. In view of Lemma 5(b) and Theorem A, we have $(H(m,1,k)^{2l-1})^{\frac{2k-1}{2l-1}} \longrightarrow H(m,1,k)^{2k-1} \longrightarrow K_m$, therefore, $\theta(H(m,1,k)^{2l-1}) \geqslant \frac{2k-1}{2l-1}$. Suppose, on the contrary, that $\theta(H(m,1,k)^{2l-1}) = t > \frac{2k-1}{2l-1}$. Choose a rational number $1 < \frac{2r+1}{2s+1}$ such that $1 < \frac{(2r+1)(2k-1)}{(2s+1)(2l-1)} < t$. Set $G \stackrel{\text{def}}{=} (H(m,1,k)^{2l-1})^{\frac{2r+1}{2s+1}}$. In view of Lemma 5(b) and definition of power thickness, one has $\chi(G^{\frac{2k-1}{2l-1}}) \leqslant m$. By Theorem 1, one has $G \longrightarrow K_m^{-\frac{2l-1}{2k-1}} \cong H(m,1,k)^{2l-1}$. Thus, $(H(m,1,k)^{2l-1})^{\frac{2r+1}{2s+1}} \longrightarrow H(m,1,k)^{2l-1}$ which contradicts Theorem 2, as claimed. ■

The next definition provides a sufficient condition for the graphs with $\theta(G) = 1$.

Definition 3. Let G be a graph with chromatic number k. G is called a colorful graph if for any proper k-coloring c of G, there exists an induced subgraph H of G such that for any vertex v of H, all colors appear in the closed neighborhood of v, i.e., $c(N[v]) = \{1, 2, \ldots, k\}$. ♠

Theorem 5. *For any non-bipartite colorful graph G, we have $\theta(G) = 1$.*

Proof. On the contrary, suppose that $\theta(G) > 1$. Choose a rational number $1 < \frac{2r+1}{2s+1} < \min\{3, \theta(G)\}$. By definition, $\chi(G^{\frac{2r+1}{2s+1}}) = \chi(G) = k$. Consider a proper k-coloring of the graph $G^{\frac{2r+1}{2s+1}}$. Since, G is a colorful graph and an induced subgraph of $G^{\frac{2r+1}{2s+1}}$, there exists an induced subgraph of $G^{\frac{2r+1}{2s+1}}$, denoted by H, such that for any vertex $v \in V(H)$, all colors appear in the closed neighborhood of v. For any edge $e = uv \in E(H)$, the vertex $(uv)_1$ (resp. $(vu)_1$) is adjacent to all the neighborhoods of the vertex u (resp. v) in H. Therefore, the color of $(uv)_1$ (resp. $(vu)_1$) should be the same as u (resp. v). By induction, one can show that the color of $(uv)_k$ (resp. $(vu)_k$) should be the same as u (resp. v) provided that $uv \in E(H)$. In view of coloring property of H, it should contain a triangle or an induced path of length three whose end vertices have the same color. Assume that H contains an induced path with vertex set $\{u, v, w, x\}$ and edge set $\{uv, vw, wx\}$ such that u and x have the same color. Consider two vertices $(uv)_s$ and $(xw)_s$. It was shown that colors of $(uv)_s$ and $(xw)_s$ should be the same as u and x, i.e, they have the same color. On the other hand, $1 < \frac{2r+1}{2s+1}$, consequently, $(uv)_s$ and $(xw)_s$ are adjacent which is a contradiction. Similarly, if H contains a triangle, we get a contradiction. ■

We know that any uniquely colorable graph is a colorful graph. Hence, the power thickness of any non-bipartite uniquely colorable graphs is one.

Corollary 3. *Let K_n be complete graph with $n \geqslant 3$ vertices. Then $\theta(K_n) = 1$.*

A less ambitious objective is to find all graphs with power thickness one. Also, we do not know whether any graph with power thickness one is colorful.

Question 1. *Is it true that any graph with power thickness one is a colorful graph?*

4 Circular Coloring

The remainder of this paper is devoted to connection between chromatic number of graph powers and circular coloring. In the next theorem we introduce an equivalent definition for the circular chromatic number of graphs.

Theorem 6. *Let G be a non-bipartite graph with chromatic number $\chi(G)$.*

a) *If $0 < \frac{2r+1}{2s+1} \leqslant \frac{\chi(G)}{3(\chi(G)-2)}$, then $\chi(G^{\frac{2r+1}{2s+1}}) = 3$. Furthermore, $\chi(G) \neq \chi_c(G)$ if and only if there exists a rational number $\frac{2r+1}{2s+1} > \frac{\chi(G)}{3(\chi(G)-2)}$ for which $\chi(G^{\frac{2r+1}{2s+1}}) = 3$.*

b) $\chi_c(G) = \inf\{\frac{2n+1}{n-t} | \chi(G^{\frac{2n+1}{3(2t+1)}}) = 3, n > t > 0\}$.

Proof. Assume that $\chi_c(G) = \frac{p}{q}$. Then

$$\begin{aligned} G^{\frac{2r+1}{2s+1}} \to K_3 &\iff G \longrightarrow K_3^{-\frac{2s+1}{2r+1}} && \text{(by Theorem 1)} \\ &\iff G \longrightarrow C_{6r+3}^{2s+1} \\ &\iff G \longrightarrow K_{\frac{6r+3}{3r+1-s}} && \text{(by Lemma 6)} \\ &\iff \frac{p}{q} \leqslant \frac{6r+3}{3r+1-s} \\ &\iff \frac{2r+1}{2s+1} \leqslant \frac{p}{3p-6q} \\ &\iff \frac{2r+1}{2s+1} \leqslant \frac{\chi_c(G)}{3(\chi_c(G)-2)}. \end{aligned}$$

One can see that $\frac{\chi(G)}{3(\chi(G)-2)} \leqslant \frac{\chi_c(G)}{3(\chi_c(G)-2)}$ and the equality holds if and only if $\chi(G) = \chi_c(G)$. Hence, Part (a) follows. To prove Part (b), we have

$$\begin{aligned} G \longrightarrow K_{\frac{2n+1}{n-t}} &\iff G \longrightarrow C_{2n+1}^{2t+1} && \text{(by Lemma 6)} \\ &\iff G^{\frac{1}{2t+1}} \longrightarrow C_{2n+1} && \text{(by Lemma 2)} \\ &\iff \chi(G^{\frac{2n+1}{3(2t+1)}}) = 3. && \text{(by Theorem B)} \end{aligned}$$

■

By the proof of Theorem 6(a), one can specify $\theta_{3-\chi(G)}(G)$ in terms of the circular chromatic number of G as follows.

Corollary 4. *Let G be a non-bipartite graph. Then*

$$\theta_{3-\chi(G)}(G) = \frac{\chi_c(G)}{3(\chi_c(G)-2)}.$$

In view of the above corollary, one can see that for any rational number $2 < \frac{p}{q} \leqslant 3$ we have $\theta(K_{\frac{p}{q}}) = \frac{p}{3p-6q}$. Specially, if $r \geqslant 1$ is a rational number where $r = \frac{n}{d}$, then $\theta(K_{\frac{6n}{3n-d}}) = r$.

Corollary 5. *For any rational number $r \geqslant 1$, there exists a graph G with $\theta(G) = r$.*

In the next theorem we show that the power thickness of the circular complete graph $K_{\frac{p}{q}}$ is greater than one provided that $q \nmid p$.

Theorem 7. *For any rational number $\frac{p}{q} > 2$ where $q \nmid p$ we have*

$$\theta(K_{\frac{p}{q}}) > 1.$$

Proof. Set $m \stackrel{\text{def}}{=} \lceil \frac{p}{q} \rceil$. Choose a positive integer d such that $\frac{p}{q} < \frac{2dm-1}{2d} < m$. We know that $K_{\frac{p}{q}} \longrightarrow K_{\frac{2dm-1}{2d}}$, hence, it is sufficient to show that there exists a positive integer s such that $(K_{\frac{2dm-1}{2d}})^{\frac{2s+1}{2s-1}} \longrightarrow K_m$.

Set $n \stackrel{\text{def}}{=} dm-1$, $t \stackrel{\text{def}}{=} d(m-2)-1$. In view of Lemma 6 and Lemma 5(b), we have

$$(K_{\frac{2dm-1}{2d}})^{\frac{2s+1}{2s-1}} \cong (C_{2n+1}^{2t+1})^{\frac{2s+1}{2s-1}} \longrightarrow (C_{2n+1})^{\frac{(2t+1)(2s+1)}{2s-1}} \cong (C_{(2n+1)(2s-1)})^{(2t+1)(2s+1)}.$$

On the other hand, Lemma 6 confirms that

$$\chi((C_{(2n+1)(2s-1)})^{(2t+1)(2s+1)}) = \lceil \frac{(2n+1)(2s-1)}{(n-t)(2s+1)-2n-1} \rceil.$$

Therefore,

$$\chi((K_{\frac{2dm-1}{2d}})^{\frac{2s+1}{2s-1}}) \leqslant \lceil \frac{(2dm-1)(2s-1)}{2d(2s+1)-2dm+1} \rceil.$$

It is easily to see that if s is sufficiently large, then $\chi((K_{\frac{2dm-1}{2d}})^{\frac{2s+1}{2s-1}}) = m$. In other words, $\theta(K_{\frac{p}{q}}) \geqslant \frac{2s+1}{2s-1} > 1$. ■

The aforementioned theorem provides a sufficient condition for the equality of the chromatic number and the circular chromatic number of graphs. In fact, if we show that power thickness of a graph G is equal to one, then $\chi(G) = \chi_c(G)$.

It is well known that $\chi_c(G) = \chi(G)$ if and only if for any proper coloring $f : V(G) \to \{1, 2, \ldots, \chi(G)\}$ there exists a cycle C_n, as a subgraph of G, with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the edge set $\{v_i v_{i+1} | \ 1 \leqslant i \leqslant n \pmod{n}\}$ such that $f(v_{i+1}) - f(v_i) = 1 \pmod{\chi(G)}$ for any $1 \leqslant i \leqslant n$. Consequently, in view of Corollary 4, the following corollary holds.

Corollary 6. *Let G be a graph with chromatic number 3. Then the following conditions are equivalent.*

a) $\theta(G) = 1$.

b) $\chi_c(G) = 3$.

c) *G is a colorful graph.*

The question of whether the circular chromatic number and the chromatic number of the Kneser graphs and the Schrijver graphs are equal has received attention and has been studied in several papers [6, 12, 15, 17, 19, 22]. Johnson, Holroyd, and Stahl [15] proved that $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n))$ if $m \leqslant 2n+2$ or $n = 2$. This shows that $\mathrm{KG}(2n+1,n)$ is a colorful graph.

Corollary 7. *Let n be a positive integer. Then $\theta(\mathrm{KG}(2n+1,n)) = 1$.*

They also conjectured that the equality holds for all Kneser graphs.

Conjecture 1. [15] *For all $m \geqslant 2n+1$, $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n))$.*

Some coloring properties of Kneser graphs have been investigated in [22, 23, 24, 25]. It is shown in [24, 25] that if c is a proper coloring of the Kneser graph $\mathrm{KG}(m,n)$ with t colors, then there exists a multicolored complete bipartite graph $K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$ with $r \stackrel{\text{def}}{=} \chi(\mathrm{KG}(m,n))$ such that r different colors occur alternating on the two sides of the bipartite graph with respect to their natural order. This result has been generalized for general Kneser graphs in [23]. It seems that Kneser graphs are colorful graphs.

Question 2. *Let m and n be positive integers where $m \geqslant 2n$. Is the Kneser graph $\mathrm{KG}(m,n)$ a colorful graph? Is it true that $\theta(\mathrm{KG}(m,n)) = 1$?*

Theorem A shows that $\theta(H(m,n,k)) \geqslant 2k-1$ whenever $m \geqslant 2n+1$. Another problem which may be of interest is the following.

Question 3. *Let m and n be positive integers where $m \geqslant 2n+1$. Is it true that $\theta(H(m,n,k)) = 2k-1$?*

Odd cycles are symmetric and they have sparse structure. Hence, it can be useful if the circular chromatic number can be expressed as homomorphism to odd cycles. Now, let G be a non-bipartite graph and t be a positive integer. Define,

$$f(G, 2t+1) \stackrel{\text{def}}{=} \max\{2n+1 | G^{\frac{1}{2t+1}} \longrightarrow C_{2n+1}\}.$$

One can see that $3 \leqslant f(G, 2t+1) \leqslant (2t+1) \times og(G)$. In view of the proof of Theorem 6(b), one can compute $f(G, 2t+1)$ in terms of the circular chromatic number of graph G and vice versa. In fact, we have

$$\chi_c(G) = \inf\{\frac{2n+1}{n-t} | G^{\frac{1}{2t+1}} \longrightarrow C_{2n+1}, n > t > 0\},$$

which leads us to the following theorem.

Theorem 8. *Let G be a non-bipartite graph and t be a positive integer. Then the maximum value of $2n+1$, for which $\mathrm{Hom}(G^{\frac{1}{2t+1}}, C_{2n+1}) \neq \varnothing$, is equal to*

$$2t+1+2\lfloor\frac{2t+1}{\chi_c(G)-2}\rfloor.$$

Also, note that there exists a necessary condition for the existence of homomorphism to symmetric graphs in terms of the eigenvalue of the Laplacian matrix. The next theorem can be useful in studying the circular chromatic number of graphs.

Theorem D. [4, 5] *Let G be a graph with $|V(G)| = m$. If $\sigma \in \mathrm{Hom}(G, C_{2n+1})$, then*

$$\lambda^{G}_{m} \geqslant \frac{2|E(G)|}{2m}\, \lambda^{C_{2n+1}}_{2n+1},$$

where λ^{G}_{m} and $\lambda^{C_{2n+1}}_{2n+1}$ stand for the largest eigenvalues of Laplacian matrices of G and C_{2n+1}, respectively.

5 Concluding Remarks

It is instructive to add some notes on the whole setup we have introduced so far. It is evident from our approach that any kind of information about power thickness of a graph has important consequences on graph homomorphism problem. There are several questions about power thickness which remain open. In fact, we do not know whether the power thickness is always a rational number.

Question 4. *Let G be a non-bipartite graph and $i \geqslant -\chi(G)+3$ be an integer. Is $\theta_i(G)$ a rational number? Also, for which real numbers $r > 1$ there exists a graph G with $\theta_i(G) = r$?*

We know that if $\chi(G) = 3$ then $\theta_{-1}(G) = 0$. Also, in view of Corollary 4, one can see that if $\chi(G) = 4$ then $\theta_{-1}(G) < 1$.

Question 5. *Let G be a non-bipartite graph. Is it true that $\theta_{-1}(G) < 1$?*

Finally, the following parameter can be studied as a natural generalization of power thickness and as a measure for graph homomorphism problem.

Definition 4. Let G and H be two graphs. Set

$$\theta_{H}(G) \stackrel{\text{def}}{=} \sup\{\frac{2r+1}{2s+1} | G^{\frac{2r+1}{2s+1}} \longrightarrow H,\ \frac{2r+1}{2s+1} < og(G)\}.$$

♠

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