On colorings avoiding a rainbow cycle and a fixed monochromatic subgraph

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Submitted: Apr 23, 2009; Accepted: Feb 7, 2010; Published: Feb 22, 2010
Mathematics Subject Classification: 05C15, 05C55

Abstract

Let H and G be two graphs on fixed number of vertices. An edge coloring of a complete graph is called (H, G)-good if there is no monochromatic copy of G and no rainbow (totally multicolored) copy of H in this coloring. As shown by Jamison and West, an (H, G)-good coloring of an arbitrarily large complete graph exists unless either G is a star or H is a forest. The largest number of colors in an (H, G)-good coloring of K_n is denoted $maxR(n, G, H)$. For graphs H which can not be vertex-partitioned into at most two induced forests, $maxR(n, G, H)$ has been determined asymptotically. Determining $maxR(n; G, H)$ is challenging for other graphs H, in particular for bipartite graphs or even for cycles. This manuscript treats the case when H is a cycle. The value of $maxR(n, G, C_k)$ is determined for all graphs G whose edges do not induce a star.

1 Introduction and main results

For two graphs G and H, an edge coloring of a complete graph is called (H, G)-good if there is no monochromatic copy of G and no rainbow (totally multicolored) copy of H in this coloring. The *mixed anti-Ramsey numbers*, $maxR(n; G, H)$, $minR(n; G, H)$ are the maximum, minimum number of colors in an (H, G)-good coloring of K_n, respectively. The number $maxR(n; G, H)$ is closely related to the classical *anti-Ramsey number* $AR(n, H)$, the largest number of colors in an edge-coloring of K_n with no rainbow copy of H introduced by Erdős, Simonovits and Sós [9]. The number $minR(n; G, H)$ is closely related to

*The first author's research supported in part by NSA grant H98230-09-1-0063 and NSF grant DMS-0901008.

the classical multicolor Ramsey number $R_k(G)$, the largest n such that there is a coloring of edges of K_n with k colors and no monochromatic copy of G. The mixed Ramsey number $minR(n; G, H)$ has been investigated in [3, 13, 11].

This manuscript addresses $maxR(n; G, H)$. As shown by Jamison and West [14], an (H, G)-good coloring of an arbitrarily large complete graph exists unless either G is a star or H is a forest. Let $a(H)$ be the smallest number of induced forests vertex-partitioning the graph H. This parameter is called a vertex arboricity. Axenovich and Iverson [3] proved the following.

Theorem 1. *Let G be a graph whose edges do not induce a star and H be a graph with $a(H) \geqslant 3$. Then $maxR(n; G, H) = \frac{n^2}{2}\left(1 - \frac{1}{a(H)-1}\right)(1 + o(1))$.*

When $a(H) = 2$, the problem is challenging and only few isolated results are known [3]. Even in the case when H is a cycle, the problem is nontrivial. This manuscript addresses this case. Since (C_k, G)-good colorings do not contain rainbow C_k, it follows that

$$maxR(n; G, C_k) \leqslant AR(n, C_k) = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1), \tag{1}$$

where the equality is proven by Montellano-Ballesteros and Neumann-Lara [16]. We show that $maxR(n; G, C_k) = AR(n; C_k)$ when G is either bipartite with large enough parts, or a graph with chromatic number at least 3. In case when G is bipartite with a "small" part, $maxR(n; G, C_k)$ depends mostly on G, namely, on the size of the "small" part. Below is the exact formulation of the main result.

If a graph G is bipartite, we let $s(G) = \min\{s : G \subseteq K_{s,r},\ s \leqslant r \text{ for some } r\}$ and $t(G) = |V(G)| - s(G)$. I.e., $s(G)$ is the sum of the sizes of smaller parts over all components of G.

Theorem 2. *Let $k \geqslant 3$ be an integer and G be a graph whose edges do not induce a star. Let $s = s(G)$ and $t = t(G)$ if G is bipartite. There are constants $n_0 = n_0(G, k)$ and $g = g(G, k)$ such that for all $n \geqslant n_0$*

$$maxR(n; G, C_k) = \begin{cases} n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1), & \text{if } \big(\chi(G) = 2 \text{ and } s \geqslant k\big) \text{ or } \big(\chi(G) \geqslant 3\big) \\ n\left(\frac{s-2}{2} + \frac{1}{s-1}\right) + g, & \text{otherwise} \end{cases}$$

Here $g = g(G, k) = ER^2\big(s+t, 3sk+t+1, k\big)$, where the number ER denotes the Erdős-Rado number stated in section 2. Note that it is sufficient to take $g(G, k) = 2^{c\ell^2 \log \ell}$, where $\ell = 3sk + t + 1$.

We give the definitions and some observations in section 2, the proof of the main theorem in section 3 and some more accurate bounds for the case when $H = C_4$ in the last section of the manuscript.

2 Definitions and preliminary results

First we shall define a few special edge colorings of a complete graph: lexical, weakly lexical, k-anticyclic, c^* and c^{**}.

Let $c : E(K_n) \to \mathbb{N}$ be an edge coloring of a complete graph on n vertices for some fixed n.

We say that c is a *weakly lexical* coloring if the vertices can be ordered $v_1, \ldots, v_n$, and the colors can be renamed such that there is a function $\lambda : V(K_n) \to \mathbb{N}$, and $c(v_i v_j) = \lambda(v_{\min\{i,j\}})$, for $1 \leqslant i, j \leqslant n$. In particular, if λ is one to one, then c is called a *lexical* coloring.

We say that c is a k-*anticyclic* coloring if there is no rainbow copy of C_k, and there is a partition of $V(K_n)$ into sets $V_0, V_1, \ldots, V_m$ with $0 \leqslant |V_0| < k-1$ and $|V_1| = \cdots = |V_m| = k-1$, where $m = \lfloor \frac{n}{k-1} \rfloor$, such that for i, j with $0 \leqslant i < j \leqslant m$, all edges between V_i and V_j have the same color, and the edges spanned by each $V_i, i = 0, \ldots, m$ have new distinct colors using pairwise disjoint sets of colors.

We denote a fixed coloring from the set of k-anticyclic colorings of K_n such that the color of any edges between V_i and V_j is $\min\{i, j\}$ by c^*.

Finally, we need one more coloring, c^{**}, of K_n. Let c^{**} be a fixed coloring from the set of the following colorings of $E(K_n)$; let the vertex set $V(K_n)$ be a disjoint union of $V_0, V_1, \ldots, V_m$ with $0 \leqslant |V_0| < s-1$, $|V_1| = \cdots = |V_{m-1}| = s-1$, and $|V_m| = k-1$, where $m - 1 = \lfloor \frac{n-k+1}{s-1} \rfloor$. Let the color of each edge between V_i and V_j for $0 \leqslant i < j \leqslant m$ be i. Color the edges spanned by each $V_i, i = 0, \ldots, m$ with new distinct colors using pairwise disjoint sets of colors.

For a coloring c, let the number of colors used by c be denoted by $|c|$. Observe that c^* is a blow-up of a lexical coloring with parts inducing rainbow complete subgraphs. Any monochromatic bipartite subgraph in c^* and c^{**} is a subgraph of $K_{k-1,t}$ and $K_{s-1,t}$ for some t, respectively. Also we easily see that if c is k-anticyclic, then

$$|c| \leqslant |c^*| = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1), \tag{2}$$

$$|c^{**}| = n\left(\frac{s-2}{2} + \frac{1}{s-1}\right) + O(1). \tag{3}$$

Let $K = K_n$. For disjoint sets $X, Y \subseteq V$, let $K[X]$ be the subgraph of K induced by X, and let $K[X, Y]$ be the bipartite subgraph of K induced by X and Y. Let $c(X)$ and $c(X, Y)$ denote the sets of colors used in $K[X]$ and $K[X, Y]$, respectively by a coloring c.

Next, we state a canonical Ramsey theorem which is essential for our proofs.

Theorem 3 (Deuber [7], Erdős-Rado [8])**.** *For any integers m, l, r, there is a smallest integer $n = ER(m, l, r)$, such that any edge-coloring of K_n contains either a monochromatic copy of K_m, a lexically colored copy of K_l, or a rainbow copy of K_r.*

The number ER is typically referred to as Erdős-Rado number, with best bound in the symmetric case provided by Lefmann and Rödl [15], in the following form: $2^{c_1 \ell^2} \leqslant ER(\ell, \ell, \ell) \leqslant 2^{c_2 \ell^2 \log \ell}$, for some constants c_1, c_2.

3 Proof of Theorem 2

If G is a graph with chromatic number at least 3, then $maxR(n; G, C_k) = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)$ as was proven in [3].

For the rest of the proof we shall assume that G is a bipartite graph, not a star, with $s = s(G)$, $t = t(G)$, and $G \subseteq K_{s,t}$. Note that $2 \leqslant s \leqslant t$. Let $K = K_n$. If $s \geqslant k$, then the lower bound on $maxR(n; G, C_k)$ is given by c^*, a special k-anticyclic coloring. The upper bound follows from (1).

Suppose $s < k$. The lower bound is provided by a coloring c^{**}. Since $maxR(n; G, C_k) \leqslant maxR(n; K_{s,t}, C_k)$, in order to provide an upper bound on $maxR(n; G, C_k)$, we shall be giving an upper bound on $maxR(n; K_{s,t}, C_k)$.

The idea of the proof is as follows. We consider an edge coloring c of $K = (V, E)$ with no monochromatic $K_{s,t}$ and no rainbow C_k, and estimate the number of colors in this coloring by analyzing specific vertex subsets: L, A, B, where L is the vertex set of the largest weakly lexically colored complete subgraph, A is the set of vertices in $V \setminus L$ which "disagrees" with coloring of L on some edges incident to the initial part of L, and B is the set of vertices in $V \setminus L$ which "disagrees" with coloring of L on some edges incident to the terminal part of L. Let $V' = V \setminus L$. We are counting the colors in the following order: first colors induced by V' which are not used on any edges incident to L or any edges induced by L, then colors used on edges between V' and L which are not induced by L, finally colors induced by L.

Now, we provide a formal proof. Assume that n is sufficiently large such that $n \geqslant ER(s + t, 3sk + t + 1, k)$. Let c be a coloring of $E(K)$ with no monochromatic copy of $K_{s,t}$ and no rainbow copy of C_k, $c : E(K) \to \mathbb{N}$. Then there is a lexically colored copy of $K_{3sk+t+1}$ by the canonical Ramsey theorem. Let L be a vertex set of a largest weakly lexically colored K_q, $q \geqslant 3sk + t + 1$, say $L = \{x_1, \dots, x_q\}$ and $c(x_i x_j) = \lambda(x_i)$ for $1 \leqslant i < j \leqslant q$, for some function $\lambda : L \to \mathbb{N}$. If $X = \{x_{i_1}, \dots, x_{i_\ell}\} \subseteq L$ and $\lambda(x_{i_1}) = \cdots = \lambda(x_{i_\ell}) = j$ for some j, then we denote $\lambda(X) = j$. We write, for $i \leqslant j$, $x_i L x_j := \{x_i, x_{i+1}, \dots, x_j\}$, and for $i > j$, $x_i L x_j := \{x_i, x_{i-1}, \dots, x_j\}$. We say that x_i precedes x_j if $i < j$.

Let T_t, T_{sk+t}, T_{2sk+t}, and T_{3sk+t} be the tails of L of size t, $sk + t$, $2sk + t$, and $3sk + t$ respectively, i.e.,

$$\begin{aligned}
T_t &:= \{x_{q-t+1}, x_{q-t+2}, \dots, x_q\}, \\
T_{sk+t} &:= \{x_{q-sk-t+1}, x_{q-sk-t+2}, \dots, x_q\}, \\
T_{2sk+t} &:= \{x_{q-2sk-t+1}, x_{q-2sk-t+2}, \dots, x_q\}, \\
T_{3sk+t} &:= \{x_{q-3sk-t+1}, x_{q-3sk-t+2}, \dots, x_q\},
\end{aligned}$$

see Figure 1.

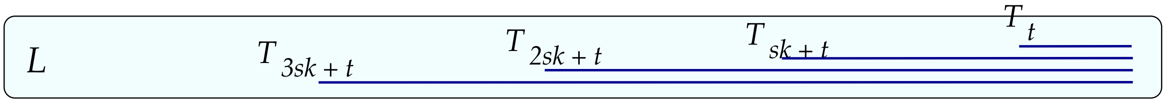

Figure 1: T_t, T_{sk+t}, T_{2sk+t}, and T_{3sk+t}

We shall use these tails to count the number of colors: the common difference, sk, of sizes of tails is from observations below(Claims 0.1–0.3). The first tail T_t is used in Claims 0.1 – 0.3 and to find monochromatic copy of $K_{s,t}$. The third tail T_{2sk+t} is the main tool used in Part 1, 2 of the proof, it helps finding rainbow copy of C_k. The other tails T_{sk+t} and T_{3sk+t} are for technical reasons used in Claim 2.1 and Claim 1.3, respectively. Note that the size of the fourth tail is used in the second parameter of Erdős-Rado number bounding n.

We start by splitting the vertices in $V \setminus L$ according to "agreement" or "disagreement" of a corresponding colors used in $L \setminus T_{2sk+t}$ and in edges between L and $V \setminus L$. Formally, let $V' = V \setminus L$, and

$$\begin{aligned} A &:= \{v \in V' \mid \text{ there exists } y \in L \setminus T_{2sk+t} \text{ such that } c(vy) \neq \lambda(y)\}, \\ B &:= \{v \in V' \mid c(vx) = \lambda(x),\, x \in L \setminus T_{2sk+t}, \\ &\qquad \text{and there exists } y \in T_{2sk+t} \setminus \{x_q\} \text{ such that } c(vy) \neq \lambda(y)\}. \end{aligned}$$

Note that $V' - A - B = \{v \in V' \mid c(vx) = \lambda(x),\, x \in L \setminus \{x_q\}\} = \emptyset$ since otherwise L is not the largest weakly colored complete subgraph. Thus

$$V = L \cup A \cup B.$$

Let $c_0 := c(L) \cup c(V', L)$. In the first part of the proof we bound $\left|\big(c(B) \cup c(B, A)\big) \setminus c_0\right| + |c(B, L) \setminus c(L)|$, in the second part we bound $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)|$.

Claim 0.1 Let $x \in L \setminus T_t$. Then $|\{y \in L \setminus T_t \mid \lambda(x) = \lambda(y)\}| \leqslant s - 1 < s$.

If this claim does not hold, the corresponding y's and T_t induce a monochromatic $K_{s,t}$.

Claim 0.2 Let $y, y' \in L \setminus T_t$ such that $|yLy'| > (s-1)\ell + 1$ for some $\ell \geqslant 0$. Then $|c(yLy')| \geqslant \ell + 1$.

It follows from Claim 0.1.

Claim 0.3 Let $v, v' \in V'$ and $y, y' \in L \setminus T_t$ such that y precedes y'. Let P be a rainbow path from v to v' in V' with $1 \leqslant |V(P)| \leqslant k-2$ and colors not from c_0. If $c(vy) \neq \lambda(y)$, $c(v'y') \notin \{c(vy), \lambda(y)\}$, and $|yLy'| > (s-1)(k - |V(P)|) + 1$, then there is a rainbow C_k induced by $V(P) \cup yLy'$.

Indeed, by Claim 0.2, $|c(yLy')| \geqslant k - |V(P)| + 1$. Hence $|c(yLy') \setminus \{c(vy), c(v'y')\}| \geqslant k - |V(P)| - 1$. So we can find a rainbow path on $k - |V(P)|$ vertices in L with endpoints y

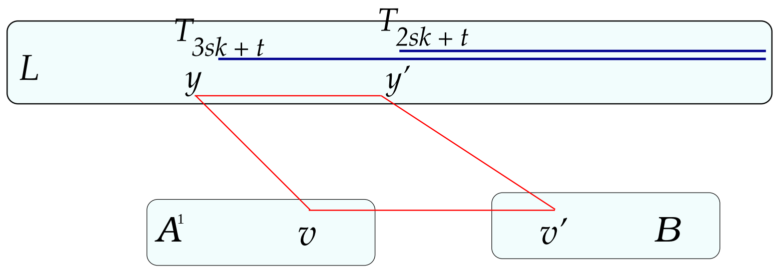

Figure 2: A rainbow C_k in Claim 1.3

and y' of colors from $c(yLy') \setminus \{c(vy), c(v'y')\}$, which together with $V(P)$ induce a rainbow C_k since colors of P are not from c_0.

PART 1

We shall show that $\left|\bigl(c(B) \cup c(B,A)\bigr) \setminus c_0\right| + |c(B,L) \setminus c(L)| \leqslant const = const(k,s,t)$.

Claim 1.1 $|B| < ER(s+t, 2sk+t+1, k)$.

Suppose $|B| \geqslant ER(s+t, 2sk+t+1, k)$. Then there is a lexically colored copy of a complete subgraph on a vertex set $Y \subseteq B$ of size $2sk+t+1$. Then $(L \cup Y) \setminus T_{2sk+t}$ is weakly lexical, which contradicts the maximality of L.

Claim 1.2 $|c(B,L) \setminus c(L)| \leqslant (2sk+t)|B|$.

$|c(B,L) \setminus c(L)| \leqslant |c(B, T_{2sk+t})| \leqslant (2sk+t)|B|$ by the definition of B.

Claim 1.3 $\left|\bigl(c(B) \cup c(B,A)\bigr) \setminus c_0\right| < \binom{ER(s+t,3sk+t+1,k)}{2}$.

Let $A = A^1 \cup A^2$, where $A^1 := \{v \in A \mid \text{ there exists } y \in L \setminus T_{3sk+t} \text{ with } c(vy) \neq \lambda(y)\}$, and $A^2 := A \setminus A^1$.

First, we show that $c(B, A^1) \subseteq c_0$. Assume that $c(v'v) \notin c_0$ for some $v \in A^1$ and $v' \in B$ with $c(vy) \neq \lambda(y)$ for some $y \in L \setminus T_{3sk+t}$ and $c(v'x) = \lambda(x)$ for any $x \in L \setminus T_{2sk+t}$. From Claim 0.1, we can find y', one of the last $2s-1$ elements in $T_{3sk+t} \setminus T_{2sk+t}$ such that $\lambda(y')$ is neither $c(vy)$ nor $\lambda(y)$. Since $\lambda(y') = c(v'y')$, we have that $c(v'y') \notin \{c(vy), \lambda(y)\}$. Moreover we have $|yLy'| > (s-1)(k-2)+1$. By Claim 0.3, there is a rainbow C_k induced by $\{v, v'\} \cup yLy'$, see Figure 2.

Second, we shall observe that $|A^2 \cup B| < ER(s+t, 3sk+t+1, k)$ by the argument similar to one used in Claim 1.1. We see that otherwise $A^2 \cup B$ contains a lexically colored complete subgraph on $3sk+t+1$ vertices, which together with $L - T_{3sk+t}$ gives a larger than L weakly lexically colored complete subgraph.

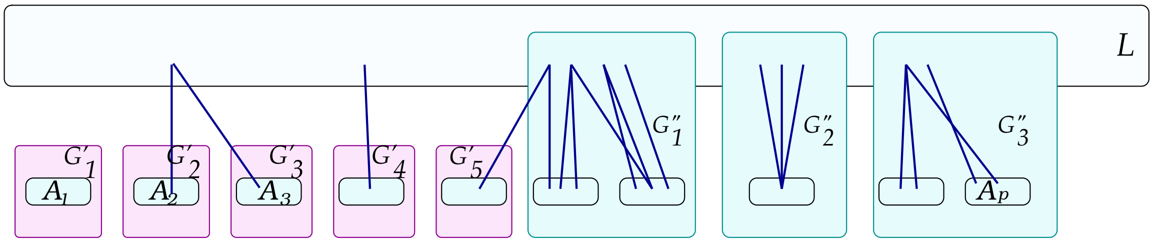

Figure 3: G_1 and G_2

PART 2

We shall show that $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)| \leqslant n\left(\frac{s-2}{2} + \frac{1}{s-1}\right)$.

In order to count the number of colors in A and (A, L), we consider a representing graph of these colors as follows. First, consider a set E' of edges from $K[A]$ having exactly one edge of each color from $c(A) \setminus c_0$. Second, consider a set of edges E'' from the bipartite graph $K[A, L]$ having exactly one edge of each color from $c(A, L) \setminus c(L)$. Let G be a graph with edge-set $E' \cup E''$ spanning A. Then $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| = |E(G)|$.

We need to estimate the number of edges in G. Let $A_1, \ldots, A_p$ be sets of vertices of the connected components of $G[A]$. Let $L_1, \ldots, L_p$ be sets of the neighbors of $A_1, \ldots, A_p$ in L respectively, i.e., for $1 \leqslant i \leqslant p$, $L_i := \{x \in L \mid \{x, y\} \in E(G) \text{ for some } y \in A_i\}$. Let

$$G_1 := \bigcup_{i\,:\,|E(G[A_i, L_i])| \leqslant 1} G[A_i],$$

$$G_2 := \bigcup_{i\,:\,|E(G[A_i, L_i])| \geqslant 2} G[A_i \cup L_i].$$

Let $G'_1, \ldots, G'_{p_1}$ be the connected components of G_1, and let $G''_1, \ldots, G''_{p_2}$ be the connected components of G_2. See Figure 3 for an example of G_1 and G_2.

Claim 2.1 We may assume that $V(G) \cap L \subseteq L \setminus T_{sk+t}$.

For a fixed $v \in A$, let ω be a color in $c(v, L) \setminus c(L)$, if such exists. Let $L(\omega) := \{x \in L \mid c(vx) = \omega\}$. Suppose $L(\omega) \subseteq T_{sk+t}$. Since $v \in A$, there exists $y \in L \setminus T_{2sk+t}$ such that $c(vy) \neq \lambda(y)$. Let $y' \in L(\omega) \subseteq T_{sk+t}$. Then $c(vy') \notin \{c(vy), \lambda(y)\}$. Since $|yLy'| > (s-1)k + 1 > (s-1)(k-1) + 1$, there is a rainbow C_k induced by $\{v\} \cup yLy'$ by Claim 0.3, see figure 4. Therefore $L(\omega) \cap (L \setminus T_{sk+t}) \neq \emptyset$. Hence we can choose edges for the edge set E'' of G only from $K[A, L \setminus T_{sk+t}]$.

Claim 2.2 For every i, $1 \leqslant i \leqslant p$, $K[A_i, T_t]$ is monochromatic; for every j, $1 \leqslant j \leqslant p_2$, $K[V(G''_j), T_t]$ is monochromatic. In particular, for every h, $1 \leqslant h \leqslant p_1$, $K[V(G'_i), T_t]$ is monochromatic.

1. Fix i, $1 \leqslant i \leqslant p$. We show that $K[A_i, T_t]$ is monochromatic. Let $v \in A_i$ and $y \in L \setminus T_{2sk+t}$ with $c(vy) \neq \lambda(y)$.

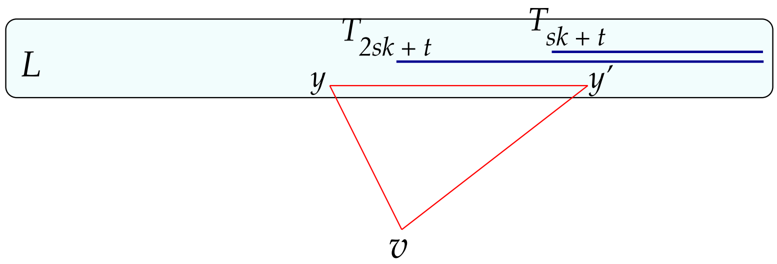

Figure 4: A rainbow C_k in Claim 2.1 and Claim 2.2-1.(1)

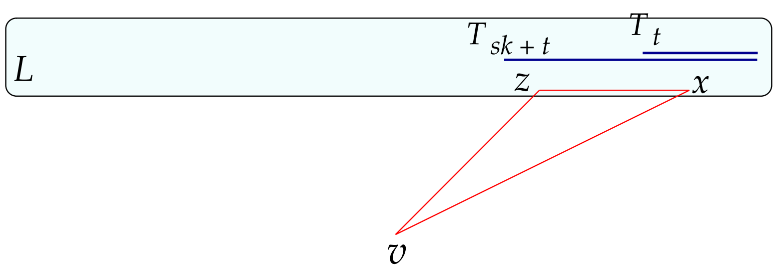

Figure 5: A rainbow C_k in Claim 2.2-1.(2)

(1) For any $y' \in T_{sk+t}$, $c(vy')$ is either $c(vy)$ or $\lambda(y)$. Indeed if $c(vy') \notin \{c(vy), \lambda(y)\}$, then there is a rainbow C_k induced by $\{v\} \cup yLy'$ by Claim 0.3, see Figure 4.

(2) $|c(v, T_t)| = 1$. Indeed, let $L^y = \{x \in T_{sk+t} \setminus T_t \mid \lambda(x) \neq c(vy) \text{ and } \lambda(x) \neq \lambda(y)\}$. Then by Claim 0.1, $|L^y| \geqslant |T_{sk+t} \setminus T_t| - 2(s-1) + 1 > (s-1)(k-3) + 1$. Hence $|c(L^y)| \geqslant k-2$ by Claim 0.2. Let z be the vertex in L^y preceding every other vertex in L^y. Suppose there is $x \in T_t$ such that $c(vx) \neq c(vz)$. Since $c(L^y) \subseteq c(zLx)$, there exists a rainbow path from z to x on $k-1$ vertices in T_{sk+t} of colors disjoint from $\{c(vy), \lambda(y)\}$. So there is a rainbow C_k induced by $\{v\} \cup zLx$, see Figure 5. Therefore for any $x \in T_t$, $c(vx) = c(vz) \in \{c(vy), \lambda(y)\}$.

(3) For any neighbor v' of v in $G[A_i]$, if such exists, $c(v', T_t) = c(v, T_t)$. Indeed, we see that for any $y' \in T_{sk+t}$, $c(v'y') \in \{c(vy), \lambda(y)\}$, otherwise there is a rainbow C_k induced by $\{v, v'\} \cup yLy'$ by Claim 0.3. Also we see that for any $x \in T_t$, $c(v'x) = c(vz) \in \{c(vy), \lambda(y)\}$, where z is defined above; otherwise there is a rainbow C_k induced by $\{v, v'\} \cup zLx$, see Figure 6. Therefore $c(v', T_t) = c(v, T_t)$.

(4) Since $G[A_i]$ is connected, $K[A_i, T_t]$ is monochromatic of color $c(vz)$.

Note that to avoid a monochromatic $K_{s,t}$, we must have that $|A_i| \leqslant s-1 \leqslant k-2$ for $1 \leqslant i \leqslant p$.

2. Fix j, $1 \leqslant j \leqslant p_2$. We show that $K[V(G''_j), T_t]$ is monochromatic.

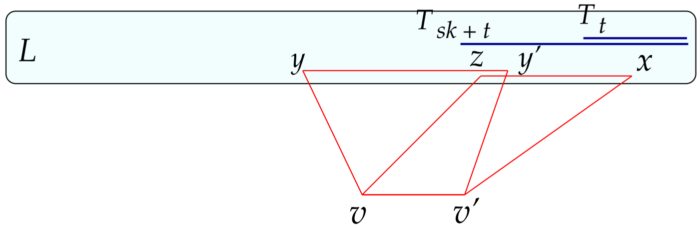

Figure 6: Rainbow C_k's in Claim 2.2-1.(3)

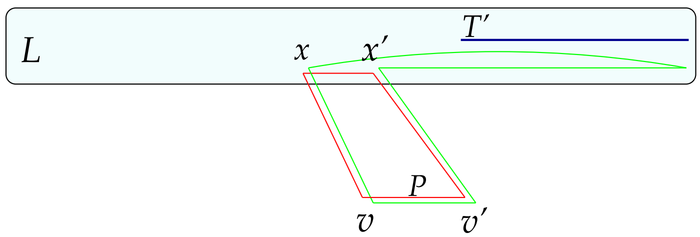

Figure 7: Rainbow C_k's in Claim 2.2-2.(1): red when $|P| = k-2$, green when $|P| < k-2$.

(1) $K[V(G''_j) \cap L, T_t]$ is monochromatic. Indeed, since G''_j, a connected component of G, is a union of $G[A_i \cup L_i]$'s satisfying $|E(G[A_i, L_i])| \geqslant 2$, by the connectivity, it is enough to show that $\lambda(x) = \lambda(x')$ for any $x, x' \in L_i$ for L_i in G''_j, where x precedes x'. From Claim 2.1, we may assume that x, x' are in $L \backslash T_{sk+t}$. Suppose $\lambda(x) \neq \lambda(x')$. Let $v, v' \in A_i$ such that $\{v, x\}$ and $\{v', x'\}$ are edges of G (possibly $v = v'$). Let P denote a set of vertices on a path from v to v' in $G[A_i]$. Then $1 \leqslant |P| \leqslant k-2$ since $|A_i| \leqslant k-2$. If $|P| = k-2$, then $P \cup \{x, x'\}$ induces a rainbow C_k, otherwise so does $P \cup \{x\} \cup x'Lx_q$ from Claim 0.3, see Figure 7. Therefore $\lambda(x) = \lambda(x')$.

(2) $K[V(G''_j), T_t]$ is monochromatic. To prove this, consider i such that $G[A_i, L_i] \subseteq G''_j$. Observe first that $K[A_i, T_t]$ and $K[L_i, T_t]$ are monochromatic by 1.(4) and 2.(1). Next, we shall show that $c(A_i, T_t) = \lambda(L_i)$. Suppose $c(A_i, T_t) \neq \lambda(L_i)$ for some i such that $G[A_i \cup L_i] \subseteq G''_j$. Let $v, v' \in A_i$ and $x, x' \in L_i$ such that $\{v, x\}$ and $\{v', x'\}$ are edges of G (possibly either $v = v'$ or $x = x'$). Since $|E(G[A_i, L_i])| \geqslant 2$, we can find such vertices. So $c(vx) \neq c(v'x')$ and $\{c(vx), c(v'x')\} \cap c(L) = \emptyset$. We may assume that $x, x' \in L \backslash T_{sk+t}$ by Claim 2.1. Since $c(A_i, T_t) \neq \lambda(L_i)$, $c(vx) = c(v'x') = c(A_i, T_t)$, otherwise there is a rainbow C_k induced by $\{v\} \cup xLx_q$ or $\{v'\} \cup x'Lx_q$ by Claim 0.3, see Figure 8. Then it contradicts the fact that $c(vx) \neq c(v'x')$.

We have that for any i such that $G[A_i, L_i] \subseteq G''_j$, $c(A_i, T_t) = \lambda(L_i)$. This implies that $K[A_i \cup L_i, T_t]$ is monochromatic of color $\lambda(L_i)$. Since G''_j is connected and A_is are disjoint, we have that for any i, i' such that $G[A_i, L_i], G[A_{i'}, L_{i'}] \subseteq G''_j$, $L_i \cap L_{i'} \neq \emptyset$, so $\lambda(L_i) = \lambda(L_{i'}) = \lambda$, for some λ. Therefore $K[V(G''_j), T_t]$ is monochromatic of color λ.

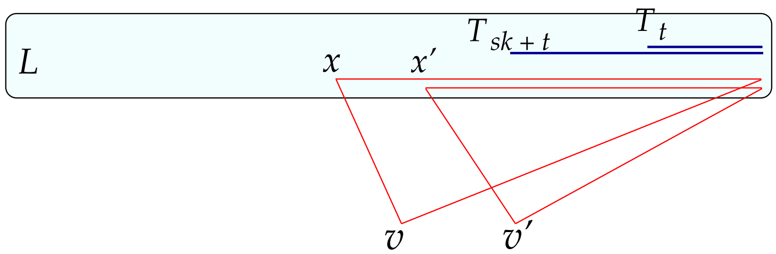

Figure 8: Rainbow C_k's for Claim 2.2-2.(2).

Claim 2.3 For $1 \leqslant i \leqslant p_1$ and $1 \leqslant j \leqslant p_2$, $1 \leqslant |V(G'_i)| \leqslant s-1$ and $1 \leqslant |V(G''_j)| \leqslant s-1$.

This claim now follows from the previous instantly.

The following claim deals with a small quadratic optimization problem we shall need.

Claim 2.4 Let $n, s \in \mathbb{N}$. Suppose n is sufficiently large and $s \geqslant 2$. Let $\xi_1, \ldots, \xi_m \in \mathbb{N}$, $1 \leqslant \xi_i \leqslant s-1$ and $\sum_{i=1}^{m} \xi_i \leqslant n$. Then

$$\sum_{i=1}^{m} \binom{\xi_i - 1}{2} \leqslant n\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big).$$

The equality holds if and only if $m = \frac{n}{s-1}$ and $\xi_1 = \cdots = \xi_m = s-1$.
See the appendix A for the proof.

Claim 2.5 $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)| = |E(G)| + |c(L)| \leqslant n(\frac{s-2}{2} + \frac{1}{s-1})$.
We have that

$$|E(G)| \leqslant \big(|E(G_1)| + p_1\big) + |E(G_2)| = \sum_{i=1}^{p_1} |E(G'_i)| + p_1 + \sum_{i=1}^{p_2} |E(G''_i)|.$$

Moreover each component G''_i of G_2 contributes at most 1 to $|c(L)|$ by Claim 2.2, and G_1 and G_2 are vertex disjoint. So

$$|c(L)| \leqslant n - |V(G_1)| - |V(G_2)| + p_2 = n - \sum_{i=1}^{p_1} |V(G'_i)| - \sum_{i=1}^{p_2} |V(G''_i)| + p_2$$

Hence we have

$$
\begin{aligned}
|c(A)\backslash c_0| + |c(A,L)\setminus c(L)| + |c(L)| &= |E(G)| + |c(L)| \\
&\leqslant \sum_{i=1}^{p_1} |E(G_i')| + p_1 + \sum_{i=1}^{p_2} |E(G_i'')| + n - \sum_{i=1}^{p_1} |V(G_i')| - \sum_{i=1}^{p_2} |V(G_i'')| + p_2 \\
&= \sum_{i=1}^{p_1} |E(G_i')| + \sum_{i=1}^{p_2} |E(G_i'')| - \sum_{i=1}^{p_1} \big(|V(G_i')| - 1\big) - \sum_{i=1}^{p_2} \big(|V(G_i'')| - 1\big) + n \\
&\leqslant \sum_{i=1}^{p_1} \binom{|V(G_i')|}{2} + \sum_{i=1}^{p_2} \binom{|V(G_i'')|}{2} - \sum_{i=1}^{p_1} \big(|V(G_i')| - 1\big) - \sum_{i=1}^{p_2} \big(|V(G_i'')| - 1\big) + n \\
&= \sum_{i=1}^{p_1} \binom{|V(G_i')| - 1}{2} + \sum_{i=1}^{p_2} \binom{|V(G_i'')| - 1}{2} + n
\end{aligned}
$$

For $1 \leqslant i \leqslant p_1 + p_2$, let

$$
\xi_i = \begin{cases} |V(G_i')|, & \text{if } 1 \leqslant i \leqslant p_1 \\ |V(G_{i-p_1}'')|, & \text{if } p_1 + 1 \leqslant i \leqslant p_1 + p_2 \end{cases} .
$$

Then $\sum_{i=1}^{p_1+p_2} \xi_i \leqslant n$ and $1 \leqslant \xi_i \leqslant s-1$ for $1 \leqslant i \leqslant p_1 + p_2$ by Claim 2.3. From Claim 2.4, we get

$$
\begin{aligned}
&|c(A) \setminus c_0| + |c(A,L) \setminus c(L)| + |c(L)| \\
&\leqslant \sum_{i=1}^{p_1+p_2} \binom{\xi_i - 1}{2} + n \leqslant n\left(\frac{s-2}{2} + \frac{1}{s-1}\right).
\end{aligned}
$$

This concludes Part 2 of the proof.

Combining Parts 1 and 2, we see that the total number of colors is at most

$$
\begin{aligned}
&\Big|\big(c(B) \cup c(B,A)\big) \setminus c_0\Big| \; + \; |c(B,L) \setminus c(L)| \; + \; |c(A) \setminus c_0| + |c(A,L) \setminus c(L)| \; + \; |c(L)| \\
&< \binom{ER(s+t, 3sk+t+1, k)}{2} + (2sk+t)ER(s+t, 2sk+t+1, k) + n\left(\frac{s-2}{2} + \frac{1}{s-1}\right) \\
&\leqslant g \; + \; n\left(\frac{s-2}{2} + \frac{1}{s-1}\right),
\end{aligned}
$$

where $g = g(s,t,k) = ER^2\big(s+t, 3sk+t+1, k\big)$.

4 More precise results for C_4

For a coloring c of $E(K_n)$ and a vertex v, let $N_c(v)$ be the set of colors between v and $V(K_n) \setminus \{v\}$, not used on edges spanned by $V(K_n) \setminus \{v\}$. Let $n_c(v) = |N_c(v)|$. Note that $c(uv) \in N_c(u) \cap N_c(v)$ if and only if the color $c(uv)$ is used only on the edge uv in the coloring c. We call this color a *unique color* in c. For a path $P = v_1 v_2 \cdots v_k$, we say that the path P is *good* if $c(v_i v_{i+1}) \in N_c(v_i)$ for $i = 1, \ldots, k-1$.

Lemma 1. *Let c be an edge-coloring of K_n with no rainbow C_k. If for all $v \in V(K_n)$, $n_c(v) \geqslant k-2$, then $(k-1) \mid n$ and c is k-anticyclic.*

Proof. Let c be an edge-coloring of K_n with no rainbow C_k. Suppose for all $v \in V(K_n)$, $n_c(v) \geqslant k-2$. Then for any $v \in V$, we can find a good path of length $k-2$ starting at v by a greedy algorithm. Let this path be $v_1 v_2 \cdots v_{k-1}$, and let $c(v_i v_{i+1}) = i$ for $i = 1, \ldots, k-2$. Let $V_0 = \{v_1, \ldots, v_{k-1}\}$.

Claim 1 For any $u \in V \setminus V_0$, $c(uv_1) = 1$ or $c(uv_1) \notin N_c(v_1)$.

Assume that $c(uv_1) \in N_c(v_1)$. If $c(uv_1) \neq 1$ then $c(uv_{k-1})$ must be the same as $c(uv_1)$, otherwise $v_1 \cdots v_{k-1} u v_1$ is a rainbow C_k. Thus, if $c(uv_1) \neq 1$ then $c(uv_1) \notin N_c(v_1)$.

Claim 2 $\{c(v_1 v_i) \mid i = 2, \ldots, k-1\}$ is a set of distinct colors from $N_c(v_1)$ and $n_c(v_1) = k-2$.

From Claim 1 we see that the colors from $N_c(v_1)$ not equal to 1 appear only on edges $v_1 v_i$ for $i = 2, \ldots, k-1$. Since $n_c(v_1) \geqslant k-2$, all these edges have distinct colors from $N_c(v_1)$ and $n_c(v_1) = k-2$.

Claim 3 For any $u \in V \setminus V_0$, $c(uv_{k-1}) \notin N_c(v_{k-1})$.

Assume otherwise, then $v_2 v_3 \cdots v_{k-1} u$ is a good path. Then $v_1 v_3 v_4 \cdots v_{k-1} u v_2 v_1$ is a rainbow C_k from Claim 2.

Claim 4 $\{c(v_i v_{k-1}) \mid i = 1, \ldots, k-2\}$ is a set of distinct colors from $N_c(v_{k-1})$ and $n_c(v_{k-1}) = k-2$.

By Claim 3, we see that all edges of colors from $N_c(v_{k-1})$ must occur on edges from $\{v_i v_{k-1} : i = 1, \ldots, k-2\}$. Since $n_c(v_{k-1}) \geqslant k-2$, edges $v_i v_{k-1}$, $i = 1, \ldots, k-2$ have distinct colors from $N_c(v_{k-1})$ and $n_c(v_{k-1}) = k-2$.

Claim 5 V_0 induces a rainbow complete subgraph with all colors unique in c. Moreover, for each v_i and each $u \notin V_0$, $c(uv_i)$ is not unique in c.

This follows from the above claims since for $i = 1, \ldots, k-1$, $v_i v_{i+1} \cdots v_{k-1} v_1 v_2 \cdots v_{i-1}$ is a good path, and $n_c(v_i) = k-2$.

Consider $u \notin V_0$ and a good path of length $k-2$ starting at u. Let the vertex set of this path be V_1. If V_0 and V_1 share a vertex, say v_i, then $v_i u$ has a unique color, a contradiction to Claim 5. Thus the graph is vertex-partitioned into copies of K_{k-1} each rainbow colored with unique colors. To avoid a rainbow C_k, any edges between two fixed parts must have the same color. Therefore $(k-1) \mid n$ and c is k-anticyclic. □

By induction on n and the above lemma with $k = 4$, we have the following results.

Corollary 4. $AR(n, C_4) = |c^*| = 4/3n + O(1)$.

Proof. We need to show that for any edge-coloring c of K_n with no rainbow C_4, $|c| \leqslant |c^*| = 4/3n + O(1)$.

We use induction on n. The statement trivially holds for $n = 3$. Let c be a coloring of $E[K_n]$ with no rainbow C_4, $n \geqslant 4$. If for all $v \in V(K_n)$, $n_c(v) \geqslant 2$, then by Lemma 1, c is 4-anticyclic. So $|c| \leqslant |c^*|$. Suppose there is a $v \in V(K_n)$ with $n_c(v) \leqslant 1$. Let $G = K_n - v$. Let c' be the coloring of $E(G)$ induced by c. Then by induction hypothesis, $|c'| \leqslant 4/3(n-1) + O(1)$. Hence $|c| \leqslant |c'| + 1 \leqslant 4/3n + O(1)$. □

Theorem 5. *Let $n \geqslant 3$. Let G be a graph whose edges do not induce a star. Let $s = s(G)$ and $t = t(G)$ if G is bipartite.*

$$maxR(n; G, C_4) = \begin{cases} \frac{4}{3}n + O(1), & \text{if } \big(\chi(G) = 2 \text{ and } s(G) \geqslant 4\big) \text{ or } \big(\chi(G) \geqslant 3\big) \\ n, & \text{otherwise} \end{cases}$$

Proof. Suppose $\big(\chi(G) = 2$ and $s(G) \geqslant 4\big)$ or $\big(\chi(G) \geqslant 3\big)$. For the lower bound, consider the 4-anticyclic coloring c^*. Each color class of c^* is either $K_{1,m}$, $K_{2,m}$, or $K_{3,m}$ for some $m \geqslant 1$, thus c^* contains no monochromatic copy of G. The upper bound follows from Corollary 4.

Suppose G is bipartite and $s(G) \leqslant 3$. We use induction on n. The statement trivially holds for $n = 3$. Let c be a coloring of $E(K_n)$ with no monochromatic G and no rainbow C_4. If $n_c(v) \geqslant 2$ for all $v \in V$, by Lemma 1 there is a color class of c that induces a $K_{3,3m}$ for some $m \geqslant 1$, which contains G. Hence we can find a $v \in V$ with $n_c(v) \leqslant 1$. Then by the induction hypothesis, $maxR(n; G, C_4) \leqslant n$. The lower bound is obtained from the coloring c^{**} with $s = s(G)$ and $k = 4$. Each color class of c^{**} is $K_{1,m}$ if $s(G) = 2$, either $K_{1,m}$ or $K_{2,m}$ if $s(G) = 3$ for some $m \geqslant 1$, thus c^{**} contains no monochromatic copy of G. The total number of colors in either cases is n. □

A Proof of Claim 2.4

Claim 2.4 Let $n, s \in \mathbb{N}$. Suppose n is sufficiently large and $s \geqslant 2$. Let $\xi_1, \ldots, \xi_m \in \mathbb{N}$, $1 \leqslant \xi_i \leqslant s-1$ and $\sum_{i=1}^{m} \xi_i \leqslant n$. Then

$$\sum_{i=1}^{m} \binom{\xi_i - 1}{2} \leqslant n\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big).$$

The equality holds if and only if $m = \frac{n}{s-1}$ and $\xi_1 = \cdots = \xi_m = s-1$.

We use induction on m. If $m = 1$, then

$$\frac{(\xi-1)(\xi-2)}{2} \leqslant \frac{(s-2)(s-3)}{2} \leqslant n\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big), \text{ for any } n \geqslant s-1,$$

where the first inequality becomes equality iff $\xi = s-1$, and the second does iff $n = s-1$. Suppose $m \geqslant 2$, $\sum_{i=1}^{m} \xi_i \leqslant n$, and $1 \leqslant \xi_i \leqslant s-1$ for $1 \leqslant i \leqslant m$. Since $\sum_{i=1}^{m-1} \xi_i \leqslant n - \xi_m$, by induction,

$$\sum_{i=1}^{m-1} \binom{\xi_i - 1}{2} \leqslant (n-\xi_m)\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big), \text{ for any } n \geqslant (m-1)(s-1) + \xi_m,$$

where the equality holds iff $m - 1 = \frac{n-\xi_m}{s-1}$ and $\xi_1 = \cdots = \xi_{m-1} = s-1$. Hence it is enough to show that $(n-\xi_m)\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big) + \binom{\xi_m - 1}{2} \leqslant n\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big)$ or equivalently $\xi_m\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big) - \binom{\xi_m-1}{2} \geqslant 0$, and the equality holds iff $\xi_m = s-1$. If $\xi_m = 1$, that is obvious. Assume $\xi_m > 1$, then

$$\begin{aligned}
\xi_m\Big(\frac{s-4}{2} + \frac{1}{s-1}\Big) - \binom{\xi_m - 1}{2} &= \xi_m \frac{(s-2)(s-3)}{2(s-1)} - \frac{(\xi_m-1)(\xi_m-2)}{2} \\
= \frac{1}{2}\left(-\xi_m^2 + \Big(s - 1 + \frac{2}{s-1}\Big)\xi_m - 2\right) &= \frac{1}{2}\Big(-\xi_m + \frac{2}{s-1}\Big)\Big(\xi_m - (s-1)\Big) \geqslant 0,
\end{aligned}$$

since $2 \leqslant \xi_m \leqslant s-1$.

Acknowledgments The authors thank the referee for a very careful reading and useful comments improving the presentation of the results.

References

[1] B. Alexeev, *On lengths of rainbow cycles*, Electron. J. Combin. 13 (2006), Research Paper 105, 14 pp. (electronic).

[2] M. Axenovich, A. Kündgen,*On a generalized anti-Ramsey problem,* Combinatorica 21 (2001), no. 3, 335–349.

[3] M. Axenovich, P. Iverson, *Edge-colorings avoiding rainbow and monochromatic subgraphs*, Discrete Math., 2008, 308(20), 4710–4723.

[4] L. Babai, *An anti-Ramsey theorem*, Graphs Combin. **1** (1985), no.1, 23–28.

[5] P. Balister, A. Gyárfás, J. Lehel, R. Schelp, *Mono-multi bipartite Ramsey numbers, designs, and matrices*, Journal of Combinatorial Theory, Series A 113 (2006), 101–112.

[6] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York, 1978.

[7] W. Deuber, *Canonization*, Combinatorics, Paul Erdős is eighty, Vol. 1, 107–123, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993.

[8] P. Erdős, R. Rado, *A combinatorial theorem*, J. London Math. Soc. 25, (1950), 249–255.

[9] P. Erdős, M. Simonovits, V. T. Sós, *Anti-Ramsey theorems*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, pp. 633–643. Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[10] L. Eroh, O. R. Oellermann, *Bipartite rainbow Ramsey numbers*, Discrete Math. 277 (2004), 57–72.

[11] J. Fox, B. Sudakov, *Ramsey-type problem for an almost monochromatic K_4* , SIAM J. of Discrete Math. 23, (2008), 155–162.

[12] V. Jungic, T. Kaiser, D. Kral, *A note on edge-colourings avoiding rainbow K_4 and monochromatic K_m*, Electron. J. Combin. 16 (2009), no. 1, Note 19, 9 pp.

[13] A. Kostochka, D. Mubayi, *When is an almost monochromatic K_4 guaranteed?*, Combinatorics, Probability and Computing 17, (2008), no. 6, 823–830.

[14] R. Jamison, D. West, *On pattern Ramsey numbers of graphs*, Graphs Combin. 20 (2004), no. 3, 333–339.

[15] H. Lefmann, V. Rödl, *On Erdős-Rado numbers*, Combinatorica **15** (1995), 85–104.

[16] J. J. Montellano-Ballesteros, V. Neumann-Lara, *An anti-Ramsey theorem on cycles*, Graphs Combin. 21 (2005), no. 3, 343–354.

[17] J. J. Montellano-Ballesteros, V. Neumann-Lara, *An anti-Ramsey theorem*, Combinatorica 22 (2002), no. 3, 445–449.

[18] D. West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996. xvi+512 pp.