Classification of Generalized Hadamard Matrices $H(6,3)$ and Quaternary Hermitian Self-Dual Codes of Length 18

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Abstract

All generalized Hadamard matrices of order 18 over a group of order 3, $H(6,3)$, are enumerated in two different ways: once, as class regular symmetric $(6,3)$-nets, or symmetric transversal designs on 54 points and 54 blocks with a group of order 3 acting semi-regularly on points and blocks, and secondly, as collections of full weight vectors in quaternary Hermitian self-dual codes of length 18. The second enumeration is based on the classification of Hermitian self-dual $[18,9]$ codes over $GF(4)$, completed in this paper. It is shown that up to monomial equivalence, there are 85 generalized Hadamard matrices $H(6,3)$, and 245 inequivalent Hermitian self-dual codes of length 18 over $GF(4)$.

1 Introduction

A *generalized Hadamard matrix* $H(\mu, g) = (h_{ij})$ of order $n = g\mu$ over a multiplicative group G of order g is a $g\mu \times g\mu$ matrix with entries from G with the property that for

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every i, j, $1 \le i < j \le g\mu$, each of the multi-sets $\{h_{is}h_{js}^{-1} \mid 1 \le s \le g\mu\}$ contains every element of G exactly μ times. It is known [12, Theorem 2.2] that if G is abelian then $H(\mu, g)^T$ is also a generalized Hadamard matrix, where $H(\mu, g)^T$ denotes the transpose of $H(\mu, g)$ (see also [5, Theorem 4.11]). This result does not generalize to non-abelian groups, as shown by Craigen and de Launey [7].

If G is an additive group and the products $h_{is}h_{js}^{-1}$ are replaced by differences $h_{is} - h_{js}$, the resulting matrices are known as *difference matrices* [2], or *difference schemes* [10]. A generalized Hadamard matrix over the multiplicative group of order two, $G = \{1, -1\}$, is an ordinary Hadamard matrix.

Permuting rows or columns, as well as multiplying rows or columns of a given generalized Hadamard matrix H over a group G with group elements changes H into another generalized Hadamard matrix. Two generalized Hadamard matrices H', H'' of order n over a group G are called *monomially equivalent* if $H'' = PH'Q$ for some monomial matrices P, Q of order n with nonzero entries from G.

All generalized Hadamard matrices over a group of order 2, that is, ordinary Hadamard matrices, have been classified up to (monomial) equivalence for all orders up to $n = 28$ [13], and all generalized Hadamard matrices over a group of order 4 (cyclic or elementary abelian) have been classified up to monomial equivalence for all orders up to $n = 16$ [9] (see also [8]).

We consider generalized Hadamard matrices over a group of order 3 in this paper. It is easy to verify that generalized Hadamard matrices $H(1, 3)$ of order 3, and $H(2, 3)$ of order 6, exist and are unique up to monomial equivalence. There are two matrices $H(3, 3)$ of order 9 [16], and one $H(4, 3)$ of order 12 up to monomial equivalence [17]. It is known [10, Theorem 6.65] that an $H(5, 3)$ of order 15 does not exist. Up to monomial equivalence, at least 11 $H(6, 3)$ of order 18 were previously known [1].

In this paper, we enumerate all generalized Hadamard matrices $H(6, 3)$ of order 18, up to monomial equivalence. We present two different enumerations, one based on combinatorial designs known as symmetric nets or transversal designs (Section 2), and a second enumeration based on the classification of Hermitian self-dual codes of length 18 over $GF(4)$ completed in Section 4.

2 Symmetric nets, transversal designs and generalized Hadamard matrices $H(6, 3)$

A *symmetric* (μ, g)-*net* is a 1-$(g^2\mu, g\mu, g\mu)$ design $\mathcal{D}$ such that both $\mathcal{D}$ and its dual design $\mathcal{D}^*$ are affine resolvable [2]: the $g^2\mu$ points of $\mathcal{D}$ are partitioned into $g\mu$ parallel classes, or *groups*, each containing g points, so that any two points which belong to the same class do not occur together in any block, while any two points which belong to different classes occur together in exactly μ blocks. Similarly, the blocks are partitioned into $g\mu$ parallel classes, each consisting of g pairwise disjoint blocks, and any two blocks which belong to different parallel classes share exactly μ points. A symmetric (μ, g)-net is also known as a *symmetric transversal design*, and denoted by $STD_\mu(g)$, or $TD_\mu(g\mu, g)$ [2], or $STD_\mu[g\mu; g]$

[17]. A symmetric (μ, g)-net is *class-regular* if it admits a group of automorphisms G of order g (called group of *bitranslations*) that acts transitively (and hence regularly) on every point and block parallel class.

Every generalized Hadamard matrix $H(\mu, g)$ over a group G of order g determines a class-regular symmetric (μ, g)-net with a group of bitranslations isomorphic to G, and conversely, every class-regular (μ, g)-net with a group of bitranslations G gives rise to a generalized Hadamard matrix $H(\mu, g)$ [2]. The $g^2\mu \times g^2\mu$ $(0, 1)$-incidence matrix of a class-regular symmetric (μ, g)-net is obtained from a given generalized Hadamard matrix $H(\mu, g) = (h_{ij})$ over a group G of order g by replacing each entry h_{ij} of $H(\mu, g)$ with a $g \times g$ permutation matrix representing $h_{ij} \in G$. This correspondence relates the task of enumerating generalized Hadamard matrices over a group of order g to the enumeration of 1-$(g^2\mu, g\mu, g\mu)$ designs with incidence matrices composed of $g \times g$ permutation submatrices. This approach was used in [9] for the enumeration of all nonisomorphic class-regular symmetric $(4, 4)$-nets over a group of order 4 and generalized Hadamard matrices $H(4, 4)$. In this paper, we use the same approach to enumerate all pairwise nonisomorphic class-regular $(6, 3)$-nets, or equivalently, symmetric transversal designs $STD_6(3)$ with a group of order 3 acting semiregularly on point and block parallel classes, and consequently, all generalized Hadamard matrices $H(6, 3)$. As in [9], the block design exploration package BDX [14], developed by Larry Thiel, was used for the enumeration.

The results of this computation can be formulated as follows.

Theorem 1. *Up to isomorphism, there are exactly* 53 *class-regular symmetric* $(6, 3)$-*nets, or equivalently,* 53 *symmetric transversal designs* $STD_6(3)$ *with a group of order* 3 *acting semiregularly on point and block parallel classes.*

The information about the 53 $(6, 3)$-nets $\mathcal{D}_i$ $(i = 1, 2, \ldots, 53)$ are listed in Table 1. In the table, $\#\mathrm{Aut}$ gives the size of the automorphism group of $\mathcal{D}_i$. The column $\mathcal{D}_i^*$ gives the number j, where $\mathcal{D}_i^*$ is isomorphic to $\mathcal{D}_j$. Incidence matrices of the 53 $(6, 3)$-nets are available at http://www.math.mtu.edu/~tonchev/sol.txt.

We note that 20 nonisomorphic $STD_6(3)$ were found by Akiyama, Ogawa, and Suetake [1]. These twenty $STD_6(3)$ are denoted by $\mathcal{D}(H_i)$ $(i = 1, 2, \ldots, 11)$ and $\mathcal{D}(H_i)^d$ $(i = 1, \ldots, 5, 7, 8, 9, 10)$ in [1, Theorem 7.3]. When $\mathcal{D}_i$ in Table 1 is isomorphic to one of the twenty $STD_6(3)$ in [1], we list the $STD_6(3)$ in the column D_{AOS} of the table.

Any generalized Hadamard matrix $H(6, 3)$ over the group $G = \{1, \omega, \omega^2 \mid \omega^3 = 1\}$ corresponds to the 54×54 $(0, 1)$-incidence matrix of a class-regular symmetric $(6, 3)$-net obtained by replacing $1, \omega$ and ω^2 with 3×3 permutation matrices I, M_3 and M_3^2, respectively, where I is the identity matrix and

$$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We note that permuting rows or columns in $H(6, 3)$ corresponds to permuting parallel classes of points or blocks in the related symmetric net, while multiplying a row or column of $H(6, 3)$ with an element α of G, corresponds to a cyclic shift (if $\alpha = \omega$) or a double

Table 1: Class-regular symmetric $(6,3)$-nets and $H(6,3)$'s

$\mathcal{D}_i$	# Aut	$\mathcal{D}_i^*$	D_{AOS}	$\overline{H(\mathcal{D}_i)}$	$\mathcal{D}_i$	# Aut	$\mathcal{D}_i^*$	D_{AOS}	$\overline{H(\mathcal{D}_i)}$
1	96	1		yes	28	162	37	$\mathcal{D}(H_1)^d$	yes
2	432	43		yes	29	54	22		no
3	864	5		yes	30	54	26		no
4	38880	4	$\mathcal{D}(H_{11})$	yes	31	432	17		no
5	864	3		yes	32	48	15		no
6	1296	19		yes	33	54	27		yes
7	3240	49	$\mathcal{D}(H_{10})$	no	34	162	53	$\mathcal{D}(H_2)$	no
8	144	46		no	35	162	50	$\mathcal{D}(H_4)$	no
9	324	44	$\mathcal{D}(H_5)$	no	36	162	51	$\mathcal{D}(H_3)$	no
10	1296	52	$\mathcal{D}(H_7)$	no	37	162	28	$\mathcal{D}(H_1)$	yes
11	180	45		no	38	1944	14	$\mathcal{D}(H_9)$	yes
12	1296	42	$\mathcal{D}(H_8)$	yes	39	972	39	$\mathcal{D}(H_6)$	yes
13	216	20		yes	40	216	21		no
14	1944	38	$\mathcal{D}(H_9)^d$	yes	41	216	16		no
15	48	32		no	42	1296	12	$\mathcal{D}(H_8)^d$	yes
16	216	41		no	43	432	2		yes
17	432	31		no	44	324	9	$\mathcal{D}(H_5)^d$	no
18	2160	23		yes	45	180	11		no
19	1296	6		yes	46	144	8		no
20	216	13		yes	47	108	24		no
21	216	40		no	48	1080	25		no
22	54	29		no	49	3240	7	$\mathcal{D}(H_{10})^d$	no
23	2160	18		yes	50	162	35	$\mathcal{D}(H_4)^d$	no
24	108	47		no	51	162	36	$\mathcal{D}(H_3)^d$	no
25	1080	48		no	52	1296	10	$\mathcal{D}(H_7)^d$	no
26	54	30		no	53	162	34	$\mathcal{D}(H_2)^d$	no
27	54	33		yes					

cyclic shift (if $\alpha = \omega^2$) of the three points or blocks of the corresponding parallel class in the related symmetric $(6,3)$-net. Thus, monomially equivalent generalized Hadamard matrices $H(6,3)$ correspond to isomorphic symmetric $(6,3)$-nets.

The inverse operation of replacing every element h_{ij} of a generalized Hadamard matrix by its inverse h_{ij}^{-1} also preserves the property of being a generalized Hadamard matrix. That is, a generalized Hadamard matrix is also obtained by replacing I, M_3 and M_3^2 with $1, \omega^2$ and ω, respectively. However, this is not considered a monomial equivalence operation. As a symmetric net, this inverse operation corresponds to replacing M_3 by M_3^2 and vice versa. The inverse operation is achievable by simultaneously interchanging rows 2 and 3 and columns 2 and 3 of the matrices I, M_3 and M_3^2. Thus, by simultaneous interchanging points 2 and 3 and blocks 2 and 3 of every parallel class of points and blocks, the inverse operator is an isomorphism operation of symmetric nets. Since the definition of isomorphic symmetric nets and monomially equivalent generalized Hadamard matrices differs only in the extra inverse operation, at most two generalized Hadamard matrices which are not monomially equivalent can arise from a symmetric net. We note that for

generalized Hadamard matrices over a cyclic group of order q, replacing every entry by its i-th power, where $\gcd(i, q) = 1$, may give a generalized Hadamard matrix which is not monomially equivalent to the original; however, their corresponding symmetric nets are isomorphic.

In order to find the number of generalized Hadamard matrices which are not monomially equivalent, we first convert the 53 nonisomorphic symmetric nets into their corresponding 53 generalized Hadamard matrices. We then create a list of 53 extra matrices by applying the inverse operation. Amongst this list of 106 matrices, we found 85 generalized Hadamard matrices $H(6, 3)$ up to monomial equivalence. As expected, the remaining 21 matrices are monomially equivalent to their "parent" before the inverse operation.

Corollary 2. *Up to monomial equivalence, there are exactly* 85 *generalized Hadamard matrices* $H(6, 3)$.

In Table 1, the column $\overline{H(\mathcal{D}_i)}$ states whether the corresponding generalized Hadamard matrix $H(\mathcal{D}_i)$ is monomially equivalent to the generalized Hadamard matrix $\overline{H(\mathcal{D}_i)}$ obtained by replacing all entries by their inverse. Thus, the set $\{H(\mathcal{D}_i), \overline{H(\mathcal{D}_j)} \mid i \in \Delta, j \in \Delta \setminus \Gamma\}$ gives the 85 generalized Hadamard matrices, where $\Delta = \{1, 2, \ldots, 53\}$ and

$$\Gamma = \{1, 2, 3, 4, 5, 6, 12, 13, 14, 18, 19, 20, 23, 27, 28, 33, 37, 38, 39, 42, 43\}.$$

Concerning the next order, $n = 21$, several examples of $STD_7(3)$ and $H(7, 3)$ are known [1], [18]. Some $STD_7(3)$'s and $H(7, 3)$'s were used in [19] as building blocks for the construction of an infinite class of quasi-residual 2-designs. An estimate based on preliminary computations with BDX suggests that it would take 500 CPU years to enumerate all $STD_7(3)$'s using one computer, or about a year of CPU if a network of 500 computers is employed.

3 Elementary divisors of generalized Hadamard matrices and Hermitian self-dual codes

Let $GF(4) = \{0, 1, \omega, \overline{\omega}\}$ be the finite field of order four, where $\overline{\omega} = \omega^2 = \omega + 1$. Codes over $GF(4)$ are often called quaternary. The *Hermitian inner product* of vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in GF(4)^n$ is defined as

$$x \cdot y = \sum_{i=1}^{n} x_i y_i^2. \tag{1}$$

The *Hermitian dual code* $C^\perp$ of a code C of length n is defined as $C^\perp = \{x \in GF(4)^n \mid x \cdot c = 0 \text{ for all } c \in C\}$. A code C is called *Hermitian self-orthogonal* if $C \subseteq C^\perp$, and *Hermitian self-dual* if $C = C^\perp$. In this section, we show that the rows of any generalized Hadamard matrix $H(6, 3)$ span a Hermitian self-dual code of length 18 and minimum weight $d \geq 4$ (Theorem 5). A consequence of this result is that all $H(6, 3)$'s can be found

as collections of vectors of full weight in Hermitian self-dual codes over $GF(4)$. This motivates us to classify all such codes as the second approach of the enumeration of all $H(6,3)$'s.

Let R be a unique factorization domain, and let p be a prime element of R. For a nonzero element $a \in R$, we denote by $\nu_p(a)$ the largest non-negative integer e such that p^e divides a.

Lemma 3. *Let R be a unique factorization domain. Suppose that the nonzero elements $a, b, c, d \in R$ satisfy $ab = cd$ and $\gcd(a,b) = 1$. Then*

$$\gcd(a,c)\gcd(a,d) = a.$$

Proof. Let p be a prime element of R dividing a. Then p does not divide b, hence

$$\nu_p(a) = \nu_p(ab) = \nu_p(c) + \nu_p(d) \geq \max\{\nu_p(c), \nu_p(d)\}.$$

Thus

$$\nu_p(\gcd(a,c)) = \min\{\nu_p(a), \nu_p(c)\} = \nu_p(c),$$
$$\nu_p(\gcd(a,d)) = \min\{\nu_p(a), \nu_p(d)\} = \nu_p(d),$$

and hence $\nu_p(a) = \nu_p(\gcd(a,c)\gcd(a,d))$. Since p is arbitrary, we obtain the assertion. $\square$

Let $\omega = \frac{-1+\sqrt{-3}}{2} \in \mathbb{C}$, where $\mathbb{C}$ denotes the complex number field. It is well known that $\mathbb{Z}[\omega]$ is a principal ideal domain. Thus we can consider elementary divisors of a matrix over $\mathbb{Z}[\omega]$. Also, $\mathbb{Z}[\omega]$ is a unique factorization domain, and 2 is a prime element. We note that $\mathbb{Z}[\omega]/2\mathbb{Z}[\omega] \cong GF(4)$.

Lemma 4. *Let H be an $n \times n$ matrix with entries in $\{1, \omega, \omega^2\}$, satisfying $H\overline{H}^T = nI$, where $\overline{H}$ denotes the complex conjugation. Let $d_1 | d_2 | \cdots | d_n$ be the elementary divisors of H over the ring $\mathbb{Z}[\omega]$. Then $d_i \overline{d_{n+1-i}}/n$ is a unit in $\mathbb{Z}[\omega]$ for all $i = 1, \ldots, n$.*

Proof. Take $P, Q \in \mathrm{GL}(n, \mathbb{Z}[\omega])$ so that $PHQ = \mathrm{diag}(d_1, \ldots, d_n)$. Since $H\overline{H}^T = nI$, we have

$$\overline{Q}^{-1} H^T \overline{P}^{-1} = n\overline{Q}^{-1}\overline{H}^{-1}\overline{P}^{-1}$$
$$= n\overline{PHQ}^{-1}$$
$$= \mathrm{diag}(n/\overline{d_1}, n/\overline{d_2}, \ldots, n/\overline{d_n}).$$

This implies that $n/\overline{d_n}, n/\overline{d_{n-1}}, \ldots, n/\overline{d_1}$ are also the elementary divisors of H. It follows from the uniqueness of the elementary divisors that $d_i \overline{d_{n+i-i}}/n$ is a unit in $\mathbb{Z}[\omega]$ for all $i = 1, \ldots, n$. $\square$

Theorem 5. *Under the same assumptions as in Lemma 4, assume further that $n \equiv 2$ (mod 4). Then the rows of H span a Hermitian self-dual code over $\mathbb{Z}[\omega]/2\mathbb{Z}[\omega] \cong GF(4)$. This Hermitian self-dual code has minimum weight at least 4.*

Proof. Let C be the code over $\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]$ spanned by the row vectors of H. Since $H\overline{H}^T \equiv 0 \pmod{2\mathbb{Z}[\omega]}$, the code C is Hermitian self-orthogonal (see also [20, Lemma 2]). Let $d_1|d_2|\cdots|d_n$ be the elementary divisors of H. Then

$$
\begin{aligned}
|C| &= |(\mathbb{Z}[\omega]/2\mathbb{Z}[\omega])^n H| \\
&= |(\mathbb{Z}[\omega]/2\mathbb{Z}[\omega])^n \operatorname{diag}(d_1,\ldots,d_n)| \\
&= \prod_{i=1}^{n} |\gcd(2,d_i)\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]| \\
&= \prod_{i=1}^{n} \frac{|\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]|}{|\mathbb{Z}[\omega]/\gcd(2,d_i)\mathbb{Z}[\omega]|} \\
&= \prod_{i=1}^{n} \frac{4}{|\gcd(2,d_i)|^2} \\
&= \prod_{i=1}^{n/2} \frac{4}{|\gcd(2,d_i)|^2} \prod_{i=n/2+1}^{n} \frac{4}{|\gcd(2,d_i)|^2} \\
&= \prod_{i=1}^{n/2} \frac{4}{|\gcd(2,d_i)|^2} \prod_{i=n/2+1}^{n} \frac{4}{|\gcd(2,n/\overline{d_{n+1-i}})|^2} \qquad \text{(by Lemma 4)} \\
&= \prod_{i=1}^{n/2} \frac{4}{|\gcd(2,d_i)|^2} \prod_{i=n/2+1}^{n} \frac{4}{|\gcd(2,n/d_{n+1-i})|^2} \\
&= \prod_{i=1}^{n/2} \frac{4}{|\gcd(2,d_i)|^2} \prod_{i=1}^{n/2} \frac{4}{|\gcd(2,n/d_i)|^2} \\
&= \prod_{i=1}^{n/2} \frac{16}{|\gcd(2,d_i)\gcd(2,n/d_i)|^2} \\
&= 4^{n/2}. \qquad \text{(by Lemma 3 since } n \equiv 2 \pmod 4)
\end{aligned}
$$

Thus, the dimension $\dim C$ is $n/2$ and C is self-dual.

If the dual code $C^\perp$ had minimum weight 2, then there exist two columns of H, one of which is a multiple by $1, \omega$, or $\overline{\omega}$ of the other, in $GF(4)$. But this implies that there exists a column of H which is a multiple by $1, \omega$, or $\overline{\omega}$ in $\mathbb{C}$. This is impossible since H is nonsingular. Hence the dual code $C^\perp$ has minimum weight at least 3. Since C is self-dual and even, C has minimum weight at least 4. $\qquad\square$

4 The classification of quaternary self-dual $[18,9]$ codes

Two codes C and C' over $GF(4)$ are *equivalent* if there is a monomial matrix M over $GF(4)$ such that $C' = CM = \{cM \mid c \in C\}$. A monomial matrix which maps C to itself is called an *automorphism* of C and the set of all automorphisms of C forms the

automorphism group $\mathrm{Aut}(C)$ of C. The number of distinct Hermitian self-dual codes of length n is given [15] by the formula:

$$N(n) = \prod_{i=0}^{n/2-1} (2^{2i+1} + 1). \tag{2}$$

It was shown in [15] that the minimum weight d of a Hermitian self-dual code of length n is bounded by $d \leq 2\lfloor n/6 \rfloor + 2$. A Hermitian self-dual code of length n and minimum weight $d = 2\lfloor n/6 \rfloor + 2$ is called *extremal*. The classification of all Hermitian self-dual codes over $GF(4)$ up to equivalence of length $n \leq 14$ was completed by MacWilliams, Odlyzko, Sloane and Ward [15], and the Hermitian self-dual codes of length 16 were classified by Conway, Pless and Sloane [6]. For example, there are 55 inequivalent Hermitian self-dual codes of length 16. For the next two lengths, 18 and 20, only partial classification was previously known, namely, the extremal Hermitian self-dual $[18, 9, 8]$ and $[20, 10, 8]$ codes were enumerated in [11] and Hermitian self-dual $[18, 9, 6]$ codes were enumerated in [4] under the weak equivalence defined at the end of this subsection.

We first consider decomposable Hermitian self-dual codes. By [15, Theorem 28], any Hermitian self-dual code with minimum weight 2 is decomposable as $C_2 \oplus C_{16}$, where C_2 is the unique Hermitian self-dual code of length 2 and C_{16} is some Hermitian self-dual code of length 16. Hence, there are 55 inequivalent Hermitian self-dual codes with minimum weight 2 [6]. In the notation of Table 4, the following codes are decomposable Hermitian self-dual codes with minimum weight 4:

$$E_8 \oplus E_{10}, E_8 \oplus B_{10}, E_6 \oplus E_{12}, E_6 \oplus C_{12}, E_6 \oplus D_{12}, E_6 \oplus F_{12}, E_6 \oplus 2E_6,$$

and there is no decomposable Hermitian self-dual code with minimum weight $d \geq 6$. In Table 2, the number $\#_{\mathrm{dec}}$ of inequivalent decomposable Hermitian self-dual codes with minimum weight d is given for each admissible value of d.

Table 2: Hermitian self-dual codes of length 18

	$d = 2$	$d = 4$	$d = 6$	$d = 8$	Total
$\#_{\mathrm{dec}}$	55	7	0	0	62
$\#_{\mathrm{indec}}$	0	152	30	1	183
Total	55	159	30	1	245

We now consider indecomposable Hermitian self-dual codes. Two self-dual codes C and C' of length n are called *neighbors* if the dimension of their intersection is $n/2 - 1$. An extremal Hermitian self-dual code S_{18} of length 18 was given in [15] and it is generated by

$$(1, \omega, \overline{\omega}, \omega, \omega, \omega, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, \omega, \omega, \overline{\omega}, \omega) \, 1$$

where the parentheses indicate that all cyclic shifts are to be used. Let $\mathrm{Nei}(C)$ denote the set of inequivalent Hermitian self-dual neighbors with minimum weight $d \geq 4$ of C. We

found that the set $\mathrm{Nei}(S_{18})$ consists of 35 inequivalent Hermitian self-dual codes, one of which is equivalent to S_{18}, 17 codes have minimum weight 6, and 17 codes have minimum weight 4. Within the set of codes

$$\{S_{18}\} \cup \mathrm{Nei}(S_{18}) \cup \mathcal{N} \cup \Big(\bigcup_{C \in \mathcal{N}} \mathrm{Nei}(C)\Big),$$

where $\mathcal{N} = \cup_{C \in \mathrm{Nei}(S_{18})} \mathrm{Nei}(C)$, we found a set $\mathcal{C}_{18}$ of 190 inequivalent Hermitian self-dual codes $C_1, \ldots, C_{190}$ with minimum weight $d \geq 4$ satisfying

$$\sum_{C \in \mathcal{C}_{18} \cup \mathcal{D}_{18}} \frac{3^{18} \cdot 18!}{\# \mathrm{Aut}(C)} = 4251538544610908358733563 = N(18), \tag{3}$$

where $\mathcal{D}_{18}$ denotes the set of the 55 inequivalent Hermitian self-dual codes of length 18 and minimum weight 2. The orders of the automorphism groups of the 245 codes in $\mathcal{C}_{18} \cup \mathcal{D}_{18}$ are listed in Table 3. The mass formula (3) shows that the set $\mathcal{C}_{18} \cup \mathcal{D}_{18}$ of codes contains representatives of all equivalence classes of Hermitian self-dual codes of length 18. Thus, the classification is complete, and Theorem 6 holds.

Theorem 6. *There are* 245 *inequivalent Hermitian self-dual codes of length* 18. *Of these, one is extremal (minimum weight* 8*),* 30 *codes have minimum weight* 6, 159 *codes have minimum weight* 4, *and* 55 *codes have minimum weight* 2.

The software package MAGMA [3] was used in the computations. Generator matrices of all Hermitian self-dual codes of length 18 can be obtained from
http://www.math.is.tohoku.ac.jp/~munemasa/selfdualcodes.htm.

In Table 2, the number $\#_{\mathrm{indec}}$ of indecomposable Hermitian self-dual codes with minimum weight d is given. In Table 4, the number $\#$ of inequivalent Hermitian self-dual codes of length n is given along with references. The largest minimum weight $d_{\max}$ among Hermitian self-dual codes of length n and the number $\#_{\max}$ of inequivalent Hermitian self-dual codes with minimum weight $d_{\max}$ are also listed along with references.

We list in Table 5 eleven Hermitian self-dual codes C_{10}, C_{14}, C_{15}, C_{17}, C_{30}, C_{38}, C_{40}, C_{83}, C_{120}, C_{147} and C_{190} of minimum weight at least 4, which are used in the next subsection. Table 5 lists the dimension $\dim$ of $S_{18} \cap C_i$, vectors $v_1, \ldots, v_{9-\dim}$ such that

$$C_i = \langle S_{18} \cap \langle v_1, \ldots, v_{9-\dim} \rangle^{\perp}, v_1, \ldots, v_{9-\dim} \rangle,$$

the numbers A_4 and A_6 of codewords of weights 4 and 6, and the order $\# \mathrm{Aut}$ of the automorphism group of C_i. By [15, Theorem 13], the weight enumerator of a Hermitian self-dual code of length 18 and minimum weight at least 4 can be written as

$$\begin{aligned}
&1 + A_4 y^4 + A_6 y^6 + (2754 + 27A_4 - 6A_6)y^8 + (18360 - 106A_4 + 15A_6)y^{10} \\
&+ (77112 + 119A_4 - 20A_6)y^{12} + (110160 - 12A_4 + 15A_6)y^{14} \\
&+ (50949 - 51A_4 - 6A_6)y^{16} + (2808 + 22A_4 + A_6)y^{18}.
\end{aligned}$$

Thus, the weight enumerator is uniquely determined by A_4 and A_6.

Table 3: Orders of the automorphism groups

d	$\# \operatorname{Aut}(C)$
2	864, 864, 1152, 1728, 2160, 2304, 2592, 6048, 6912, 6912, 10368, 13824, 13824, 17280, 20736, 41472, 82944, 82944, 82944, 82944, 103680, 110592, 124416, 235872, 248832, 311040, 331776, 497664, 580608, 829440, 995328, 995328, 1327104, 2073600, 2177280, 2488320, 3110400, 4478976, 12192768, 13436928, 18662400, 37324800, 39191040, 69672960, 89579520, 92897280, 139968000, 179159040, 195084288, 313528320, 671846400, 3023308800, 3762339840, 36279705600, 3656994324480
4	24, 24, 24, 24, 24, 24, 24, 36, 48, 48, 48, 48, 72, 72, 72, 72, 72, 72, 96, 96, 96, 96, 96, 96, 96, 144, 144, 144, 144, 144, 192, 192, 192, 192, 192, 192, 192, 192, 192, 288, 288, 288, 288, 288, 288, 288, 288, 288, 384, 384, 384, 384, 384, 384, 432, 504, 576, 576, 576, 768, 768, 864, 864, 1152, 1152, 1152, 1152, 1152, 1152, 1152, 1152, 1152, 1152, 1152, 1536, 1728, 2304, 2304, 2304, 2304, 2304, 3072, 3456, 3456, 4608, 4608, 4608, 5760, 6144, 6912, 6912, 6912, 6912, 6912, 6912, 6912, 9216, 10368, 10368, 12960, 13824, 13824, 13824, 13824, 13824, 13824, 14040, 17280, 17280, 18432, 18432, 18432, 20736, 27648, 27648, 34560, 48384, 51840, 55296, 55296, 55296, 55296, 62208, 69120, 82944, 82944, 103680, 124416, 138240, 138240, 145152, 207360, 207360, 221184, 221184, 248832, 248832, 248832, 414720, 518400, 552960, 725760, 967680, 1105920, 1658880, 2032128, 3110400, 3732480, 4147200, 4147200, 11197440, 11664000, 23224320, 32659200, 74649600, 87091200, 278691840, 7558272000
6	6, 12, 12, 12, 12, 12, 18, 24, 24, 27, 36, 36, 36, 36, 36, 54, 54, 72, 96, 180, 180, 216, 216, 288, 648, 1080, 1152, 1296, 2916, 23328
8	24480

In the above classification, we employed monomial matrices over $GF(4)$ in the definition for equivalence of codes. To define a weaker equivalence, one could consider a conjugation γ of $GF(4)$ sending each element to its square in the definition of equivalence, that is, two codes C and C' are *weakly* equivalent if there is a monomial matrix M over $GF(4)$ such that $C' = CM$ or $C' = CM\gamma$ (see [11]).

We have verified that the equivalence classes of self-dual codes of lengths up to 16 are the same under both definitions. For length 18, there are 230 classes under the weaker equivalence. More specifically, the following codes are weakly equivalent:

$$(C_8, C_9), (C_{10}, C_{11}), (C_{19}, C_{20}), (C_{24}, C_{25}), (C_{26}, C_{27}),$$
$$(C_{28}, C_{29}), (C_{30}, C_{31}), (C_{50}, C_{51}), (C_{56}, C_{57}), (C_{73}, C_{74}),$$
$$(C_{89}, C_{90}), (C_{92}, C_{93}), (C_{94}, C_{95}), (C_{113}, C_{114}), (C_{118}, C_{119}).$$

5   A classification of generalized Hadamard matrices $H(6, 3)$ based on codes

Let $G = \langle \omega \rangle$ be the cyclic group of order 3 being the multiplicative group of $GF(4)$. Assume that $H(6, 3)$ is a generalized Hadamard matrix of order 18 over G. By Theorem

Table 4: Hermitian self-dual codes

n	#	References	$d_{\max}$	$\#_{\max}$	References
2	1	[15]	2	1	C_2 in [15]
4	1	[15]	2	1	$2C_2$ in [15]
6	2	[15]	4	1	E_6 in [15]
8	3	[15]	4	1	E_8 in [15]
10	5	[15]	4	2	E_{10}, B_{10} in [15]
12	10	[15]	4	5	$E_{12}, C_{12}, D_{12}, F_{12}, 2E_6$ in [15]
14	21	[15]	6	1	[15]
16	55	[6]	6	4	[6]
18	245	Section 4	8	1	[11]
20	?		8	2	[11]

5, the code $C(H(6,3))$ generated by the rows of $H(6,3)$ is a Hermitian self-dual code over $GF(4)$ of length 18 and minimum weight at least 4.

Let C be a Hermitian self-dual code of length 18 and minimum weight at least 4. We define a simple undirected graph $\Gamma(C)$, whose set V of vertices is the set of codewords $x = (1, x_2, \ldots, x_{18})$ of weight 18 in C, with two vertices $x, y \in V$ being adjacent if $(n_1, n_\omega, n_{\overline{\omega}}) = (6, 6, 6)$, where $n_\alpha = \#\{i \mid x_i y_i^2 = \alpha\}$.

The following statement was obtained by computations using MAGMA.

Lemma 7. *Let C be a Hermitian self-dual code of length* 18. *The graph $\Gamma(C)$ has a* 18-*clique if and only if C is equivalent to C_{10}, C_{11}, C_{14}, C_{15}, C_{17}, C_{30}, C_{31}, C_{38}, C_{40}, C_{83}, C_{120}, C_{147}, or C_{190}.*

Note that the eleven codes other than C_{11}, C_{31} can be found in Table 5, while the codes C_{11} and C_{31} are obtained as $C_{10}\gamma$ and $C_{30}\gamma$, respectively.

The 18-cliques in the graph $\Gamma(C)$ are generalized Hadamard matrices $H(6,3)$. It is clear that $\mathrm{Aut}(C)$ acts on the graph $\Gamma(C)$ as a (not necessarily full) group of automorphisms. If two 18-cliques in $\Gamma(C)$ are in the same $\mathrm{Aut}(C)$-orbit of the set of 18-cliques in $\Gamma(C)$, then the two generalized Hadamard matrices corresponding to the two 18-cliques are equivalent. Hence, we found generalized Hadamard matrices corresponding to representatives of 18-cliques in $\Gamma(C)$ up to the action of $\mathrm{Aut}(C)$. Then we verified whether two generalized Hadamard matrices are equivalent by a method similar to that given in Section 2. For each code C_i, we list in Table 6 the number # of generalized Hadamard matrices $H(6,3)$ which are not monomially equivalent, obtained in this way. In Table 6, we also list corresponding generalized Hadamard matrices given in Section 2. Therefore, we have an alternative classification of the generalized Hadamard matrices $H(6,3)$, given in Corollary 2.

Table 5: The codes C_i $(i = 10, 14, 15, 17, 30, 38, 40, 83, 120, 147, 190)$

i	dim	$v_1,\ldots,v_{9-\dim}$	A_4	A_6	# Aut
10	8	$(1,1,1,\omega,1,1,1,1,1,\overline{\omega},\overline{\omega},\overline{\omega},\omega,\overline{\omega},\overline{\omega},\overline{\omega},0,0)$	0	45	180
14	8	$(1,1,1,1,1,1,1,1,1,1,\overline{\omega},\omega,0,0,0,0,0,0)$	0	27	2916
15	8	$(1,\omega,1,1,1,1,1,1,1,1,\omega,\overline{\omega},0,\overline{\omega},0,1,\omega,\omega)$	0	27	648
17	8	$(1,1,1,1,1,1,1,1,1,0,\omega,0,\omega,\overline{\omega},\omega,\omega,\omega,\omega)$	0	99	1080
30	8	$(1,1,1,1,1,1,1,1,1,\omega,\omega,0,\overline{\omega},0,0,0,0,0)$	9	36	2304
38	7	$(1,1,1,1,1,1,1,1,1,0,0,\overline{\omega},\omega,\overline{\omega},\omega,\omega,\omega,\omega)$ $(1,\omega,1,1,1,1,1,1,1,0,0,1,\overline{\omega},0,\overline{\omega},0,\overline{\omega},\overline{\omega})$	0	108	23328
40	7	$(1,1,1,1,1,1,1,1,1,1,\omega,\omega,1,1,\omega,1,1,1)$ $(\omega,1,1,1,1,1,1,1,1,1,\overline{\omega},\omega,\omega,\overline{\omega},0,\omega,0,\omega,1)$	0	72	216
83	7	$(1,1,1,1,1,1,1,1,1,0,0,0,\overline{\omega},1,\omega,0,0,0)$ $(\omega,1,1,1,1,1,1,1,1,0,\overline{\omega},\overline{\omega},\omega,0,\omega,\overline{\omega},1,\omega)$	9	72	62208
120	7	$(1,1,1,1,1,1,1,1,1,1,0,0,\omega,\omega,\omega,1,1,1)$ $(\omega,1,1,1,1,1,1,1,1,1,1,\omega,0,0,1,\overline{\omega},1,\omega)$	27	18	248832
147	7	$(1,1,1,1,1,1,1,1,1,1,\overline{\omega},\overline{\omega},0,\omega,\overline{\omega},\omega,1,0)$ $(\omega,1,1,1,1,1,1,1,1,\overline{\omega},\overline{\omega},1,\overline{\omega},0,1,1,\overline{\omega},0)$	45	90	$2^7 3^6 5^3$
190	6	$(1,1,1,1,1,1,1,1,1,0,\overline{\omega},0,0,\overline{\omega},\overline{\omega},0,0,0)$ $(\omega,1,1,1,1,1,1,1,1,0,\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},1,0,0,0)$ $(1,\omega,1,1,1,1,1,1,1,0,1,0,0,\overline{\omega},\overline{\omega},1,0,\overline{\omega})$	135	54	$2^{10} 3^{10} 5^3$

Table 6: Generalized Hadamard matrices in C_i

i	#	generalized Hadamard matrices
10	1	$H(\mathcal{D}_{45})$
11	1	$\overline{H(\mathcal{D}_{45})}$
14	4	$H(\mathcal{D}_i), \overline{H(\mathcal{D}_i)}$ $(i=44,53)$
15	3	$H(\mathcal{D}_{19}),\ H(\mathcal{D}_{21}), \overline{H(\mathcal{D}_{21})}$
17	8	$H(\mathcal{D}_{23}),\ H(\mathcal{D}_{27}),\ H(\mathcal{D}_i), \overline{H(\mathcal{D}_i)}$ $(i=24,25,26)$
30	2	$H(\mathcal{D}_{32}), \overline{H(\mathcal{D}_{46})}$
31	2	$H(\mathcal{D}_{46}), \overline{H(\mathcal{D}_{32})}$
38	9	$H(\mathcal{D}_i)$ $(i=37,38,39),\ H(\mathcal{D}_j), \overline{H(\mathcal{D}_j)}$ $(j=34,35,36)$
40	3	$H(\mathcal{D}_{20}),\ H(\mathcal{D}_{22}), \overline{H(\mathcal{D}_{22})}$
83	12	$H(\mathcal{D}_i)$ $(i=28,33,42,43),\ H(\mathcal{D}_j), \overline{H(\mathcal{D}_j)}$ $(j=30,31,51,52)$
120	9	$H(\mathcal{D}_i)$ $(i=1,2,3),\ H(\mathcal{D}_j), \overline{H(\mathcal{D}_j)}$ $(j=15,16,17)$
147	14	$H(\mathcal{D}_i), \overline{H(\mathcal{D}_i)}$ $(i=29,40,41,47,48,49,50)$
190	17	$H(\mathcal{D}_i)$ $(i=4,5,6,12,13,14,18),\ H(\mathcal{D}_j), \overline{H(\mathcal{D}_j)}$ $(j=7,8,9,10,11)$

Acknowledgments

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