Minimally Intersecting Set Partitions of Type B

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Abstract

Motivated by Pittel's study of minimally intersecting set partitions, we investigate minimally intersecting set partitions of type B. Our main result is a formula for the number of minimally intersecting r-tuples of B_n-partitions. As a consequence, it implies the formula of Benoumhani for the Dowling number in analogy to Dobiński's formula.

1 Introduction

This paper is primarily concerned with the meet structure of the lattice of type B_n partitions of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$. The lattice of type B_n set partitions has been studied by Reiner [8]. It can be regarded as a representation of the intersection lattice of the type B Coxeter arrangements, see Björner and Wachs [3], Björner and Brenti [2] and Humphreys [6].

A set partition of type B_n is a partition π of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ into blocks satisfying the following conditions:

(i) For any block B of π, its opposite $-B$ obtained by negating all elements of B is also a block of π;

(ii) There is at most one zero-block, which is defined to be a block B such that $B = -B$.

We call $\pm B$ a block pair of π if B is a non-zero-block of π. For example,

$$\pi_1 = \{\{\pm 1, \pm 2, \pm 5, \pm 8, \pm 12\},\ \pm\{3, 11\},\ \pm\{4, -7, 9, 10\},\ \pm\{6\}\}$$

is a B_{12}-partition consisting of 3 block pairs and the zero-block $\{\pm 1, \pm 2, \pm 5, \pm 8, \pm 12\}$.

Our main result is a formula for the number of r-tuples of minimally intersecting B_n-partitions. We have used similar ideas in Pittel [7], but the variable setting for type B does not seem to be a straightforward generalization.

Let us give a precise formulation of Pittel's results. Let Π_n be the lattice of partitions of $[n] = \{1, 2, \ldots, n\}$. The minimum element in Π_n is

$$\hat{0} = \{\{1\}, \{2\}, \ldots, \{n\}\}.$$

The partitions $\pi_1, \pi_2, \ldots, \pi_r$ are said to intersect minimally if

$$\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_r = \hat{0}.$$

Let π be a partition of the set $[n]$, and let $i_1, \ldots, i_k$ be the sizes of the blocks of π listed in any order. Given $l > 1$, the number $N(\pi, l)$ of partitions with exactly l blocks that minimally intersect π equals

$$N(\pi, l) = \frac{\mathbf{i}!}{l!} \left[\mathbf{x}^{\mathbf{i}}\right] \left(\prod_{\alpha \in [k]} (1 + x_\alpha) - 1 \right)^l, \tag{1.1}$$

where

$$\mathbf{i}! = \prod_{\alpha \in [k]} i_\alpha!,$$

and $\left[\mathbf{x}^{\mathbf{i}}\right]$ stands for the coefficient of $\mathbf{x}^{\mathbf{i}}$ in the power series expansion. As pointed out by Pittel, the expression (1.1) reduces to Dobiński's formula. In other words, setting $\pi = \hat{0}$ one obtains

$$B_n = e^{-1} \sum_{k \geqslant 0} \frac{k^n}{k}, \tag{1.2}$$

where B_n denotes the Bell number. Moreover, in view of (1.1), Pittel deduced that the number $N(\pi)$ of partitions that minimally intersect π equals

$$N(\pi) = \mathbf{i}! \left[\mathbf{x}^{\mathbf{i}}\right] \exp \left(\prod_{\alpha \in [k]} (1 + x_\alpha) - 1 \right). \tag{1.3}$$

Pittel also obtained the number $N_2(k)$ of ordered pairs (π, π') of minimally intersecting partitions such that π consists of exactly k blocks, that is,

$$N_2(k) = e^{-1} \frac{n!}{k!} [x^n] \sum_{l \geqslant 0} \frac{1}{l!} \left[(1 + x)^l - 1\right]^k. \tag{1.4}$$

Using the above formula, he further derived the following expression for the number N_{2n} of ordered pairs of minimally intersecting partitions

$$N_{n,2} = e^{-2} \sum_{k,l \geqslant 0} \frac{(kl)_n}{k! l!}, \tag{1.5}$$

where $(m)_n = m(m-1)\cdots(m-n+1)$ denotes the falling factorial. By the same method, Pittel generalized (1.5) and showed that the number $N_{n,r}$ of r-tuples ($r \geqslant 2$) of minimally intersecting partitions equals

$$N_{n,r} = \frac{1}{e^r} \sum_{k_1,\,\ldots,\,k_r \geqslant 0} \frac{(k_1\,k_2\,\cdots\,k_r)_n}{k_1!\,k_2!\,\cdots\,k_r!}. \tag{1.6}$$

Canfield [4] found a formula connecting the generating functions of $N_{n,r}$ and the r-th power of Bell numbers.

The set of partitions of type B on $\{\pm 1, \pm 2, \ldots, \pm n\}$ forms a lattice under refinement, denoted Π_n^B, with the minimal element

$$\hat{0}^B = \{\pm\{1\},\, \pm\{2\},\, \ldots,\, \pm\{n\}\}.$$

The B_n-partitions $\pi_1, \pi_2, \ldots, \pi_r$ are said to be minimally intersecting if

$$\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_r = \hat{0}^B.$$

We shall study the meet structure of Π_n^B in analogy with Pittel's formulas. Our main result is the following theorem.

Theorem 1.1 *Let $r \geqslant 2$. The number of minimally intersecting r-tuples $(\pi_1, \pi_2, \ldots, \pi_r)$ of B_n-partitions equals*

$$N_{n,r}^B = \frac{2^n}{e^{r/2}} \sum_{k_1,\,\ldots,\,k_r \geqslant 0} \frac{(f_r)_n}{(2k_1)!!\,(2k_2)!!\,\cdots\,(2k_r)!!}, \tag{1.7}$$

where

$$f_r = \frac{1}{2}\left(\prod_{t\in[r]} (2k_t + 1) - 1\right).$$

The proof of the above formula leads to a formula of Benoumhani [1] for the number of B_n-partitions, called the Dowling number [5]. This paper is organized as follows. In the next section, we derive type B analogues of the formulas from (1.1) to (1.6) and we give a proof of Theorem 1.1. In Section 3, we shall consider the corresponding problems with respect to B_n-partitions without zero-block.

2 Minimally intersecting B_n-partitions

The main objective of this section is to derive a formula for the number of minimally intersecting r-tuples of B_n-partitions. If $\pi \in \Pi_n^B$ has a zero-block $Z = \{\pm r_1, \pm r_2, \ldots, \pm r_k\}$, we say that Z is of half-size k. Let $\mathbf{j} = (j_1, j_2, \ldots, j_k)$ be a composition of n. Let π be a B_n-partition consisting of k block pairs and a zero-block of half-size i_0. We often assume that the block pairs of π are ordered subject to certain convention for the purpose of enumeration. We say that π is of type $(i_0; \mathbf{j})$ if the block pairs of π are ordered such that the i-th block pair is of length j_i.

We first consider the problem of counting the number of B_n-partitions with l block pairs which minimally intersect a given B_n-partition.

Theorem 2.1 *Let π be a B_n-partition consisting of a zero-block of half-size i_0 (allowing $i_0 = 0$) and k block pairs of sizes $i_1, i_2, \ldots, i_k$ ($k \geqslant 1$) listed in any order. For any $l \geqslant 1$, the number of B_n-partitions π' containing exactly l block pairs that minimally intersect π equals*

$$N^B(\pi; l) = \frac{\mathbf{i}!}{(2l-2i_0)!!} \sum_{\mathbf{i}'} \left[\mathbf{x}^{\mathbf{i}'}\right] \left(\prod_{\alpha \in [k]} (1+x_\alpha)^2 - 1 \right)^{l-i_0} \prod_{\alpha \in [k]} (1+x_\alpha)^{2i_0}, \tag{2.1}$$

where $\mathbf{i}'$ ranges over all vectors $(i'_1, i'_2, \ldots, i'_k)$ such that $i'_\alpha \in \{i_\alpha, i_\alpha - 1\}$ for any $\alpha \in [k]$.

For example, Π_2^B contains 6 partitions:

$$\hat{0}^B,\ \{\{\pm 1, \pm 2\}\},\ \{\pm\{1\}, \{\pm 2\}\},\ \{\pm\{2\}, \{\pm 1\}\},\ \{\pm\{1,2\}\},\ \{\pm\{1,-2\}\}.$$

Let $\pi = \{\pm\{1\}, \{\pm 2\}\}$. We have $i_0 = 1$, $k = 1$, and $i_1 = 1$. For $l = 1$, by (2.1),

$$N^B(\pi; 1) = \sum_{i=0}^{1} \left[x^i\right] (1+x)^2 = 3.$$

The three B_2-partitions which contain exactly 1 block pair and intersect π minimally are $\{\pm\{2\}, \{\pm 1\}\}$, $\{\pm\{1,2\}\}$, and $\{\pm\{1,-2\}\}$. Recall that Pittel [7] characterized the intersecting structure of two partitions in terms of 01-matrices. He used the coefficient

$$\left[\mathbf{x}^{\mathbf{i}}\mathbf{y}^{\mathbf{j}}\right] \prod_{\alpha \in [k],\, \beta \in [l]} (1 + x_\alpha y_\beta) \tag{2.2}$$

to represent the number of ways to assign 0 or 1 to all kl pairwise intersections of blocks of two minimally intersecting ordinary partitions. We will use a similar idea to deal with the intersecting structure of B_n-partitions.

Proof of Theorem 2.1. Let Z_1 be the zero-block of π, and $\pm B_1, \pm B_2, \ldots, \pm B_k$ the block pairs of π. Let Z_2 be the zero-block of π', and $\pm B'_1, \pm B'_2, \ldots, \pm B'_l$ the block pairs of π'.

To ensure that π and π' are minimally intersecting, it is necessary to characterize the intersecting relations for all pairs (B, B') where B is a block of π and B' is a block of π'. Since π and π' intersect minimally, we observe that each $B \cap B'$ contains at most one element, where both B and B' may be the zero-block. So we have four cases.

- $B = Z_1$ and $B' = Z_2$. We have $Z_1 \cap Z_2 = \emptyset$ since the cardinality of $Z_1 \cap Z_2$ is even.

- $B \neq Z_1$ and $B' = Z_2$. We introduce the variable z_2 to represent the zero-block Z_2, and the variable x_α to represent the block B_α. The intersection $B_\alpha \cap Z_2$ can be represented by $x_\alpha z_2$ if it is of cardinality 1. In this case, the intersection $(-B_\alpha) \cap Z_2$ can be ignored since
$$(-B_\alpha) \cap Z_2 = -(B_\alpha \cap Z_2)\,.$$
- $B = Z_1$ and $B' \neq Z_2$. We introduce the variable z_1 to represent the zero-block Z_1, and the variable w_β to represent the block B'_β. Then $Z_1 \cap B'_\beta$ can be represented by $z_1 w_\beta$ if it is of cardinality 1. In this case, the intersection $Z_1 \cap (-B'_\beta)$ can be disregarded since
$$Z_1 \cap (-B'_\beta) = -\left(Z_1 \cap B'_\beta\right).$$
- $B \neq Z_1$ and $B' \neq Z_2$. In this case, we introduce the variable y_β (resp. $\bar{y}_\beta$) to represent the block B'_β (resp. $-B'_\beta$). Then $B_\alpha \cap B'_\beta$ (resp. $B_\alpha \cap (-B'_\beta)$) can be represented by $x_\alpha y_\beta$ (resp. $x_\alpha \bar{y}_\beta$) if it is of cardinality 1. Note that it is not necessary to consider the intersection involving the block $-B_\alpha$ since
$$(-B_\alpha) \cap (\pm B'_\beta) = -\left(B_\alpha \cap (\mp B'_\beta)\right).$$

Combining the above four cases, we can represent the meet $\pi \wedge \pi'$ by
$$F(k;l) \prod_{\alpha \in [k]} (1 + x_\alpha z_2) \prod_{\beta \in [l]} (1 + z_1 w_\beta), \tag{2.3}$$
where
$$F(k;l) = \prod_{\alpha \in [k],\, \beta \in [l]} (1 + x_\alpha y_\beta)(1 + x_\alpha \bar{y}_\beta). \tag{2.4}$$

Notice that the expression (2.3) is analogous to
$$\prod_{\alpha \in [k],\, \beta \in [l]} (1 + x_\alpha y_\beta)$$
in (2.2). Now we are going to introduce an operator for (2.3) which corresponds to $\left[\mathbf{x}^{\mathbf{i}} \mathbf{y}^{\mathbf{j}}\right]$ in (2.2). In this way, we can express the number of ways to assign cardinalities 0 or 1 to all pairwise intersections of blocks of two minimally intersecting B_n-partitions.

Let j_0 be a nonnegative integer and $\mathbf{j} = (j_1, j_2, \ldots, j_l)$ a composition of $n - j_0$. Denote by $N^B(\pi; j_0, \mathbf{j})$ the number of B_n-partitions π' of type $(j_0; \mathbf{j})$ such that π' minimally meets π. In the above notation, we have
$$N^B(\pi; j_0, \mathbf{j}) = c \cdot \sum_{\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{j}} \left[\mathbf{x}^{\mathbf{i}} z_1^{i_0} z_2^{j_0} \mathbf{w}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}} \bar{\mathbf{y}}^{\mathbf{c}}\right] F(k;l) \prod_{\alpha \in [k]} (1 + x_\alpha z_2) \prod_{\beta \in [l]} (1 + z_1 w_\beta), \tag{2.5}$$
where
$$c = \mathbf{i}! \cdot \frac{(2i_0)!!}{(2l)!!}, \tag{2.6}$$

and

$$\mathbf{x} = (x_1, x_2, \dots, x_k), \qquad \mathbf{i} = (i_1, i_2, \dots, i_k), \qquad \mathbf{x}^{\mathbf{i}} = \prod_{\alpha\in[k]} x_\alpha^{i_\alpha};$$

$$\mathbf{w} = (w_1, w_2, \dots, w_l), \qquad \mathbf{a} = (a_1, a_2, \dots, a_l), \qquad \mathbf{w}^{\mathbf{a}} = \prod_{\beta\in[l]} w_\beta^{a_\beta};$$

$$\mathbf{y} = (y_1, y_2, \dots, y_l), \qquad \mathbf{b} = (b_1, b_2, \dots, b_l), \qquad \mathbf{y}^{\mathbf{b}} = \prod_{\beta\in[l]} y_\beta^{b_\beta};$$

$$\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_l), \qquad \mathbf{c} = (c_1, c_2, \dots, c_l), \qquad \bar{\mathbf{y}}^{\mathbf{c}} = \prod_{\beta\in[l]} \bar{y}_\beta^{c_\beta}.$$

Here we give a combinatorial explanation for the coefficient c in (2.6). In fact, for the partition π', by permuting the l block pairs or interchanging the two blocks in a common block pair, we still have the same partition. This explains the denominator $(2l)!!$. On the other hand, for any block B_α, every block of π' contains at most one element of B_α. Considering the assignment of an element to the intersection $B_\alpha \cap B'$, where B' is a block of π', we are led to the factor $\mathbf{i}!$. Similarly, the factor $(2i_0)!!$ is associated with the assignment of elements in Z_1 to the blocks of π'.

Denote by $\binom{S}{m}$ the collection of all m-subsets of S. Since

$$\left[z_2^{j_0}\right] \prod_{\alpha\in[k]} (1 + x_\alpha z_2) = \sum_{X\in\binom{[k]}{j_0}} \prod_{\alpha\in X} x_\alpha, \tag{2.7}$$

$$\left[z_1^{i_0}\right] \prod_{\beta\in[l]} (1 + z_1 w_\beta) = \sum_{Y\in\binom{[l]}{i_0}} \prod_{\beta\in Y} w_\beta, \tag{2.8}$$

substituting (2.7) and (2.8) into (2.5), we obtain that

$$\begin{aligned} N^B(\pi; j_0, \mathbf{j}) &= c \cdot \sum_{\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{j}} \left[\mathbf{x}^{\mathbf{i}}\mathbf{w}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}\bar{\mathbf{y}}^{\mathbf{c}}\right] \left(\sum_{Y\in\binom{[l]}{i_0}} \prod_{\beta\in Y} w_\beta \right) \left(\sum_{X\in\binom{[k]}{j_0}} \prod_{\alpha\in X} x_\alpha \right) F(k; l) \\ &= c \cdot \sum_{X, Y, \mathbf{b}} \left[\mathbf{y}^{\mathbf{b}} \prod_{\alpha\in[k]} x_\alpha^{i_\alpha - \chi(\alpha\in X)} \prod_{\beta\in[l]} \bar{y}_\beta^{j_\beta - b_\beta - \chi(\beta\in Y)} \right] F(k; l), \end{aligned}$$

where χ is defined by $\chi(P) = 1$ if P is true, and $\chi(P) = 0$ otherwise. Therefore the number of B_n-partitions π' containing exactly l block pairs that intersect π minimally equals

$$N^B(\pi; l) = \sum_{\substack{j_0+j_1+\cdots+j_l=n \\ j_0\geqslant 0,\, j_1,\dots,j_l\geqslant 1}} N^B(\pi; j_0, \mathbf{j}) = c \cdot \sum_{j_0,\, X} \left[\prod_{\alpha} x_\alpha^{i_\alpha - \chi(\alpha\in X)} \right] \sum_{\substack{j_0+j_1+\cdots+j_l=n \\ j_1,\dots,j_l\geqslant 1}} f(\mathbf{j}), \tag{2.9}$$

where

$$f(\mathbf{j}) = \sum_{Y, \mathbf{b}} \left[\mathbf{y}^{\mathbf{b}} \prod_{\beta} \bar{y}_\beta^{j_\beta - b_\beta - \chi(\beta\in Y)} \right] F(k; l).$$

In view of the expression (2.4), the total degree of x_α in $F(k;l)$ agrees with the sum of the degrees of y_β and $\bar{y}_\beta$. Concerning (2.9), we find

$$\sum_{\alpha\in[k]} i_\alpha - \chi(\alpha\in X) = \sum_{\beta\in[l]} b_\beta + (j_\beta - b_\beta - \chi(\beta\in Y)),$$

that is,

$$j_0 + j_1 + \cdots + j_l = i_0 + i_1 + \cdots + i_k = n.$$

So we may drop this condition in the inner summation of (2.9). In order to reduce the factor $\sum_{j_1,\ldots,j_l\geqslant 1} f(\mathbf{j})$, we introduce

$$S(A) = \sum_{\substack{j_1,\ldots,j_l\geqslant 0\\ j_\beta=0 \text{ if } \beta\notin A}} f(\mathbf{j}) = \sum_{Y}\sum_{\substack{b_\gamma,j_\gamma\geqslant 0\\ \gamma\in A}} \left[\prod_{\gamma\in A} y_\gamma^{b_\gamma}\bar{y}_\gamma^{j_\gamma-b_\gamma-\chi(\gamma\in Y)}\right] F(k;A)$$

for any $A\subseteq[l]$, where

$$F(k;A) = \prod_{\alpha\in[k],\,\gamma\in A} (1+x_\alpha y_\gamma)(1+x_\alpha\bar{y}_\gamma).$$

Since j_γ and b_γ run over all nonnegative integers, the exponent $j_\gamma - b_\gamma - \chi(\gamma\in Y)$ can be considered as a summation index. It follows that

$$S(A) = \sum_{Y\in\binom{A}{i_0}}\sum_{b_\gamma,c_\gamma\geqslant 0,\,\gamma\in A}\left[\prod_{\gamma\in A} y_\gamma^{b_\gamma}\bar{y}_\gamma^{c_\gamma}\right]F(k;A) = \binom{|A|}{i_0}\prod_{\alpha\in[k]}(1+x_\alpha)^{2|A|}.$$

By the principle of inclusion-exclusion, we have

$$\begin{aligned}\sum_{j_1,\ldots,j_l\geqslant 1} f(\mathbf{j}) &= \sum_{A\subseteq[l]}(-1)^{l-|A|}S(A) = \sum_m \binom{l}{m}(-1)^{l-m}\binom{m}{i_0}\prod_{\alpha\in[k]}(1+x_\alpha)^{2m}\\ &= \binom{l}{i_0}\prod_{\alpha\in[k]}(1+x_\alpha)^{2i_0}\left(\prod_{\alpha\in[k]}(1+x_\alpha)^2-1\right)^{l-i_0}.\end{aligned}$$

Now, employing (2.9) we find that $N^B(\pi;l)$ equals

$$\frac{\mathbf{i}!}{(2l-2i_0)!!}\sum_{X\subseteq[k]}\left[\prod_{\alpha\in[k]} x_\alpha^{i_\alpha-\chi(\alpha\in X)}\right]\prod_{\alpha\in[k]}(1+x_\alpha)^{2i_0}\left(\prod_{\alpha\in[k]}(1+x_\alpha)^2-1\right)^{l-i_0}, \qquad (2.10)$$

which can be rewritten in the form of (2.1). This completes the proof. ∎

Summing (2.1) over $l \geqslant i_0$, we obtain the following formula.

Corollary 2.2 *The number $N^B(\pi)$ of B_n-partitions that minimally intersect π is*

$$N^B(\pi) = \frac{\mathbf{i}!}{\sqrt{e}} \sum_{\mathbf{i}'} \left[\mathbf{x}^{\mathbf{i}'}\right] F(\mathbf{x}), \tag{2.11}$$

where

$$F(\mathbf{x}) = \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_0} \right) \exp\left(\frac{1}{2} \prod_{\alpha \in [k]} (1 + x_\alpha)^2 \right). \tag{2.12}$$

Setting $\pi = \hat{0}^B$ in (2.11), we get $i_0 = 0$ and

$$N^B(\hat{0}^B) = \frac{1}{\sqrt{e}} \sum_{i'_\alpha \in \{0,1\}} \left[x_1^{i'_1} \cdots x_n^{i'_n}\right] \sum_{j \geqslant 0} \frac{1}{(2j)!!} \prod_{\alpha=1}^{n} (1 + x_\alpha)^{2j}.$$

This immediately reduces to Benoumhani's formula for the Dowling number

$$\left|\Pi_n^B\right| = \frac{1}{\sqrt{e}} \sum_{k \geqslant 0} \frac{(2k+1)^n}{(2k)!!}, \tag{2.13}$$

in analogy to Dobiński's formula (1.2). In fact, the number $N^B(\pi)$ can also be written as an infinite sum.

Corollary 2.3

$$N^B(\pi) = \frac{1}{\sqrt{e}} \sum_{j \geqslant 0} \frac{(2i_0 + 2j + 1)!^k}{(2j)!!} \prod_{\alpha \in [k]} \frac{1}{(2i_0 + 2j + 1 - i_\alpha)!}. \tag{2.14}$$

Proof. From (2.12) it follows that

$$F(x) = \sum_{j \geqslant 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2(i_0 + j)}.$$

Hence

$$\begin{aligned} N^B(\pi) &= \frac{\mathbf{i}!}{\sqrt{e}} \sum_{j \geqslant 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} \left(\binom{2(i_0+j)}{i_\alpha} + \binom{2(i_0+j)}{i_\alpha - 1} \right) \\ &= \frac{\mathbf{i}!}{\sqrt{e}} \sum_{j \geqslant 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} \binom{2(i_0+j)+1}{i_\alpha}, \end{aligned}$$

which gives (2.14). This completes the proof. ∎

Corollary 2.4 *Let $N^B_{n,2}(i_0;k)$ denote the number of ordered pairs (π,π') of minimally intersecting B_n-partitions such that π consists of exactly k block pairs and a zero-block of half-size i_0 (allowing $i_0=0$). Then*

$$N^B_{n,2}(i_0;k) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}}\left[x^{n-i_0}\right]\sum_{j\geqslant 0}\frac{1}{(2j)!!}\left((1+x)^{2i_0+2j+1}-1\right)^k. \tag{2.15}$$

Proof. By a simple combinatorial argument, we see that the number of B_n-partitions of type $(i_0;i_1,\ldots,i_k)$ equals

$$c = \binom{n}{i_0,i_1,\ldots,i_k}\frac{2^{n-i_0-k}}{k!} = \frac{(2n)!!}{(2i_0)!!(2k)!!}\cdot\frac{1}{\mathbf{i}!}.$$

Thus by (2.11), we have

$$N^B_{n,2}(k) = \sum_{\substack{i_0+i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\geqslant 1}} c\cdot N^B(\pi) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}}\sum_{\substack{i_0+i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\geqslant 1}}\sum_{\mathbf{i}'}\left[\mathbf{x}^{\mathbf{i}'}\right]F(\mathbf{x}). \tag{2.16}$$

For any $A\subseteq[k]$, consider

$$S(A) = \sum_{\substack{i_0+i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\geqslant 0\\ i_\alpha=0 \text{ if } \alpha\notin A}}\sum_{\mathbf{i}'}\left[\mathbf{x}^{\mathbf{i}'}\right]F(\mathbf{x}) = \sum_{\substack{i_0+\sum_{\alpha\in A} i_\alpha=n\\ i_\alpha\geqslant 0,\,\alpha\in A}}\sum_{\mathbf{i}'|_A}\left[\mathbf{x}^{\mathbf{i}'}\big|_A\right]F\left(\mathbf{x}\big|_A\right),$$

where $\mathbf{x}\big|_A$ (resp. $\mathbf{i}'|_A$) denotes the vector obtained by removing all x_α (resp. i'_α) such that $\alpha\notin A$ from the vector $\mathbf{x}$ (resp. $\mathbf{i}'$). Let t be the number of α's such that $i'_\alpha = i_\alpha - 1$ in the inner summation. Noting that

$$F\left(\mathbf{x}\big|_A\right) = \left(\prod_{\alpha\in A}(1+x_\alpha)^{2i_0}\right)\exp\left(\frac{1}{2}\prod_{\alpha\in A}(1+x_\alpha)^2\right),$$

$S(A)$ can be written as

$$\begin{aligned} S(A) &= \left(\sum_t\binom{|A|}{t}\left[x^{n-i_0-t}\right]\right)(1+x)^{2i_0|A|}\exp\left(\frac{1}{2}(1+x)^{2|A|}\right)\\ &= \left[x^{n-i_0}\right](1+x)^{(2i_0+1)|A|}\exp\left(\frac{1}{2}(1+x)^{2|A|}\right). \end{aligned}$$

In view of the principle of inclusion-exclusion, we deduce from (2.16) that

$$N^B_{n,2}(k) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}}\sum_{A\subseteq[k]}(-1)^{k-|A|}S(A),$$

which gives (2.15). This completes the proof. ∎

Summing over $0 \leqslant k \leqslant n - i_0$ and $0 \leqslant i_0 \leqslant n$, we obtain the number of ordered pairs of minimally intersecting B_n-partitions.

Corollary 2.5 *The number $N^B_{n,2}$ of ordered pairs (π, π') of minimally intersecting B_n-partitions is given by*

$$N^B_{n,2} = \frac{2^n}{e} \sum_{k,l \geqslant 0} \frac{(2kl + k + l)_n}{(2k)!!(2l)!!}.$$

For example, $N^B_{1,2} = 3$, $N^B_{2,2} = 23$, $N^B_{3,2} = 329$. For general r, we have Theorem 1.1. We now proceed to give a proof as a direct generalization of the proof of Corollary 2.5.

Proof of Theorem 1.1. For any $s \in [r]$, let i_s be an nonnegative integer and $\mathbf{j}_s = (j_{s,1}, j_{s,2}, \ldots, j_{s,k_s})$ be a composition of n. Let π_s be a B_n-partition of type $(i_s; \mathbf{j}_s)$, with the zero-block Z_s and block pairs

$$\pm B_{s,1}, \ \pm B_{s,2}, \ \ldots, \ \pm B_{s,k_s}. \tag{2.17}$$

Suppose that $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting. Let B_s be a block of π_s $(1 \leqslant s \leqslant r)$. It may be either the zero-block Z_s or any one of the $2k_s$ blocks in (2.17). We shall consider each intersection

$$B_1 \cap B_2 \cap \cdots \cap B_r. \tag{2.18}$$

Since $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting, each intersection (2.18) contains at most one element. We consider the number $t \in \{0, 1, \ldots, r+1\}$ such that

$$B_1 = Z_1, \ B_2 = Z_2, \ \ldots, \ B_{t-1} = Z_{t-1}, \ B_t \neq Z_t.$$

In particular, the case $t = 0$ (resp. $t = r + 1$) implies that all B_s's are non-zero-blocks (resp. zero-blocks). Note that

$$\bigcap_{s \in [t-1]} Z_s \cap (-B_t) = -\left(\bigcap_{s \in [t-1]} Z_s \cap B_t \right).$$

So the intersection in the form of (2.18) can be excluded when $B_t = -B_{t,i}$ for some $i \in [k_t]$.

We now assume that $B_t = B_{t,i}$ for some i. We use the variable z_s to represent Z_s for all $s \in [r]$, and use $x_{t,i}$ to represent the block $B_{t,i}$. For $p \geqslant t + 1$, we use the variable $y_{p,i}$ (resp. $\bar{y}_{p,i}$) to represent the block $B_{p,i}$ (resp. $-B_{p,i}$), where $i \in [k_p]$. So we can represent the intersection property by a factor

$$f_t = 1 + z_1 \cdots z_{t-1} x_{t,\alpha_t} Y_{t+1} \cdots Y_r, \tag{2.19}$$

where $\alpha_t \in [k_t]$ and

$$Y_p \in \left\{ z_p, \ y_{p,1}, \ \bar{y}_{p,1}, \ \ldots, \ y_{p,k_p}, \ \bar{y}_{p,k_p} \right\}$$

for any $p \geqslant t+1$. Let

$$\mathbf{x}_s = (x_{s,1},\, \ldots,\, x_{s,k_s}), \qquad \mathbf{a}_s = (a_{s,1},\, \ldots,\, a_{s,k_s}), \qquad \mathbf{x}_s^{\mathbf{a}_s} = \prod_{i\in[k_s]} x_{s,i}^{a_{s,i}};$$
$$\mathbf{y}_s = (y_{s,1},\, \ldots,\, y_{s,k_s}), \qquad \mathbf{b}_s = (b_{s,1},\, \ldots,\, b_{s,k_s}), \qquad \mathbf{y}_s^{\mathbf{b}_s} = \prod_{i\in[k_s]} y_{s,i}^{b_{s,i}};$$
$$\bar{\mathbf{y}}_s = (\bar{y}_{s,1},\, \ldots,\, \bar{y}_{s,k_s}), \qquad \mathbf{c}_s = (c_{s,1},\, \ldots,\, c_{s,k_s}), \qquad \bar{\mathbf{y}}_s^{\mathbf{c}_s} = \prod_{i\in[k_s]} \bar{y}_{s,i}^{c_{s,i}}.$$

Denote by $N^B(\pi_1; i_2, \mathbf{j}_2; \ldots; i_r, \mathbf{j}_r)$ the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of B_n-partitions such that π_s $(2 \leqslant s \leqslant r)$ is of type $(i_s, \mathbf{j}_s)$ and $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally. In the notation of f_t in (2.19), we get

$$N^B(\pi_1; i_2, \mathbf{j}_2; \ldots; i_r, \mathbf{j}_r) = c\left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{\substack{\mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s=\mathbf{j}_s \\ 2\leqslant s\leqslant r}} \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_r,$$

where

$$c = \mathbf{j}_1! \cdot (2i_1)!! \prod_{2\leqslant s\leqslant r} (2k_s)!!^{-1}, \tag{2.20}$$
$$F_r = \prod_{t\in[r]} \prod_{\alpha_t\in[k_t]} \prod_{\substack{Y_p\in\left\{z_p, y_{p,1}, \bar{y}_{p,1}, \ldots, y_{p,k_p}, \bar{y}_{p,k_p}\right\} \\ t+1\leqslant p\leqslant r}} f_t.$$

The value of the coefficient c in (2.20) can be explained similar to the one in (2.6). We omit the explanation here.

Now, let $N^B(\pi_1, k_2, \ldots, k_r)$ be the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of B_n-partitions such that π_s contains exactly k_s block pairs and $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally. Then

$$N^B(\pi_1, k_2, \ldots, k_r) = \sum_{\substack{i_s\geqslant 0,\, j_{s,1}, \ldots, j_{s,k_s}\geqslant 1 \\ j_{s,1}+\cdots+j_{s,k_s}+i_s=n}} N^B(\pi_1; i_2, \mathbf{j}_2; \ldots; i_r, \mathbf{j}_r). \tag{2.21}$$

We claim that the conditions $j_{s,1} + \cdots + j_{s,k_s} + i_s = n$ can be dropped in the above summation. In fact, for any $i \in \{1, 2, \ldots, r\}$, the sum of the degrees of $\mathbf{x}_i$, $\mathbf{y}_i$, $\bar{\mathbf{y}}_i$, and z_i is 0 or 1 in the factor f_t. More importantly, this sum is independent of i. In particular, the sum for $i = 1$ equals the sum for any $2 \leqslant s \leqslant r$, that is,

$$j_{s,1} + \cdots + j_{s,k_s} + i_s = j_{1,1} + \cdots + j_{1,k_1} + i_1 = n. \tag{2.22}$$

Hence we can ignore the conditions (2.22) in (2.21). This implies that

$$N^B(\pi_1, k_2, \ldots, k_r) = c\left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{\substack{i_s\geqslant 0,\, \mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s\geqslant \mathbf{1} \\ 2\leqslant s\leqslant r}} \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_r,$$

where $\mathbf{a}_s + \mathbf{b}_s + \mathbf{c}_s \geqslant \mathbf{1}$ indicates that $a_{s,h_s} + b_{s,h_s} + c_{s,h_s} \geqslant 1$ for any $1 \leqslant h_s \leqslant k_s$. We will compute $\sum \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_r$ for $s = 2, 3, \ldots, r$ by the following procedure. First, for $s = 2$, we have

$$\sum_{i_2 \geqslant 0,\, \mathbf{a}_2+\mathbf{b}_2+\mathbf{c}_2 \geqslant \mathbf{1}} \left[\mathbf{x}_2^{\mathbf{a}_2}\mathbf{y}_2^{\mathbf{b}_2}\bar{\mathbf{y}}_2^{\mathbf{c}_2} z_2^{i_2}\right] F_r = \sum_{l_2} \binom{k_2}{l_2} (-1)^{k_2-l_2} F_{r,2},$$

where $F_{r,2}$ equals

$$\prod_{\alpha_1, Y_p} (1 + x_{1,\alpha_1} Y_3 \cdots Y_r)^{2l_2+1} \prod_{Y_p} (1 + z_1 Y_3 \cdots Y_r)^{l_2} \prod_{t \geqslant 3,\, \alpha_t,\, Y_p} (1 + z_1 z_3 \cdots z_{t-1} x_{t,\alpha_t} Y_{t+1} \cdots Y_r).$$

So $N^B(\pi_1, k_2, \ldots, k_r)$ equals

$$c \left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{l_2} \binom{k_2}{l_2} (-1)^{k_2-l_2} \sum_{\substack{i_s \geqslant 0,\, \mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s \geqslant \mathbf{1} \\ 3 \leqslant s \leqslant r}} \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_{r,2}. \tag{2.23}$$

To compute the inner summation, let

$$g_s = \frac{1}{2} \left(\prod_{2 \leqslant i \leqslant s} (2l_i + 1) - 1 \right).$$

For any $s \geqslant 2$, it is clear that

$$(2l_{s+1} + 1) g_s + l_{s+1} = g_{s+1}.$$

Starting with (2.23), we can continue the above procedure to deduce that for any $2 \leqslant h \leqslant r - 1$, $N^B(\pi_1, k_2, \ldots, k_r)$ equals

$$c \left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{l_2, \ldots, l_h} \prod_{2 \leqslant i \leqslant h} \binom{k_i}{l_i} (-1)^{k_i-l_i} \sum_{\substack{i_s \geqslant 0,\, \mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s \geqslant \mathbf{1} \\ h+1 \leqslant s \leqslant r}} \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_{r,h},$$

where

$$\begin{aligned} F_{r,h} = & \prod_{\alpha_1, Y_p} (1 + x_{1,\alpha_1} Y_{h+1} \cdots Y_r)^{\prod_{2 \leqslant i \leqslant h} (2l_i+1)} \prod_{Y_p} (1 + z_1 Y_{h+1} \cdots Y_r)^{g_h} \\ & \cdot \prod_{t \geqslant h+1,\, \alpha_t,\, Y_p} (1 + z_1 z_{h+1} \cdots z_{t-1} x_{t,\alpha_t} Y_{t+1} \cdots Y_r). \end{aligned}$$

In particular, for $h = r - 1$, we have

$$N^B(\pi_1, k_2, \ldots, k_r) = c \left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{l_2, \ldots, l_{r-1}} \left(\prod_{2 \leqslant i \leqslant r-1} \binom{k_i}{l_i} (-1)^{k_i-l_i} \right) G, \tag{2.24}$$

where

$$G = \sum_{\mathbf{a}_r+\mathbf{b}_r+\mathbf{c}_r \geqslant \mathbf{1}} \left[\mathbf{x}_r^{\mathbf{a}_r}\mathbf{y}_r^{\mathbf{b}_r}\bar{\mathbf{y}}_r^{\mathbf{c}_r}\right] \prod_{\alpha_1, Y_p} (1+x_{1,\alpha_1})^{\prod_{2\leqslant i\leqslant r-1}(2l_i+1)} \prod_{Y_p} (1+z_1)^{g_{r-1}} \prod_{\alpha_r} (1+z_1 x_{r,\alpha_r})$$

$$= \sum_{l_r} \binom{k_r}{l_r} (-1)^{k_r-l_r} (1+z_1)^{g_r} \prod_{\alpha_1} (1+x_{1,\alpha_1})^{\prod_{2\leqslant i\leqslant r}(2l_i+1)}.$$

Since the number of B_n-partitions of type $\mathbf{j}_1$ equals

$$c' = \binom{n}{i_1}\binom{n-i_1}{\mathbf{j}_1}\frac{2^{n-i_1-k_1}}{k_1!} = \frac{(2n)!!}{(2i_1)!!(2k_1)!!\mathbf{j}_1!},$$

by (2.24), we obtain

$$N_{n,r}^B = \sum_{\substack{j_{1,1},\ldots,j_{1,k_1}\geqslant 1 \\ i_1+j_{1,1}+\cdots+j_{1,k_1}=n}} c' \sum_{k_2,\ldots,k_r} N^B(\pi_1, k_2, \ldots, k_r)$$

$$= (2n)!! \sum_{\substack{k_2,\ldots,k_r \\ l_2,\ldots,l_r}} \left(\prod_{2\leqslant s\leqslant r} \binom{k_s}{l_s} \frac{(-1)^{k_s-l_s}}{(2k_s)!!}\right) \sum_{i_1,k_1} \frac{1}{(2k_1)!!} \left[z_1^{i_1}\right] (1+z_1)^{g_r} H, \tag{2.25}$$

where

$$H = \sum_{\substack{i_1+j_{1,1}+\cdots+j_{1,k_1}=n \\ j_{1,1},\, j_{1,2},\, \ldots,\, j_{1,k_1}\geqslant 1}} \left[\mathbf{x}_1^{\mathbf{j}_1}\right] \prod_{\alpha_1} (1+x_{1,\alpha_1})^{\prod_{2\leqslant i\leqslant r}(2l_i+1)}$$

$$= \sum_{l_1} \binom{k_1}{l_1} (-1)^{k_1-l_1} \left[x^{n-i_1}\right] (1+x)^{l_1 \prod_{2\leqslant i\leqslant r}(2l_i+1)}.$$

Using the identity

$$\sum_k \binom{k}{l} \frac{(-1)^{k-l}}{(2k)!!} = \frac{e^{-1/2}}{(2l)!!}, \tag{2.26}$$

we can simplify the summation over $k_1, k_2, \ldots, k_r \geqslant 0$ in (2.25) in the following way.

$$N_{n,r}^B = (2n)!! \sum_{\substack{k_1,k_2,\ldots,k_r \\ l_1,l_2,\ldots,l_r}} \left(\prod_{t\in[r]} \binom{k_t}{l_t} \frac{(-1)^{k_t-l_t}}{(2k_t)!!}\right) \sum_{i_1} \left[x^{n-i_1} z_1^{i_1}\right] (1+z_1)^{g_r} (1+x)^{l_1 \prod_{2\leqslant i\leqslant r}(2l_i+1)}$$

$$= \frac{(2n)!!}{e^{r/2}} \sum_{l_1,l_2,\ldots,l_r} \frac{1}{(2l_1)!!(2l_2)!!\cdots(2l_r)!!} \left[x^n\right] (1+x)^{g_r + l_1 \prod_{2\leqslant i\leqslant r}(2l_i+1)}. \tag{2.27}$$

To further simplify the above summation, we observe that

$$g_r + l_1 \prod_{2\leqslant i\leqslant r} (2l_i+1) = \frac{1}{2}\left(\prod_{t\in[r]} (2l_t+1) - 1\right). \tag{2.28}$$

Substituting (2.28) into (2.27), we arrive at (1.7). This completes the proof. ∎

For example, we have $N_{1,r} = 2^r - 1$ and $N_{2,3}^B = 187$.

3 Minimally intersecting B_n-partitions without zero-block

In this section, we consider B_n-partitions without zero-block and give analogous results for the minimally intersecting problems which was investigated in the last section. Clearly B_n-partitions without zero-block form a meet-semilattice under refinement. The minimal B_n-partition without zero-block is still $\hat{0}^B$. We will omit the redundant proofs.

Inspecting the proof of Theorem 2.1, we can restrict our attention to the B_n-partitions without zero-block by setting $i_0 = 0$ and $X = \emptyset$ in (2.10). Concretely speaking, let π be a B_n-partition consisting of k block pairs of sizes $i_1, i_2, \ldots, i_k$ listed in any order. For a given $l \geqslant 1$, the number $N^D(\pi; l)$ of B_n-partitions π' consisting of l block pairs, which intersect π minimally, is equal to

$$N^D(\pi; l) = \frac{\mathbf{i}!}{(2l)!!} \left[\mathbf{x}^{\mathbf{i}}\right] \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^l. \tag{3.1}$$

The number of B_n-partitions without zero-block that intersect π minimally is given by

$$N^D(\pi) = \frac{\mathbf{i}!}{\sqrt{e}} \left[\mathbf{x}^{\mathbf{i}}\right] \exp\left(\frac{1}{2} \prod_{\alpha \in [k]} (1 + x_\alpha)^2 \right). \tag{3.2}$$

For example, let $n = 3$, $\pi = \{\pm\{2\}, \pm\{1, -3\}\}$ and $l = 2$. Then (3.1) yields $N^D(\pi; 2) = 5$. In fact, the B_n-partitions consisting of 2 block pairs which intersect π minimally are exactly the 5 partitions consisting of two block pairs except for π itself.

Let N_n be the number of B_n-partitions without zero-block. Taking $\pi = \hat{0}^B$ in (3.2), we obtain that

$$N_n = \frac{1}{\sqrt{e}} \sum_{k \geqslant 0} \frac{(2k)^n}{(2k)!!}. \tag{3.3}$$

Let $N_n(k)$ denote the number of B_n-partitions containing k block pairs but no zero-block. It should be noted that the formula (3.3) can be easily deduced from the relation

$$N_n(k) = 2^{n-k} S(n, k), \tag{3.4}$$

where $S(n, k)$ are the Stirling numbers of the second kind, and the following identity on the Bell polynomials [9, 10]:

$$\sum_{k=0}^{n} S(n, k) x^k = \frac{1}{e^x} \sum_{k \geqslant 0} \frac{k^n}{k!} x^k.$$

Inspecting the proof of Corollary 2.4, we obtain the following result. Let $N^D_{n,2}(k)$ denote the number of ordered pairs (π, π') of minimally intersecting B_n-partitions without zero-block such that π consists of exactly k block pairs. Then

$$N^D_{n,2}(k) = \frac{(2n)!!}{(2k)!!\sqrt{e}} \left[x^n\right] \sum_{j \geqslant 0} \frac{1}{(2j)!!} \left[(1 + x)^{2j} - 1\right]^k. \tag{3.5}$$

The number $N^D_{n,2}$ of ordered pairs (π, π') of minimally intersecting B_n-partitions without zero-block is given by

$$N^D_{n,2} = \frac{2^n}{e} \sum_{k,\, l \geqslant 0} \frac{(2kl)_n}{(2k)!!\,(2l)!!}. \tag{3.6}$$

For example, $N^D_{1,2} = 1$, $N^D_{2,2} = 7$, $N^D_{3,2} = 75$.

The following theorem is an analogue of Theorem 1.1 with respect to the meet-semilattice of B_n-partitions without zero-block.

Theorem 3.1 *For $r \geqslant 2$, the number of minimally intersecting r-tuples $(\pi_1, \pi_2, \ldots, \pi_r)$ of B_n-partitions without zero-block equals*

$$N^D_{n,r} = \frac{2^n}{e^{r/2}} \sum_{k_1,\, \ldots,\, k_r \,\geqslant 0} \frac{(2^{r-1}\, k_1\, k_2\, \cdots\, k_r)_n}{(2k_1)!!\,(2k_2)!!\, \cdots\, (2k_r)!!}. \tag{3.7}$$

Proof. Let $1 \leqslant t \leqslant r$. Let $\mathbf{j}_t = (j_{t,1}, j_{t,2}, \ldots, j_{t,k_t})$ be a composition of n. Assume that π_t is of type $(0; \mathbf{j}_t)$. Let $N^D(\pi_1, \mathbf{j}_2, \ldots, \mathbf{j}_r)$ be the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of such B_n-partitions such that $(\pi_1, \pi_2, \ldots, \pi_r)$ is minimally intersecting. By the argument in the proof of Theorem 2.1, we find

$$N^D(\pi_1, \mathbf{j}_2, \ldots, \mathbf{j}_r) = c \cdot \left[\mathbf{x}^{\mathbf{j}_1}\right] \sum_{\mathbf{b}_s + \mathbf{c}_s = \mathbf{j}_s} \left[\mathbf{y}_2^{\mathbf{b}_2} \bar{\mathbf{y}}_2^{\mathbf{c}_2} \cdots \mathbf{y}_r^{\mathbf{b}_r} \bar{\mathbf{y}}_r^{\mathbf{c}_r}\right] f(\mathbf{j}), \tag{3.8}$$

where

$$c = \mathbf{j}_1! \prod_{2 \leqslant s \leqslant r} (2k_s)!!^{-1},$$

$$f(\mathbf{j}) = \prod_{\substack{\alpha \in [k_1] \\ Y_s \in \left\{ y_{s,1},\, \bar{y}_{s,1},\, \ldots,\, y_{s,k_s},\, \bar{y}_{s,k_s} \right\}}} \left(1 + x_\alpha Y_2 Y_3 \cdots Y_r\right).$$

Let $N^D(\pi_1, k_2, \ldots, k_r)$ be the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of B_n-partitions such that π_s consists of k_s block pairs, and $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting. It follows from (3.8) that

$$\begin{aligned} N^D(\pi_1, k_2, \ldots, k_r) &= c \cdot \left[\mathbf{x}^{\mathbf{j}_1}\right] \sum_{\mathbf{b}_s + \mathbf{c}_s = \mathbf{j}_s \geqslant \mathbf{1}} \left[\mathbf{y}_2^{\mathbf{b}_2} \cdots \bar{\mathbf{y}}_r^{\mathbf{c}_r}\right] f(\mathbf{j}) \\ &= \mathbf{j}_1! \sum_{l_2, \ldots, l_r} \left(\left[\mathbf{x}^{\mathbf{j}_1}\right] \prod_{\alpha \in [k_1]} (1 + x_\alpha)^{2^{r-1} l_2 \cdots l_r} \right) \prod_{2 \leqslant s \leqslant r} \binom{k_s}{l_s} \frac{(-1)^{k_s - l_s}}{(2k_s)!!}. \end{aligned}$$

Consequently,

$$\begin{aligned} N^D_{n,r} &= \sum_{k_1} \frac{1}{(2k_1)!!} \sum_{\substack{j_{1,1} + \cdots + j_{1,k_1} = n \\ j_{1,1}, \ldots, j_{1,k_1} \geqslant 1}} \frac{2^n n!}{\mathbf{j}_1!} \sum_{k_2, \ldots, k_r} N^D(\pi_1, k_2, \ldots, k_r) \\ &= (2n)!! \sum_{\substack{k_1, k_2, \ldots, k_r \\ l_1, l_2, \ldots, l_r}} \prod_{1 \leqslant s \leqslant r} \binom{k_s}{l_s} \frac{(-1)^{k_s - l_s}}{(2k_s)!!} [x^n](1 + x)^{2^{r-1} l_1 l_2 \cdots l_r}. \end{aligned}$$

Applying (2.26), we can restate the above formula in the form of (3.7). This completes the proof. ∎

For example, when $n = 2$ and $r = 3$, by (3.7) we find that $N_{2,3}^D = 25$. In fact, there are 3 B_2-partitions without zero-block, that is,

$$0^B, \ \pi_1 = \{\pm\{1, 2\}\}, \ \pi_2 = \{\pm\{1, -2\}\}.$$

Among all 27 3-tuples of B_2-partitions without zero-block, there are only two partitions (π_1, π_1, π_1) and (π_2, π_2, π_2) that are not minimally intersecting.

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