Degree distributions in general random intersection graphs

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Abstract

We study $G(n, m, F, H)$, a variant of the standard random intersection graph model in which random weights are assigned to both vertex types in the bipartite structure. Under certain assumptions on the distributions of these weights, the degree of a vertex is shown to depend on the weight of that particular vertex and on the distribution of the weights of the other vertex type.

1 Introduction

Random intersection graphs, denoted by $G(n, m, p)$, are introduced in [9, 14] as opposed to classical Erdős-Rényi random graphs. Let us consider a set V with n vertices and another universal set W with m elements. Define a bipartite graph $B(n, m, p)$ with independent vertex sets V and W. Edges between $v \in V$ and $w \in W$ exist independently with probability p. The random intersection graph $G(n, m, p)$ derived from $B(n, m, p)$ is defined on the vertex set V with vertices $v_1, v_2 \in V$ adjacent if and only if there exists some $w \in W$ such that both v_1 and v_2 are adjacent to w in $B(n, m, p)$.

To get an interesting graph structure and bounded average degree, the work [15] sets $m = \lfloor n^\alpha \rfloor$ and $p = cn^{-(1+\alpha)/2}$ for some α, $c > 0$ and determines the distribution of the degree of a typical vertex. Some related properties for this model are recently investigated; for example, independent sets [11] and component evolution [1, 10]. A generalized random intersection graph is introduced in [5] by allowing a more general connection probability in the underlying bipartite graph. The corresponding vertex degrees are also studied by some authors, see e.g. [2, 7, 8], and shown to be asymptotically Poisson distributed.

In this paper, we consider a variant model of random intersection graphs, where each vertex and element are associated with a random weight, in order to obtain a larger class of degree distributions. Our model, referred to as $G(n, m, F, H)$, is defined as follows.

Definition 1. *Let us consider a set $V = [n]$ of n vertices and a set $W = [m]$ of m elements. Define $m = \lfloor \beta n^{\alpha} \rfloor$ with $\alpha, \beta > 0$. Let $\{A_i\}_{i=1}^{n}$ be an independent, identically distributed sequence of positive random variables with distribution F. For brevity, F is assumed to have mean 1 if the mean is finite. The sequence $\{B_i\}_{i=1}^{m}$ is defined analogously with distribution H, which is independent with F and assumed to have mean 1 if the mean is finite. For some $i \in V$, $j \in W$ and $c > 0$, set*

$$p_{ij} = \left(cA_iB_jn^{-(1+\alpha)/2}\right) \wedge 1. \tag{1}$$

Define a bipartite graph $B(n, m, F, H)$ with independent vertex sets V and W. Edges between $i \in V$ and $j \in W$ exist independently with probability p_{ij}. Then, $G(n, m, F, H)$ is constructed by taking V as the vertex set and drawing an edge between two distinct vertices $i, j \in V$ if and only if they have a common adjacent element $k \in W$ in $B(n, m, F, H)$.

If every element in W has a unit weight, i.e. H is a shifted Heaviside function, our model reduces to that treated in [4]. Compared with Theorem 1.1 in [4], our result (see Theorem 1 below) provides more flexibility. A similar mechanism of assigning random weights has been utilized for Erdős-Rényi graphs in [3] to generate random graphs with prescribed degree distribution.

The rest of the paper is organized as follows. Our main results are presented in Section 2 and we give proofs in Section 3.

2 The results

Let B be a random variable with distribution H and suppose B is independent with $\{B_i\}$. The following result concerns the asymptotic expected degree of a vertex under appropriate moment conditions on F and H.

Proposition 1. *Let D_i denote the degree of vertex $i \in V$ in a general random intersection graph $G(n, m, F, H)$ with $m = \lfloor \beta n^{\alpha} \rfloor$ and p_{ij} as in (1). If F has finite mean and H has finite moment of order 2, then, for all values of $\alpha > 0$, we have that*

$$E(D_i|A_i) \to c^2A_i\beta E(B^2)$$

almost surely, as $n \to \infty$.

Our main theorem, which can be viewed as a generalization of Theorem 2 in [15] and Theorem 1.1 in [4], reads as follows.

Theorem 1. *Let D_i be the degree of vertex $i \in V$ in a general random intersection graph $G(n, m, F, H)$ with $m = \lfloor \beta n^{\alpha} \rfloor$ and p_{ij} as in (1). Assume that F has finite mean.*

(i) If $\alpha < 1$, H has finite moment of order $(2\alpha/(1-\alpha)) + \varepsilon$ for some $\varepsilon > 0$, then, as $n \to \infty$, the degree D_i converges in distribution to a point mass at 0.

(ii) If $\alpha = 1$, H has finite mean, then D_i converges in distribution to a sum of a Poisson$(cA_i\beta)$ distributed number of Poisson(cB) variables, where all variables are independent.

(iii) If $\alpha > 1$, H has finite moment of order 2, then D_i is asymptotically Poisson$(c^2A_i\beta)$ distributed.

The basic idea of proof is similar with that in [4], but some significant modifications and new methods are adopted to tackle the non-homogeneous connection probability involved here.

3 Proofs

Let $|S|$ denote the cardinality of a set S. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers with $y_n > 0$ for all n, we write $x_n \sim y_n$ if $\lim_{n\to\infty} x_n/y_n = 1$; and if X and Y are two random variables, we write $X \stackrel{d}{=} Y$ for equivalence in distribution. Without loss of generality, we prove the results for vertex $i = 1$.

Proof of Proposition 1. We introduce cut-off versions of the weight variables. For $i = 2, \cdots, n$, let $A_i' = A_i 1_{[A_i \leqslant n^{1/4}]}$ and $A_i'' = A_i - A_i'$. Let D_1' and D_1'' be the degrees of vertex 1 when the weights $\{A_i\}_{i\neq 1}$ are replaced by $\{A_i'\}$ and $\{A_i''\}$, respectively; that is, D_1' is the number of neighbors of 1 with weight less than or equal to $n^{1/4}$ and D_1'' is the number of neighbors with weight larger than $n^{1/4}$. For $j \in W$, write p_{1j}' and p_{1j}'' for the analog of (1) based on the truncated weights.

For $i \in V$ and $i \neq 1$, we observe that

$$1 - \prod_{j=1}^{m}(1 - p_{1j}p_{ij}'') \leqslant \sum_{j=1}^{m} p_{1j}p_{ij}'' \leqslant cA_1 n^{-(1+\alpha)/2} \sum_{j=1}^{m} B_j p_{ij}''.$$

Hence, we have

$$E(D_1''|A_1) = \sum_{i=2}^{n} E\Big(1 - \prod_{j=1}^{m}(1 - p_{1j}p_{ij}'')\Big) \leqslant c\beta A_1 n^{(\alpha-1)/2} \sum_{i=2}^{n} \left(\frac{\sum_{j=1}^{m} B_j E p_{ij}''}{m}\right).$$

Since F and H have finite means, it follows that $(\sum_{j=1}^{m} B_j)/m \to EB_1 = 1$ almost surely, by the strong law of large numbers, and

$$Ep_{ij}'' \leqslant cn^{-(1+\alpha)/2} EA_i'' EB_j = cn^{-(1+\alpha)/2} P(A_i > n^{1/4}) \leqslant cn^{-(1+\alpha)/2} \frac{EA_i}{n^{1/4}},$$

by using the Markov inequality. Therefore, $E(D_1''|A_1) \to 0$ almost surely, as $n \to \infty$.

As for D_1', we observe that

$$1 - \prod_{j=1}^{m}(1 - p_{1j}p_{ij}') = c^2 A_1 A_i' \Big(\sum_{j=1}^{m} B_j^2\Big) n^{-(1+\alpha)} + O\Big(A_1^2 A_i'^2 \Big(\sum_{k\neq l, k,l=1}^{m} B_k^2 B_l^2\Big) n^{-2(1+\alpha)}\Big),$$

and therefore,

$$E(D_1'|A_1) = c^2 A_1 \beta n^{-1}\Big(\sum_{i=2}^{n} EA_i'\Big)\Big(\frac{\sum_{j=1}^{m} E(B_j^2)}{m}\Big) + n^{-2(1+\alpha)} O\Big(A_1^2 E(A_i'^2)\Big(\sum_{k\neq l,k,l=1}^{m} E(B_k^2)E(B_l^2)\Big)\Big). \quad (2)$$

The first term on the right-hand side of (2) converges to $c^2 A_1 \beta E(B^2)$ almost surely as $n \to \infty$ since $\big(\sum_{j=1}^{m} E(B_j^2)\big)/m \to E(B^2)$ and $EA_i' = EA_i P(A_i \leqslant n^{1/4}) \to EA_i = 1$. The fact that $A_i'^2 \leqslant n^{1/2}$ implies the second term on the right-hand side of (2) is $O(n^{-2(1+\alpha)} n^{1/2} m^2) = o(1)$. The proof is thus completed by noting that $D_1 = D_1' + D_1''$. □

Proof of Theorem 1. Let $N_1 = \{j \in W|\ j$ is adjacent to $1 \in V$ in $B(n, m, F, H)\}$. Therefore, (i) follows if we prove that $P(|N_1| = 0) \to 1$ as $n \to \infty$ for $\alpha < 1$. Conditional on A_1, $B_1, \cdots, B_m$, we have

$$P(|N_1| = 0|\ A_1, B_1, \cdots, B_m) = \prod_{k=1}^{m}(1 - p_{1k}) = 1 - O\Big(\sum_{k=1}^{m} p_{1k}\Big). \quad (3)$$

From (1) we observe that

$$\sum_{k=1}^{m} p_{1k} \leqslant \sum_{k=1}^{m} cA_1 B_k n^{-(1+\alpha)/2} \leqslant m \max_k\{B_k\} cA_1 n^{-(1+\alpha)/2} = \beta c A_1 n^{(\alpha-1)/2} \max_k\{B_k\}.$$

By the Markov inequality, for $\eta > 0$

$$\begin{aligned} P\big(n^{(\alpha-1)/2} \max_k\{B_k\} > \eta\big) &\leqslant mP\big(n^{(\alpha-1)/2} B_k > \eta\big) \\ &= \beta n^{\alpha} P\big(n^{-\alpha+\varepsilon(\alpha-1)/2} B_k^{(2\alpha/(1-\alpha))+\varepsilon} > \eta^{(2\alpha/(1-\alpha))+\varepsilon}\big) \\ &\leqslant \frac{\beta E(B_k^{(2\alpha/(1-\alpha))+\varepsilon})}{\eta^{(2\alpha/(1-\alpha))+\varepsilon} n^{\varepsilon(1-\alpha)/2}} \end{aligned}$$

It then follows immediately from (3) that $P(|N_1| = 0|\ A_1, B_1, \cdots, B_m) \to 1$ in probability, as $n \to \infty$. Bounded convergence then gives that $P(|N_1| = 0) = EP(|N_1| = 0|\ A_1, B_1, \cdots, B_m) \to 1$, as desired.

Next, to prove (ii) and (iii), we first note that $ED_1'' \to 0$ as is proved in Proposition 1. The inequality $P(D_1'' > 0) \leqslant ED_1''$ implies that D_1'' converges to zero in probability, and then it suffices to show that the generating function of D_1' converges to that of the claimed limiting distribution. We condition on the variable A_1, which is assumed to be fixed in the sequel. For $i = 2, \cdots, n$, let $X_i' = \{j \in W|\ j$ is adjacent to both $i \in V$ and $1 \in V$ in $B(n, m, F, H)\}$. Then by definition, we may write $D_1' = \sum_{i=2}^{n} 1_{[|X_i'| \geqslant 1]}$. Conditional on $N_1, A_2', \cdots, A_n', B_1, \cdots, B_m$, it is clear that $\{|X_i'|\}$ are independent random variables and $X_i' \stackrel{d}{=} \text{Bernoulli}(p'_{ij_1}) + \cdots + \text{Bernoulli}(p'_{ij_{|N_1|}})$, where the Bernoulli variables involved

here are independent and we assume $N_1 = \{j_1, \cdots, j_{|N_1|}\} \subseteq W$. For $t \in [0,1]$, the generating function of D_1' can be expressed as

$$\begin{aligned} E\big(t^{D_1'}\big) &= E\Big(\prod_{i=2}^{n} E\big(t^{1_{[|X_i'|\geqslant 1]}}\big|\ N_1, A_2', \cdots, A_n', B_1, \cdots, B_m\big)\Big) \\ &= E\Big(\prod_{i=2}^{n} \big(1+(t-1)P(|X_i'|\geqslant 1|\ N_1, A_2', \cdots, A_n', B_1, \cdots, B_m)\big)\Big). \end{aligned}$$

Observe similarly as in Proposition 1 that

$$P(|X_i'|\geqslant 1|\ N_1, A_2', \cdots, A_n', B_1, \cdots, B_m) = 1-\prod_{k=1}^{|N_1|}(1-p_{ij_k}') = \sum_{k=1}^{|N_1|} p_{ij_k}' + O\Big(\sum_{k\neq l, k,l=1}^{|N_1|} p_{ij_k}' p_{ij_l}'\Big).$$

Thereby, we have

$$\begin{aligned} &\prod_{i=2}^{n} \big(1+(t-1)P(|X_i'|\geqslant 1|\ N_1, A_2', \cdots, A_n', B_1, \cdots, B_m)\big) \\ &\quad = \exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p_{ij_k}' + O\Big(\sum_{i=2}^{n}\sum_{k,l=1}^{|N_1|} p_{ij_k}' p_{ij_l}'\Big)\Big) \\ &\quad = \exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p_{ij_k}'\Big) + R(n), \end{aligned}$$

where

$$R(n) := \exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p_{ij_k}'\Big)\cdot\Big(\exp\Big(O\Big(\sum_{i=2}^{n}\sum_{k,l=1}^{|N_1|} p_{ij_k}' p_{ij_l}'\Big)\Big)-1\Big).$$

Note that $E\big(t^{D_1'}\big) \in [0,1]$ and $\exp\big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p_{ij_k}'\big) \in [0,1]$ since $t \in [0,1]$. Thus we have $R(n) \in [-1,1]$.

We then aim to prove the following three statements.

(a) $E\big(\exp\big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p_{ij_k}'\big)\big) \to e^{cA_1\beta(\tau-1)}$, if $\alpha = 1$;

(b) $E\big(\exp\big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p_{ij_k}'\big)\big) \to e^{c^2A_1\beta(t-1)}$, if $\alpha > 1$;

(c) $R(n) \to 0$ in probability, if $\alpha \geqslant 1$,

where $\tau = \tau(t)$ is the generating function of a Poi(cB) variable. The above limits in (a) and (b) are the generating functions for the desired compound Poisson and Poisson distributions in (ii) and (iii) of Theorem 1, respectively. By the bounded convergence theorem, (c) yields $E(R(n)) \to 0$, which together with (a) and (b) concludes the proof.

For $\alpha = 1$, we have $|N_1| \stackrel{d}{=} \text{Bernoulli}(p_{11}) + \cdots + \text{Bernoulli}(p_{1m})$ and all m variables involved here are independent. By employing the strong law of large numbers, we get

$$\sum_{k=1}^{m} p_{1k} = cA_1\beta\frac{\sum_{j=1}^{m} B_j}{\beta n} \to cA_1\beta \quad a.e.$$

Then the Poisson paradigm (see e.g.[13]) readily gives $|N_1| \stackrel{d}{=} \text{Poisson}(cA_1\beta)$. We have

$$E\Big(\exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|}p'_{ij_k}\Big)\Big) = E\Big(E\Big(\exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|}p'_{ij_k}\Big)\Big|\ A'_2,\cdots,A'_n\Big)\Big)$$
$$= E\Big(\sum_{s=0}^{m}\exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{s}p'_{ik}\Big)\cdot P(|N_1|=s)\Big). \quad (4)$$

Since for any k it follows that $EA'_i \to EA_i = 1$ and $\sum_{i=2}^{n} p'_{ik} = cB_k(\sum_{i=2}^{n} A'_i)/n \to cB_k$ almost surely,

$$\sum_{s=0}^{m}\exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{s}p'_{ik}\Big)\cdot P(|N_1|=s) \sim \sum_{s=0}^{m}\exp\Big((t-1)c\sum_{k=1}^{s}B_k\Big)e^{-cA_1\beta}\frac{(cA_1\beta)^s}{s!}.$$

Therefore, we obtain

$$E\Big(\sum_{s=0}^{m}\exp\Big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{s}p'_{ik}\Big)\cdot P(|N_1|=s)\Big)$$
$$\sim E\Big(\sum_{s=0}^{m}\exp\Big((t-1)c\sum_{k=1}^{s}B_k\Big)e^{-cA_1\beta}\frac{(cA_1\beta)^s}{s!}\Big)$$
$$= \sum_{s=0}^{m}\Big(\prod_{k=1}^{s}E\big(e^{(t-1)cB_k}\big)\Big)e^{-cA_1\beta}\frac{(cA_1\beta)^s}{s!}$$
$$= e^{-cA_1\beta}\sum_{s=0}^{m}\frac{(\tau cA_1\beta)^s}{s!}$$
$$\to e^{cA_1\beta(\tau-1)}$$

as $n \to \infty$. Combining this with (4) gives (a).

For $\alpha > 1$, we also have $|N_1| \stackrel{d}{=} \text{Bernoulli}(p_{11}) + \cdots + \text{Bernoulli}(p_{1m})$ and all m variables involved here are independent. From the strong law of large numbers, it yields

$$\sum_{k=1}^{m} p_{1k} = cA_1\beta n^{(\alpha-1)/2}\frac{\sum_{j=1}^{m}B_j}{\beta n^{\alpha}} \sim cA_1\beta n^{(\alpha-1)/2} \quad a.e. \quad (5)$$

Note that

$$\sum_{k=1}^{m} p_{1k}^2 = \frac{\beta c^2 A_1^2}{n}\cdot\frac{\sum_{j=1}^{m}B_j^2}{\beta n^{\alpha}} \to 0 \quad a.e. \quad (6)$$

as $n \to \infty$, since H has finite moment of order 2. By (5), (6) and a coupling argument of Poisson approximation (see Section 2.2 [6]), we obtain $|N_1| \stackrel{d}{=} \text{Poisson}(cA_1\beta n^{(\alpha-1)/2})$.

We have that

$$\sum_{i=2}^{n}\sum_{k=1}^{|N_1|}p'_{ij_k} = c\sum_{i=2}^{n}\sum_{k=1}^{|N_1|}A'_iB_{j_k}n^{-(1+\alpha)/2}$$
$$= c\cdot\frac{|N_1|}{n^{(\alpha-1)/2}}\cdot\frac{\sum_{i=2}^{n}A'_i}{n}\cdot\frac{\sum_{k=1}^{|N_1|}B_k}{|N_1|}. \quad (7)$$

Here $|N_1|$ is distributed as the sum of $n^{(\alpha-1)/2}$ i.i.d. Poisson$(cA_1\beta)$ variables, implying that the first fraction converges to $cA_1\beta$ almost surely. The second fraction converges to 1 since $EA_i' \to 1$ as is proved in Proposition 1. To determine the convergence of the last fraction in (7), we note that (see e.g. Lemma 1.4 [12])

$$P\Big(\big||N_1| - cA_1\beta n^{(\alpha-1)/2}\big| \geqslant \frac{1}{2}(cA_1\beta)^{3/4}n^{3(\alpha-1)/8}\Big) \leqslant \exp\Big(-\frac{1}{9}(cA_1\beta)^{1/2}n^{(\alpha-1)/4}\Big).$$

By the Borel-Cantelli lemma, $n_1^- \leqslant |N_1| \leqslant n_1^+$ almost surely, where

$$n_1^{\pm} := cA_1\beta n^{(\alpha-1)/2} \pm \frac{1}{2}(cA_1\beta)^{3/4}n^{3(\alpha-1)/8}.$$

Hence, we have

$$\begin{aligned}\frac{\sum_{k=1}^{n_1^-} B_k}{n_1^-} \cdot \frac{n_1^-}{cA_1\beta n^{(\alpha-1)/2}} \cdot \frac{cA_1\beta n^{(\alpha-1)/2}}{|N_1|} &\leqslant \frac{\sum_{k=1}^{|N_1|} B_k}{|N_1|} \\ &\leqslant \frac{\sum_{k=1}^{n_1^+} B_k}{n_1^+} \cdot \frac{n_1^+}{cA_1\beta n^{(\alpha-1)/2}} \cdot \frac{cA_1\beta n^{(\alpha-1)/2}}{|N_1|},\end{aligned}$$

and by the strong law of large numbers and $EB_k = 1$, $\frac{\sum_{k=1}^{|N_1|} B_k}{|N_1|} \to 1$ almost surely. Therefore, by bounded convergence, we have

$$E\Big(\exp\big((t-1)\sum_{i=2}^{n}\sum_{k=1}^{|N_1|} p'_{ij_k}\big)\Big) \to e^{c^2A_1\beta(t-1)}$$

as desired.

It remains to show (c). First note it suffices to show

$$\sum_{i=2}^{n}\sum_{k,l=1}^{|N_1|} p'_{ik}p'_{il} \to 0 \quad \text{in probability} \tag{8}$$

as $n \to \infty$. Recalling that $A_i' \leqslant n^{1/4}$, we have for $\alpha \geqslant 1$ that

$$\sum_{k,l=1}^{|N_1|}\sum_{i=2}^{n} p'_{ik}p'_{il} \leqslant \sum_{k,l=1}^{|N_1|} c^2 n^{-(1+\alpha)} B_k B_l \sum_{i=2}^{n} A_i'^2 \leqslant c^2 n^{(1/2)-\alpha}\Big(\sum_{k=1}^{|N_1|} B_k\Big)^2.$$

For any $\eta > 0$, we have

$$P\Big(n^{(1/4)-\alpha/2}\sum_{k=1}^{|N_1|} B_k > \eta\Big) \leqslant \frac{E\big(\sum_{k=1}^{|N_1|} B_k\big)}{\eta n^{(\alpha/2)-1/4}} = \frac{(E|N_1|)(EB_1)}{\eta n^{(\alpha/2)-1/4}} \leqslant \frac{c\beta}{\eta n^{1/4}}$$

by using the Markov inequality, the Wald equation (see e.g. [13]), $E|N_1| \leqslant c\beta n^{(\alpha-1)/2}$ and $EB_1 = 1$, proving the claim (8) as it stands. □

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