Constructing 5-configurations with chiral symmetry

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Submitted: Mar 16, 2009; Accepted: Dec 11, 2009; Published: Jan 5, 2010
Mathematics Subject Classification: 05B30, 51E30

Abstract

A 5-configuration is a collection of points and straight lines in the Euclidean plane so that each point lies on five lines and each line passes through five points. We describe how to construct the first known family of 5-configurations with chiral (that is, only rotational) symmetry, and prove that the construction works; in addition, the construction technique produces the smallest known geometric 5-configuration.

In recent years, there has been a resurgence in the study of k-configurations with high degrees of geometric symmetry; that is, in the study of collections of points and straight lines in the Euclidean plane where each point lies on k lines and each line passes through k points, with a small number of symmetry classes of points and lines under Euclidean isometries that map the configuration to itself. 3-configurations have been studied since the late 1800s (see, e.g., [15, Ch. 3], and more recently [9, 12, 13]), and there has been a great deal of recent investigation into 4-configurations (e.g., see [1, 2, 5, 8, 14]). However, there has been little investigation into k-configurations for $k > 4$.

Following [10], we say that a geometric k-configuration is *polycyclic* if a rotation by angle $\frac{2\pi i}{m}$ for some integers i and m is a symmetry operation that partitions the points and lines of the configuration into equal-sized symmetry classes (orbits), where each orbit contains m points. If $n = dm$, then there are d orbits of points and lines under the rotational symmetry. The group of symmetries of such a configuration is at least cyclic. In many cases, the full symmetry group is dihedral; this is the case for most known polycyclic 4-configurations.

A k-configuration is *astral* if it has $\lfloor \frac{k+1}{2} \rfloor$ symmetry classes of points and of lines under rotations and reflections of the plane that map the configuration to itself. It has been conjectured that there are no *astral* 5-configurations, which would have 3 symmetry classes of points and lines [6], [11, Conj. 4.1.1]; support for this conjecture was given in [3], where it was shown that there are no astral 5-configurations with dihedral symmetry. The existence of astral 5-configurations with only cyclic symmetry is still unsettled but is highly unlikely. Until recently, there were no known families of 5-configurations with a high degree of symmetry in the Euclidean plane. There were a few recently discovered examples in the extended Euclidean plane [12], [11, Section 4.1], but these are not polycyclic, since not all of the symmetry classes have the same number of points. The 5-configurations described in this paper have four symmetry classes of points and lines and chiral symmetry (that is, they have no mirrors of reflective symmetry); it is likely that they are as symmetric as possible.

1 2-astral configurations

The construction of the 5-configurations begins with *astral* 4-configurations. Such a configuration, also known as a 2-astral configuration, may be described by the symbol

$$m\#(a,b;c,d),$$

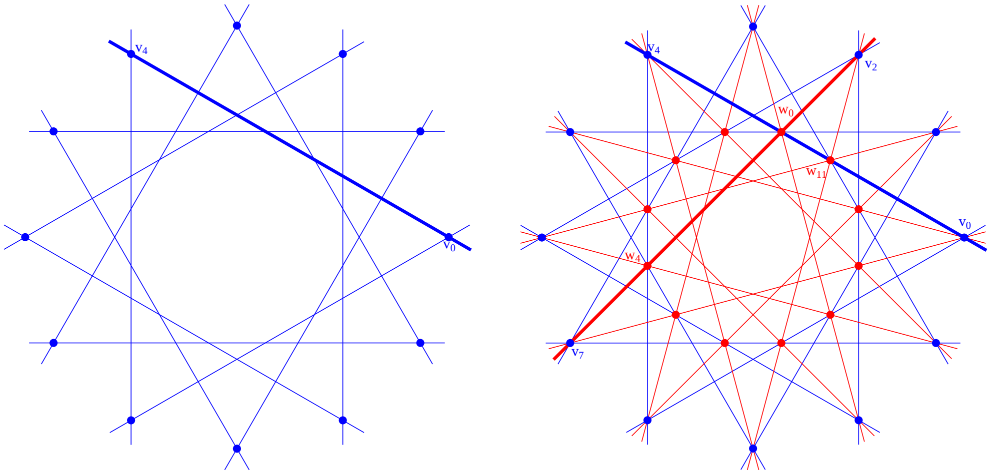
where $m = 6k$ for some k. These configurations are the smallest case of a general class of configurations with high degrees of geometric symmetry called *multiastral* or *h-astral* configurations [11, Section 3.5–3.8] (called *celestial* configurations in [5]), which in general have symbol

$$m\#(s_1,t_1;\ldots;s_h,t_h);$$

that is, the configuration $m\#(a,b;c,d)$ would be written as $m\#(s_1,t_1;s_2,t_2)$ in that notation. In [8, 12], it was shown that there are two infinite families of 2-astral configurations, of the form $6k\#(3k-j,2k;j,3k-2j)$ and $6k\#(2k,j;3k-2j,3k-j)$,for $k \geqslant 2$, $1 \leqslant j < 3k/2$, with $j \neq k$, along with 27 sporadic configurations in the case when $m = 30, 42, 60$ (plus their disconnected multiples). These configurations have been discussed in detail in other places (e.g., [7, 8, 10, 14]). Note that in some of these references, the configuration $m\#(a,b;c,d)$ is denoted as $m\#a_b\,d_c$. In [10], the configuration $m\#(a,b;c,d)$ is denoted $\mathcal{C}_4\left(m,(a,c),(b,d),\frac{a+c-b-d}{2}\right)$.

In this section, we simply will describe the construction technique for constructing a 2-astral configuration with symbol $m\#(a,b;c,d)$.

Given a configuration symbol, the corresponding configuration is constructed as follows. For more details on this construction, see [5], where this construction is discussed in the more general context of h-astral configurations; more details on why the construction method produces 4-configurations may be found in [11, Section 3.5]. Typically in the literature (again, see [11, Section 3.5] for a recent description), the construction of multiastral configurations has been described geometrically, by constructing collections of diagonals of regular m-gons of a particular "span" and then constructing subsequent points of the configuration by considering particular intersection points of those diagonals with each other. In what follows, we will continue to use this approach, but we also will explicitly determine the points and lines under discussion. An example of the construction is shown



(a) (b)

Figure 1: The 2-astral configuration $12\#(4,1;4,5)$, and its construction. (a) The vertices v_i and lines B_i of span 4 with respect to these vertices. The thick line has label B_0. (b) Adding the vertices w_i and lines R_i to complete the configuration. The thick red line has label R_0 and the thick blue line has label B_0. Note that R_0 contains points w_0, w_4, $v_\sigma = v_7$, and $v_{\sigma-5} = v_2$, while B_0 contains points v_0, v_4, w_0 and $w_{-1} = w_{11}$, where $\sigma = \frac{1}{2}(a+b+c+d)$.

in Figure 1 for the configuration $12\#(4,1;4,5)$. The configuration $12\#(4,1;4,5)$ is the smallest astral 4-configuration, and its picture has appeared in many places, including as Figure 18 in [10] and Figure 1 of [8].

Given points P and Q and lines ℓ_1 and ℓ_2, denote the line containing P and Q as $P \vee Q$ and the point of intersection of lines ℓ_1 and ℓ_2 as $\ell_1 \wedge \ell_2$.

1. Construct the vertices of a regular convex m-gon centered at the origin, with circumcircle of radius r, which is offset from horizontal by an angle ϕ (that is, the angle between horizontal and Ov_0 is ϕ), cyclically labelled as $v_0, v_1, \ldots, v_{m-1}$; in general,
$$v_i = \left(r\cos\left(\frac{2\pi i}{m} + \phi\right), r\sin\left(\frac{2\pi i}{m} + \phi\right)\right), \tag{1}$$
although typically, we take $r = 1$ and $\phi = 0$.

2. Construct lines $B_i = v_i \vee v_{i+a}$. These lines are said to be of *span* a with respect to the v_i. (In Figure 1(a), these are the blue lines.)

3. Construct the points w_i on the lines B_i which are the b-th intersection of this line with the other span a lines: more precisely, define $w_i = B_i \wedge B_{i+b}$. With this definition,
$$w_i = r\frac{\cos\left(\frac{\pi a}{m}\right)}{\cos\left(\frac{\pi b}{m}\right)} \left(\cos\left((a+b+2i)\frac{\pi}{m} + \phi\right), \sin\left((a+b+2i)\frac{\pi}{m} + \phi\right)\right). \tag{2}$$

Figure 2 gives a geometric argument for the determination of the coordinates of w_i.

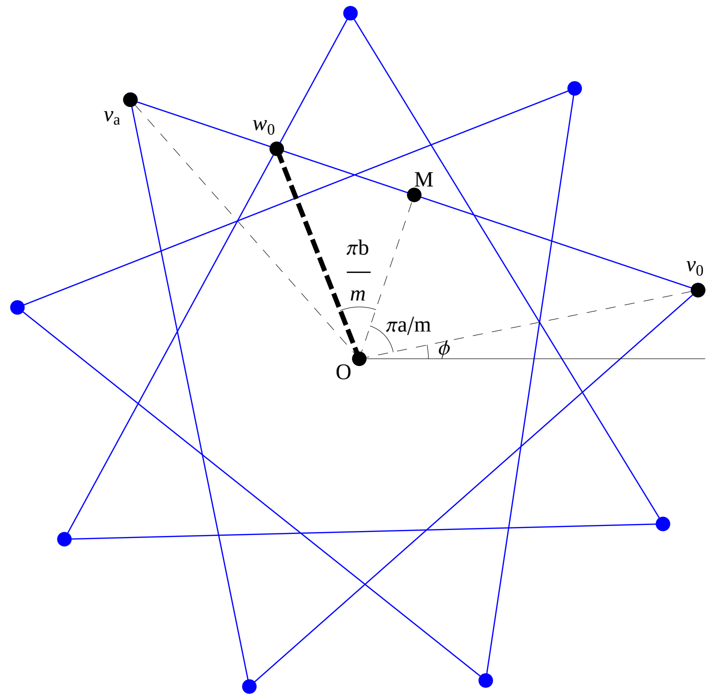


Figure 2: Determining the coordinates of w_0 with respect to a regular convex m-gon of radius r with vertices $v_0, v_1, \ldots, v_{m-1}$, where the angle between v_0 and horizontal is ϕ. Point v_a has coordinates $\left(r\cos\left(\frac{2a\pi}{m}+\phi\right), r\sin\left(\frac{2a\pi}{m}+\phi\right)\right)$, so $\angle v_0OM = \frac{a\pi}{m}$, where M is the foot of the perpendicular from the center O to the line $B_0 = v_0 \vee v_a$. If $Ov_0 = r$, then since $\cos(\angle MOv_0) = \frac{OM}{Ov_0}$, it follows that $OM = r\cos(a\pi/m)$. Since $w_0 = B_0 \wedge B_b$, $\angle MOw_0 = \frac{b\pi}{m}$. Therefore, $\cos(\angle MOw_0) = \frac{OM}{Ow_0}$, so $Ow_0 = r \cdot \frac{\cos(a\pi/m)}{\cos(b\pi/m)}$, and $\angle v_0Ow_0 = \frac{\pi(a+b)}{m}$. In the diagram, $m = 9$, $a = 3$ and $b = 2$, and $\phi = 0.3$.

4. Construct lines R_i of span c with respect to the vertices w_i: that is, $R_i = w_i \vee w_{i+c}$. (In Figure 1(b), these are the red lines.) If the configuration symbol is valid, then the points which are the d-nd intersection points of the R_i must coincide with the points v_i; in particular, $R_i \wedge R_{i+d} = v_{i+\sigma}$, where $\sigma = \frac{1}{2}(a+b+c+d)$.

A necessary condition for a configuration symbol $m\#(a,b;c,d)$ to be valid is that $a+b+c+d$ is even (see [11, p. 196, (A6)] for details). Therefore $\sigma = \frac{1}{2}(a+b+c+d)$ is always an integer. Notationally, a point which is the t-th intersection of a span s line with other span s lines is given label $(s//t)$. Thus, the points w_i have label $(a//b)$ with respect to the span a lines B_i and the points v_i. The points v_i, on the other hand, have label $(d//c)$ with respect to the points w_i and the lines R_i of span d. (In using the notation $(s//t)$, we follow the most current notation, introduced in [11, Chapter 3]; in [4, 5, 12] the notation $[[s,t]]$ was used instead of $(s//t)$.) Table 1 lists the specific point-line incidences in $m\#(a,b;c,d)$.

Table 1: Point-line incidence for the points and lines in $m\#(a,b;c,d)$; $\sigma = \frac{1}{2}(a+b+c+d)$.

Element	Contains			
B_i	v_i	v_{i+a}	w_i	w_{i-b}
R_i	$v_{i+\sigma}$	$v_{i+\sigma-d}$	w_i	w_{i+c}
v_i	B_i	B_{i-a}	$R_{i-\sigma}$	$R_{i-\sigma+d}$
w_i	B_i	B_{i+b}	R_i	R_{i-c}

2 Constructing 5-configurations

We begin with a 2-astral configuration with symbol $m\#(a,b;c,d)$ constructed as above, where the first ring of vertices is labelled v_i and the second is labelled w_i, and the (blue) lines of span a with respect to the v_i are labelled B_i and the (red) lines of span d with respect to the v_i (which are span c with respect to the w_i) are labelled R_i. In particular, we set

$$v_i = \left(\cos\left(\frac{2\pi i}{m}\right), \sin\left(\frac{2\pi i}{m}\right)\right) \tag{3}$$

$$w_i = \frac{\cos\left(\frac{\pi a}{m}\right)}{\cos\left(\frac{\pi b}{m}\right)}\left(\cos\left(\frac{\pi(a+b+2i)}{m}\right),\ \sin\left(\frac{\pi(a+b+2i)}{m}\right)\right) \tag{4}$$

We extend this configuration to an incidence structure called the *associated subfiguration* $\mathcal{S}(m,(a,b;c,d),\lambda)$, as follows. Each subfiguration is determined by five discrete parameters m,a,b,c,d and one continuous parameter, λ.

1. Determine a point p_i uniquely on each line B_i, by defining

   $$p_i = (1-\lambda)v_i + \lambda v_{i+a};$$

   these points p_i have explicit coordinates

   $$p_i = \left(\lambda\cos\left(\frac{2\pi(a+i)}{m}\right) + (1-\lambda)\cos\left(\frac{2\pi i}{m}\right),\right.$$
   $$\left.\lambda\sin\left(\frac{2\pi(a+i)}{m}\right) + (1-\lambda)\sin\left(\frac{2\pi i}{m}\right)\right) \tag{5}$$

   for a particular value of λ. Note that the points p_i form a regular convex m-gon.

2. Using these p_i as the initial m-gon (here, $\phi = \arctan\left(\frac{\lambda \sin\left(\frac{2\pi a}{m}\right)}{\lambda \cos\left(\frac{2\pi a}{m}\right) - \lambda + 1}\right)$), construct the 2-astral configuration with symbol $m\#(d, a; b, c)$. Label the second set of vertices formed in this construction as q_i, the (green) span d lines with respect to the p_i as G_i and the (magenta) span b lines with respect to the q_i as M_i. That is, define $G_i = p_i \vee p_{i+d}$, $q_i = G_i \wedge G_{i+a}$, and $M_i = q_i \vee q_{i+b}$.

The subfiguration $\mathcal{S}(12, (4, 1; 4, 5), 0.1)$ is shown in Figure 3.

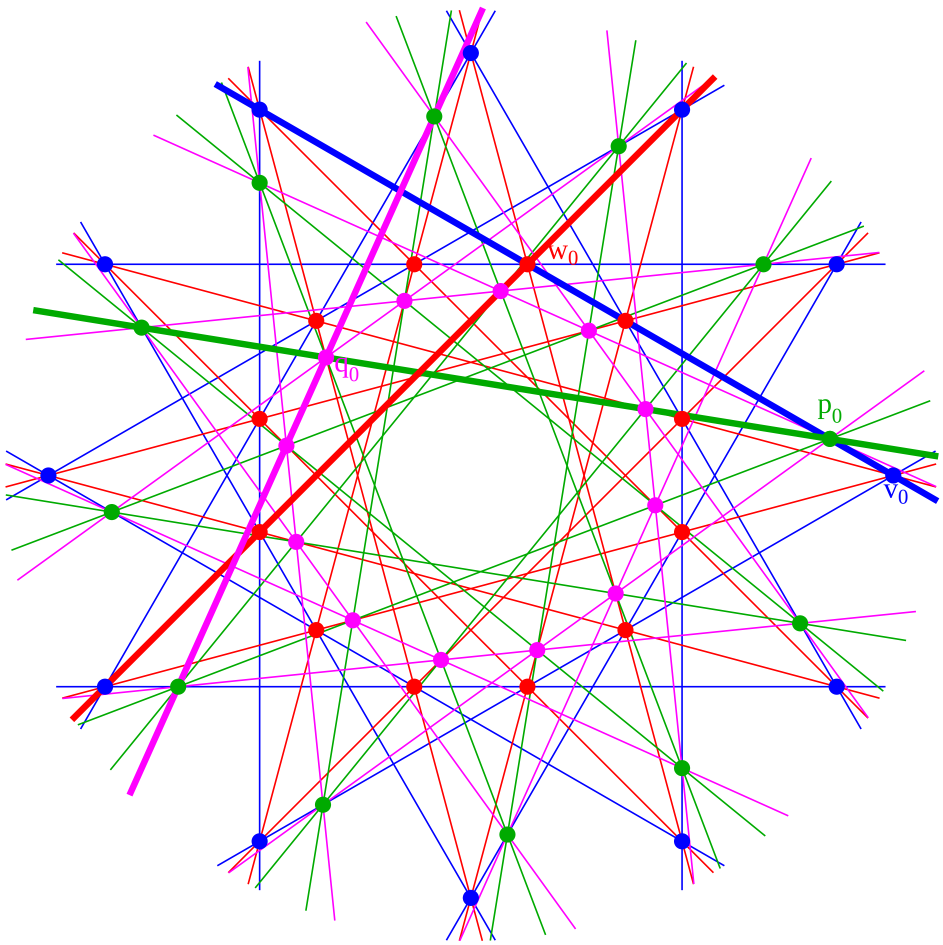


Figure 3: The subfiguration $\mathcal{S}(12, (4, 1; 4, 5), 0.1)$. The lines B_0 (blue), R_0 (red), G_0 (green) and M_0 (magenta) are shown thick, and the points v_0, w_0, p_0 and q_0 are labelled.

The following lemma was proved in [5] (in a restated form); see Figure 4 for an illustration.

Theorem 1 (Crossing Spans Lemma). *Given a regular m-gon $\mathcal{M}$ with vertices $u_0, u_1, \ldots, u_{m-1}$ and diagonals $\Theta_i = u_i \vee u_{i+\alpha}$ of span α and $\Psi_i = u_i \vee u_{i+\beta}$ of span β, suppose that*

$x_0 = (1-\lambda)u_0 + \lambda u_\alpha$ is an arbitrary point on Θ_0, and construct other points x_i to be the rotations of x_0 through $\frac{2\pi i}{m}$ (so that $x_i = (1-\lambda)u_i + \lambda u_{i+a}$), forming a second regular, convex m-gon $\mathcal{N}$. Construct diagonals Γ_i of span β with respect to the x_i: that is, let $\Gamma_i = x_i \vee x_{i+\beta}$. Let $y_i = \Gamma_i \wedge \Psi_i$ and let $y_i' = \Gamma_{i-a} \wedge \Psi_i$. Then $y_i = y_i'$.

That is, begin with a set of diagonals of span α and span β of a regular convex m-gon $\mathcal{M}$. Place an arbitrary point x_0 on a diagonal of span α, and using x_0, construct another regular convex m-gon $\mathcal{N}$ whose vertices are the rotated images of x_0 through angles of $\frac{2\pi i}{m}$. Then construct diagonals of span β using $\mathcal{N}$. Two of these span β diagonals intersect each other and a span α diagonal of $\mathcal{M}$ in the same point, and the intersection points are precisely the points labeled $(\beta//\alpha)$ with respect to $\mathcal{N}$.

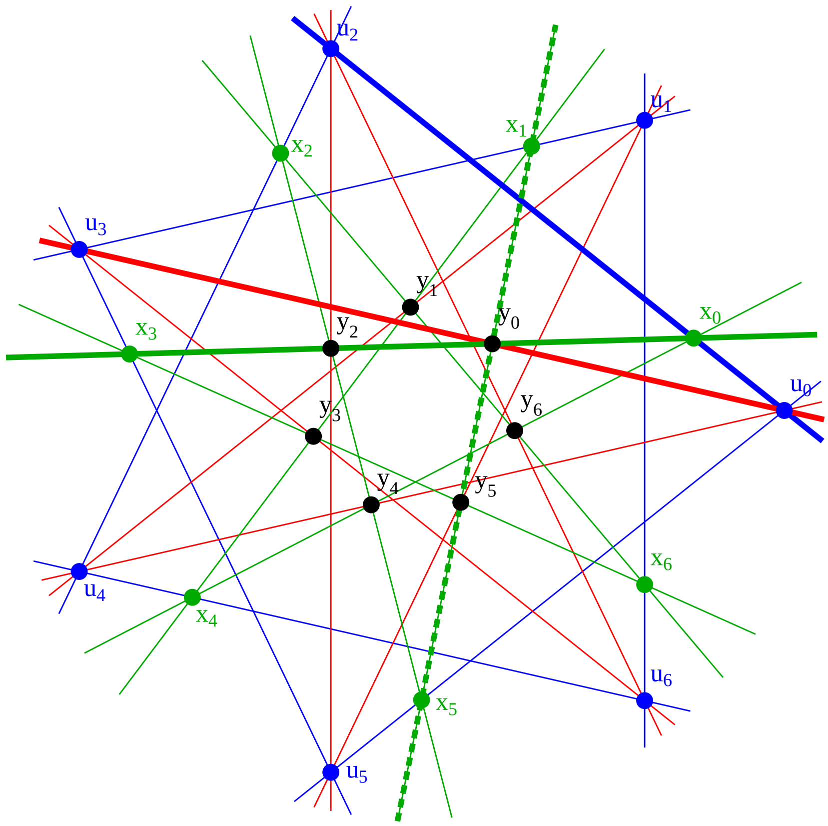


Figure 4: Illustration of the Crossing Spans Lemma with $m = 7$, $\alpha = 2$, $\beta = 3$. The outer, blue points are the original m-gon $\mathcal{M}$ with vertices u_i, the middle, green points are the "arbitrary" points x_i forming $\mathcal{N}$, and the inner, black points are the intersection points y_i with label $(\beta//\alpha)$ with respect to $\mathcal{N}$. The lines Θ_i are blue, Ψ_i are green, and Γ_i are are red. Lines Θ_0, Ψ_0, and Γ_0 are shown bold and thick, while line $\Gamma_{-\alpha}$ is shown bold and dashed.

Using this theorem we can show the following:

Theorem 2. *Given a subfiguration $\mathcal{S} = \mathcal{S}(m,(a,b;c,d),\lambda)$ with vertices v_i, w_i, p_i, and q_i and lines B_i, R_i, G_i and M_i, each point q_i lies on five lines.*

Proof. We apply the Crossing Spans Lemma, with points $\{u_i\} = \{v_i\} = \mathcal{M}$ and $\{x_i\} = \{p_i\} = \mathcal{N}$, and lines $\Theta_i = B_i$, of span $\alpha = a$ with respect to $\mathcal{M}$, $\Psi_i = R_{i-\sigma+d}$, of span $\beta = d$ with respect to $\mathcal{M}$, and $\Gamma_i = G_i$, of span $\beta = d$ with respect to $\mathcal{N}$. The Crossing Spans Lemma allows us to conclude that each point $y_i = q_{i-a}$ lies on lines G_i, $R_{i-\sigma+d}$ and G_{i-a}. However, each point q_{i-a} also lies on two magenta lines, M_{i-a} and M_{i-b-a}. Therefore, each point q_i lies on five lines. □

Table 2 gives the specific point-line incidences in a subfiguration $\mathcal{S}(m,(a,b;c,d),\lambda)$. Notice that the lines B_i and R_i each contain five points, and the points p_i and q_i have five lines passing through them. However, the lines G_i and M_i only contain four points, and the points v_i and w_i only have four lines passing through them.

Table 2: point-line incidence for the points and lines in $\mathcal{S}(m,(a,b;c,d),\lambda)$; $\sigma = \frac{1}{2}(a+b+c+d)$.

Element	Contains				
B_i	v_i	v_{i+a}	w_i	w_{i-b}	p_i
R_i	$v_{i+\sigma}$	$v_{i+\sigma-d}$	w_i	w_{i+c}	$q_{i-a-d+\sigma}$
G_i	p_i	p_{i+d}	q_i	q_{i-a}	
M_i	$p_{i+\sigma}$	$p_{i+\sigma-c}$	q_i	q_{i+b}	
v_i	B_i	B_{i-a}	$R_{i-\sigma}$	$R_{i-\sigma+d}$	
w_i	B_i	B_{i+b}	R_i	R_{i-c}	
p_i	G_i	G_{i-a}	$M_{i-\sigma}$	$M_{i-\sigma+c}$	B_i
q_i	G_i	G_{i+a}	M_i	M_{i-b}	$R_{i-\sigma+a+d}$

The points p_i were placed on the blue lines B_i arbitrarily, and each line B_i passes through the vertices v_i and v_{i+a}. We can attempt to vary the position of p_i so that the green lines G_i, which are constructed as the span d lines through the p_i, pass through the set of points labelled w_i. More precisely: since

$$p_0 = (1-\lambda)v_0 + \lambda\, v_a,$$

we can try to find λ so that the line $G_0 = p_0 \vee p_d$ passes through the vertex labelled w_k for some $k = 0, 1, 2, \ldots m-1$ of our choosing. That is, we solve for a value of λ so that p_0, p_d, and w_k are collinear.

More precisely, if $p_i(x)$ and $p_i(y)$ (respectively, $w_i(x), w_i(y)$) are the x and y-coordinates of p_i (respectively, w_i), in order for p_0, p_d, w_k to be collinear we need, using the coordinates from (4) and (5),

$$0 = \det \begin{pmatrix} p_0(x) & p_0(y) & 1 \\ p_d(x) & p_d(y) & 1 \\ w_k(x) & w_k(y) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda \cos\left(\frac{2a\pi}{m}\right) + (1-\lambda) & \lambda \sin\left(\frac{2a\pi}{m}\right) & 1 \\ (1-\lambda)\cos\left(\frac{2d\pi}{m}\right) + \lambda \cos\left(\frac{2(a+d)\pi}{m}\right) & (1-\lambda)\sin\left(\frac{2d\pi}{m}\right) + \lambda \sin\left(\frac{2(a+d)\pi}{m}\right) & 1 \\ \frac{\cos\left(\frac{a\pi}{m}\right)}{\cos\left(\frac{b\pi}{m}\right)} \cos\left(\frac{(a+b+2k)\pi}{m}\right) & \frac{\cos\left(\frac{a\pi}{m}\right)}{\cos\left(\frac{b\pi}{m}\right)} \sin\left(\frac{(a+b+2k)\pi}{m}\right) & 1 \end{pmatrix} \tag{6}$$

which is a quadratic polynomial in λ. (Note while writing out the polynomial is notationally cumbersome, for particular choices of m, a, b, k, d it is straightforward to use a computer to solve the equation.) In general, there are two possible values of λ values for a given w_k, although in particular cases, there is no real solution, or the solution exists but produces a subfiguration with some of the sets of points v_i, w_i, p_i, q_i coinciding (a *degenerate* subfiguration). Table 3 shows all solutions for the subfiguration $\mathcal{S}(12, (4, 1; 4, 5), \lambda)$; note that nondegenerate subfigurations of this type exist only for $k = 1$ and $k = 3$. For $k = 0, 2, 4, 5, 6, 10, 11$ the resulting configurations have two of the rings of points p_i, q_i, w_i, v_i coinciding, while for $k = 7, 8, 9$ there are no real solutions to (6).

Table 3: Values of λ for which $\mathcal{S}(12, (4, 1; 4, 5), \lambda)$ has the points p_0, p_d, w_k collinear, for $k = 0, 1, \ldots, 11$. The note DNE indicates that the corresponding configuration does not exist.

k	λ_0	note	λ_1	note
0	$-\frac{1}{\sqrt{3}}$	$q_i = v_{i+4}$	$\frac{1}{\sqrt{3}}$	$p_i = w_i$
1	$\frac{1-\sqrt{3+2\sqrt{3}}}{3+\sqrt{3}}$		$\frac{1+\sqrt{3+2\sqrt{3}}}{3+\sqrt{3}}$	
2	0	$p_i = v_i$	1	$p_i = v_{i+4}$
3	$\frac{3+2\sqrt{3}-\sqrt{9+6\sqrt{3}}}{3\left(1+\sqrt{3}\right)}$		$\frac{3+2\sqrt{3}+\sqrt{9+6\sqrt{3}}}{3\left(1+\sqrt{3}\right)}$	
4	$1-\frac{1}{\sqrt{3}}$	$p_{i+1} = w_i$	$1+\frac{1}{\sqrt{3}}$	$v_i = q_{i+3}$
5	$\frac{1}{\sqrt{3}}$	$p_i = w_i$	$1+\frac{1}{\sqrt{3}}$	$v_i = q_{i+3}$
6	1	$p_i = v_{i+4}$	1	$p_i = v_{i+4}$
7	complex	DNE	complex	DNE
8	complex	DNE	complex	DNE
9	complex	DNE	complex	DNE
10	0	$p_i = v_i$	0	$p_i = v_i$
11	$-\frac{1}{\sqrt{3}}$	$q_i = v_{i+4}$	$1-\frac{1}{\sqrt{3}}$	$p_i = w_{i-1}$

Suppose λ_0 and λ_1 are the two real solutions to Equation (6) which force p_0, p_d, and w_k to be collinear; by convention, we set $\lambda_0 \leqslant \lambda_1$, and suppose that $\chi \in \{0, 1\}$. Define $\mathcal{C}(m, (a, b; c, d), k, \chi)$ to be the subfiguration $\mathcal{S}(m, (a, b; c, d), \lambda_\chi)$, .

Theorem 3. *The subfiguration* $\mathcal{C} = \mathcal{C}(m, (a, b; c, d), k, \chi)$ *has the property that the line* $M_{i-2\sigma+c+d-k}$ *passes through point* v_i*.*

Proof. Again, apply the Crossing Spans Lemma. Let $\mathcal{M} = \{u_i\} = \{p_i\}$, and let $\Theta_i = G_i$ (of span $\alpha = d$ with respect to $\mathcal{M}$) and $\Psi_i = M_{i-\sigma+c}$ (of span $\beta = c$ with respect to $\mathcal{M}$). By the choice of λ_χ in the construction of $\mathcal{C}(m, (a, b; c, d), k, \chi)$, we have forced p_0, p_d and w_k to be collinear, so for each i, in fact, point w_{i+k} lies on line $\Theta_i = G_i$, since $G_i = p_i \vee p_{i+d}$.

Define $\mathcal{N} = \{x_i\} = \{w_{i+k}\}$. The lines R_i are of span c with respect to $\mathcal{N}$; thus, define $\Gamma_i = R_{i+k} = w_{i+k} \vee w_{(i+k)-c}$. By the Crossing Spans Lemma, we conclude that the lines $\Gamma_i = R_{i+k}$, $\Psi_i = M_{i-\sigma+c}$, and $\Gamma_{i-a} = R_{i+k-d}$ are coincident.

However, because $m\#(a, b; c, d)$ is a valid configuration symbol, for each j, $R_{j-\sigma} \wedge R_{j-\sigma+d} = v_j$. Therefore, the lines

$$\Gamma_i = R_{i+k} = R_{(i+k+\sigma-d)-\sigma+d} \quad \text{and} \quad \Gamma_{i-a} = R_{i+k-d} = R_{(i+k+\sigma-d)-\sigma}$$

intersect at the point $v_{i+k+\sigma-d}$, so $v_{i+k+\sigma-d}$ also lies on $M_{i-\sigma+c}$.

That is, each point v_i lies on the five lines B_i, B_{i-a}, $R_{i-\sigma}$, $R_{i-\sigma+d}$, and $M_{i-2\sigma+c+d-k}$. □

If k is chosen so that $\mathcal{C}(m, (a, b; c, d), k, \chi)$ exists and the points v_i, w_i, p_i, and q_i are all distinct, then we say that $\mathcal{C}(m, (a, b; c, d), k, \chi)$ is a nondegenerate 5-configuration. The precise point-line incidences are shown in Table 4.

Corollary 4. *Every nondegenerate* $\mathcal{C}(m, (a, b; c, d), k, \chi)$ *is a* $((4m)_5)$ *geometric configuration.*

Table 4: point-line incidence for the points and lines in $\mathcal{C}(m, (a, b; c, d), k, \chi)$; $\sigma = \frac{1}{2}(a + b + c + d)$.

Element	Contains				
B_i	v_i	v_{i+a}	w_i	w_{i-b}	p_i
R_i	$v_{i+\sigma}$	$v_{i+\sigma-d}$	w_i	w_{i+c}	$q_{i-a-d+\sigma}$
G_i	p_i	p_{i+d}	q_i	q_{i-a}	w_{i+k}
M_i	$p_{i+\sigma}$	$p_{i+\sigma-c}$	q_i	q_{i+b}	$v_{i+2\sigma-c-d+k}$
v_i	B_i	B_{i-a}	$R_{i-\sigma}$	$R_{i-\sigma+d}$	$M_{i-2\sigma+c+d-k}$
w_i	B_i	B_{i+b}	R_i	R_{i-c}	G_{i-k}
p_i	G_i	G_{i-a}	$M_{i-\sigma}$	$M_{i-\sigma+c}$	B_i
q_i	G_i	G_{i+a}	M_i	M_{i-b}	$R_{i-\sigma+a+d}$

The (48_5) configurations $\mathcal{C}(12, (4, 1; 4, 5), 1, 1)$, which is shown in Figure 5, and $\mathcal{C}(12, (4, 1; 4, 5), 3, 1)$, form the smallest known examples of 5-configurations. The configuration $\mathcal{C}(12, (4, 1; 4, 5), 3, 0)$ appears as Figure 4.1.4 in [11, p. 238], although the colors used are

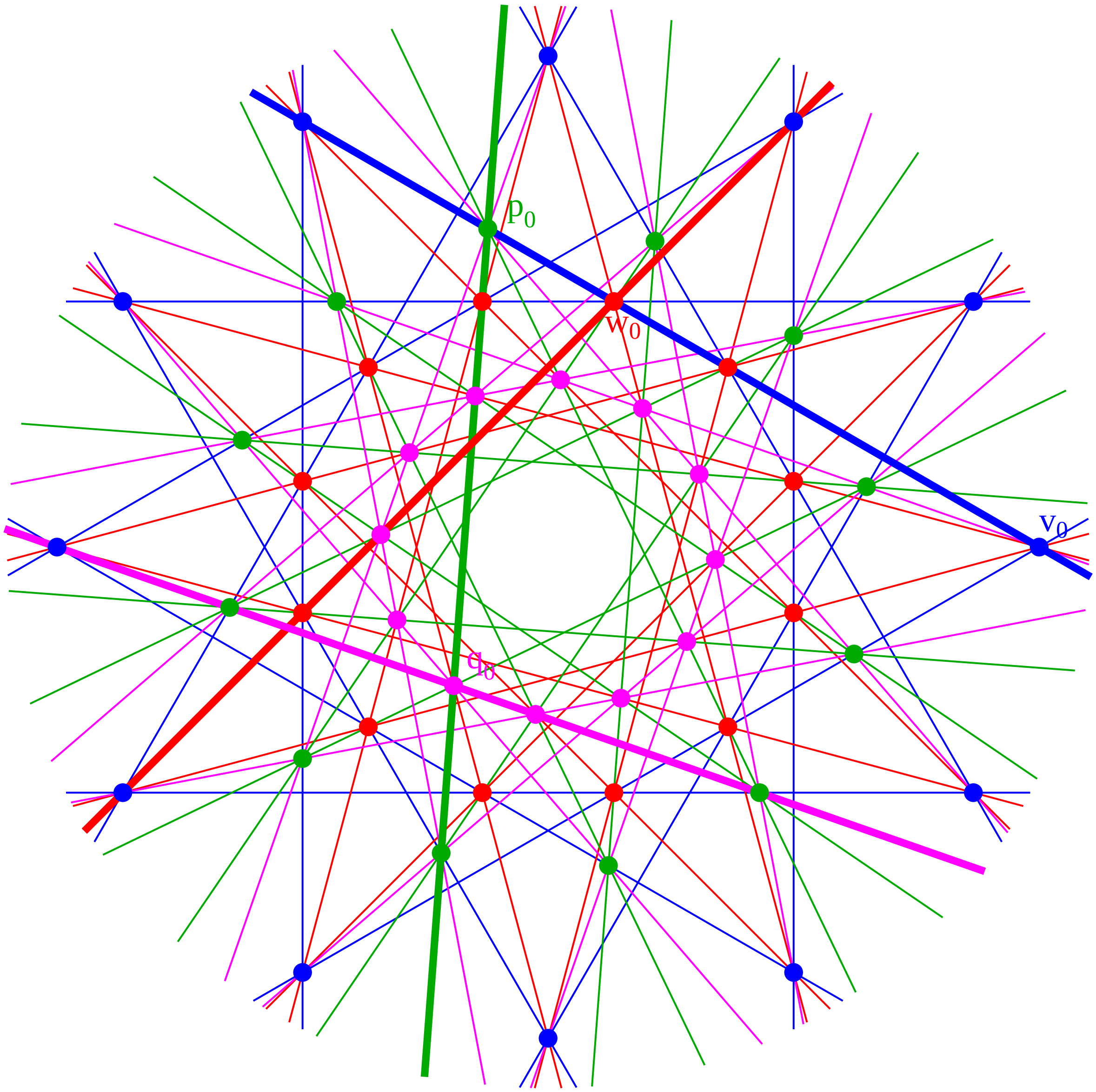


Figure 5: The 5-configuration $\mathcal{C}(12,(4,1,4,5),1,1)$.

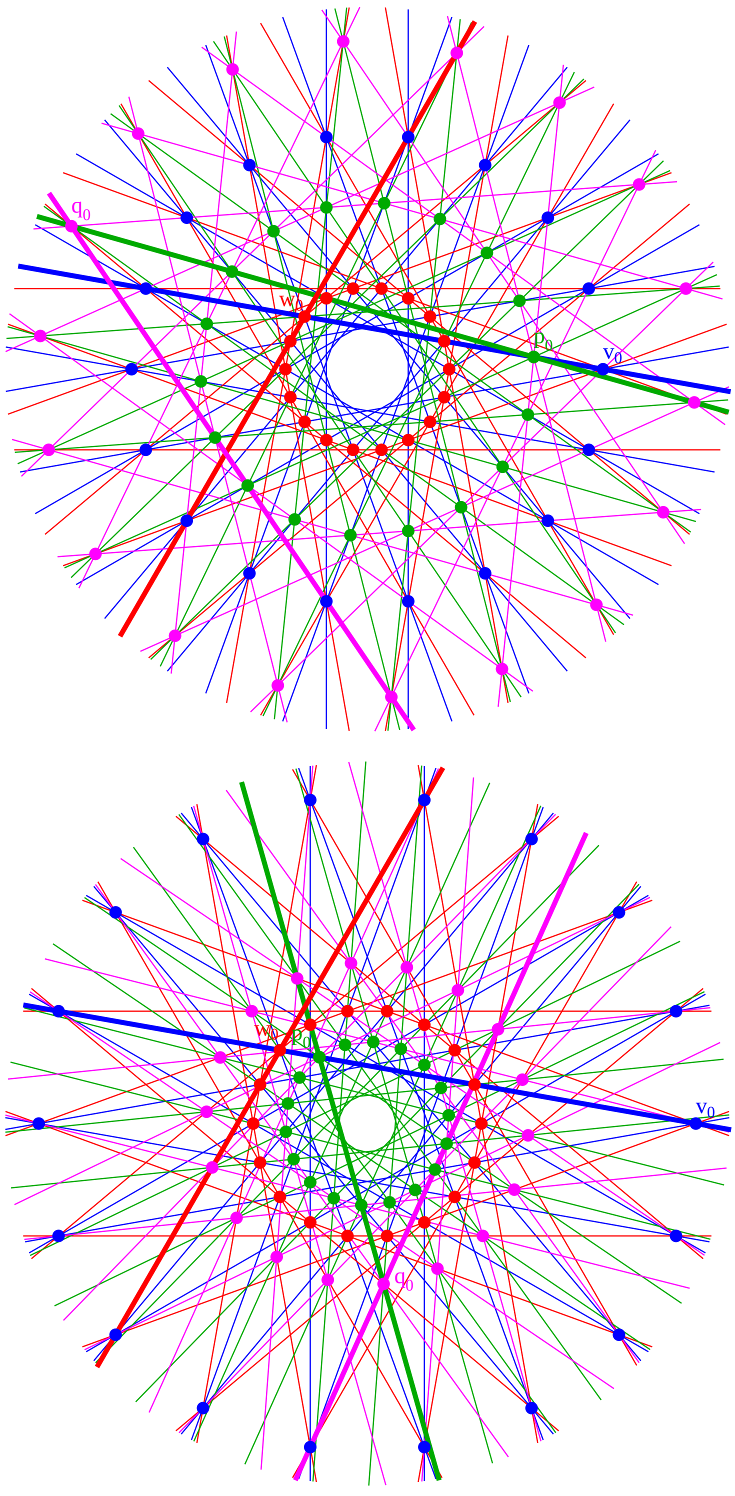


Figure 6: The configurations $\mathcal{C}(18,(8,6;1,7),17,0)$, with $\lambda_0 \approx 0.151564$ and $\mathcal{C}(18,(8,6;1,7),17,1)$, with $\lambda_1 \approx 0.5906625$.

different from the conventions in this paper. The configurations $\mathcal{C}(18,(8,6;1,7),17,0)$ and $\mathcal{C}(18,(8,6;1,7),17,1)$ are shown in Figure 6.

It appears to be nontrivial to determine for a given configuration $m\#(a,b;c,d)$ which values of k produce 5-configurations, which produce degenerate subfigurations, and which correspond to only complex solutions of Equation (6). This is related to an analogous open problem of completely characterizing chiral astral 3-configurations; see, e.g., [11, Section 2.7], which also reduces to determining the solutions to an equation formed by taking the determinant of three points which is quadratic in a continuous variable λ. With respect to that problem, Grünbaum commented that "the complete characterization of all possible symbols is, in principle, determinable by the non-negativity of the discriminant of that quadratic eqaution" but that in practice "...no amount of effort, on the computer or manually, was successful in explicitly describing the necessary and sufficient integer parameters..." [11, p. 116].

However, there are a few values of k for which it is easy to tell that $\mathcal{C}(m,(a,b;c,d),k,\chi)$ exists but is degenerate. For example, the points v_0, w_0 and v_a are collinear by the construction of the initial 2-astral configuration, and p_0 is chosen to lie on that line by construction. If $k=0$, then p_0, p_k and w_0 will certainly be collinear if λ is chosen so that $p_0 = w_0$.

3 Open Problems

1. Given a valid 4-configuration $m\#(a,b;c,d)$, for which k does $\mathcal{C}(m,(a,b;c,d),k,\chi)$ exist? For which k does $\mathcal{C}(m,(a,b;c,d),k,\chi)$ exist as a degenerate 5-configuration?

2. The configuration in Figure 5 and the two configurations in Figure 6 are self-polar. The two configurations in Figure 6 are isomorphic to each other (the coloring indicates the isomorphism), so they are dual to each other although they are not polar to each other. Are all configurations $\mathcal{C}(m,(a,b;c,d),k,\chi)$ self-polar? What can be said about the relationship between $\mathcal{C}(m,(a,b;c,d),k,0)$ and $\mathcal{C}(m,(a,b;c,d),k,1)$?

3. For a given m, how many distinct 5-configurations $\mathcal{C}(m,(a,b;c,d),k,\chi)$ are there? That is, are all the configurations different, for different k?

4. There are other h-astral configurations. Does this construction method generalize to produce other interesting configurations?

5. Find other families of 5-configurations with geometric symmetry. Are there families with dihedral symmetry?

The authors acknowledge the generous support of the Ursinus College Summer Fellows program in facilitating this research. They also appreciate the very helpful feedback provided by the anonymous referee.

References

[1] Berman, Leah Wrenn and Branko Grünbaum. Deletion constructions of symmetric 4-configurations. Part I. *Contributions to Discrete Mathematics.* In press.

[2] Berman, Leah Wrenn, Jürgen Bokowski, Branko Grünbaum and Tomaž Pisanski. Geometric "floral" configurations. *Canadian Mathematical Bulletin.* 52 (3): 327–341, 2009.

[3] Berman, Leah Wrenn and Jürgen Bokowski. Linear astral (n_5) configurations with dihedral symmetry. *European Journal of Combinatorics.* 29 (8): 1831-1842, 2008.

[4] Berman, Leah Wrenn. A new class of movable (n_4) configurations. *Ars Mathematica Contemporanea.* 1 (1): 44–50, 2008.

[5] Berman, Leah Wrenn. Movable (n_4) configurations. *The Electronic Journal of Combinatorics.* 13: #R104, 2006.

[6] Berman, Leah Wrenn. Some results on odd astral configurations. *The Electronic Journal of Combinatorics.* 13: #R27, 2006.

[7] Berman, Leah Wrenn. Even astral configurations. *The Electronic Journal of Combinatorics.* 11: #R37, 2004.

[8] Berman, Leah Wrenn. A characterization of astral (n_4) configurations. *Discrete and Computational Geometry.* 26 (4): 603 – 612, 2001.

[9] Betten, A., Brinkmann, G., Pisanski, T. Counting symmetric configurations v_3. *Discrete and Applied Mathematics* 99: 331 – 338, 2000.

[10] Boben, M. and T. Pisanski. Polycyclic configurations. *European Journal of Combinatorics*. 24 (4):431 – 457, 2003.

[11] Grünbaum, Branko. *Configurations of points and lines.* American Mathematical Society. 2009.

[12] Grünbaum, Branko. Configurations of points and lines. In *The Coxeter Legacy: Reflections and Projections.*, Chandler Davis and Erich W. Ellers, eds. American Mathematical Society, 179 – 225, 2006.

[13] Gropp, H. Configurations and their realization. *Discrete Mathematics* 174: 137 – 151, 1997.

[14] Grünbaum, B. Astral (n_4) configurations. *Geombinatorics* 9: 127 – 134, 2000,.

[15] Hilbert, D. and S. Cohn-Vossen. *Geometry and the Imagination*. American Mathematical Society, 1999.