Factorisation of Snarks

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Abstract

We develop a theory of factorisation of snarks — cubic graphs with edge-chromatic number 4 — based on the classical concept of the dot product. Our main concern are *irreducible* snarks, those where the removal of every nontrivial edge-cut yields a 3-edge-colourable graph. We show that if an irreducible snark can be expressed as a dot product of two smaller snarks, then both of them are irreducible. This result constitutes the first step towards the proof of the following "unique-factorisation" theorem:

Every irreducible snark G can be factorised into a collection $\{H_1, \ldots, H_n\}$ of cyclically 5-connected irreducible snarks such that G can be reconstructed from them by iterated dot products. Moreover, such a collection is unique up to isomorphism and ordering of the factors regardless of the way in which the decomposition was performed.

The result is best possible in the sense that it fails for snarks that are close to being irreducible but themselves are not irreducible. Besides this theorem, a number of other results are proved. For example, the unique-factorisation theorem is extended to the case of factorisation with respect to a preassigned subgraph K which is required to stay intact during the whole factorisation process. We show that if K has order at least 3, then the theorem holds, but is false when K has order 2.

1 Introduction

In the study of various important and difficult problems in graph theory (such as the Cycle Double Cover Conjecture and the 5-Flow Conjecture) one encounters an interesting but

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somewhat mysterious variety of graphs called *snarks*. In spite of their simple definition — a snark is just a "nontrivial" cubic graph with edge-chromatic number 4 — and over a century long investigation, their properties and structure are largely unknown.

Since their first occurrence in the 19th century [10, 13] much attention has been paid to developing constructive methods of investigation of snarks (see [15]). This approach has resulted in such fundamental concepts as the dot product [6] and superposition [7], among others. Our paper follows a different line of research, one which grows from the ideas of Goldberg [4] and Cameron, Chetwynd and Watkins [2] and focuses on the composite structure of snarks.

Our study is closely related to the problem of nontriviality of snarks, a phenomenon which has been recurring since the very outset of their study. It is based on the observation that there are some situations in which a snark can naturally be regarded trivial and hence uninteresting. The simplest instance of this situation appears to be the occurrence of a bridge in a snark, that is, the existence of a 1-edge-cut, for there is no cubic graph with a bridge that can be 3-edge-coloured. However, this is not the only case. Snarks with independent edge-cuts of size 2 or 3 can be split into two smaller graphs, at least one of which cannot be 3-edge-coloured. Since these cuts are all cycle-separating, a "nontrivial" snark should be at least cyclically 4-edge-connected.

Besides having small cycle-separating cuts there may be other features that make a snark trivial. A snark should also be considered trivial when it is merely a "simple variation" of another smaller snark. For instance, adding or removing a digon or a triangle from a snark does not change its uncolourable property. The same holds when a quadrilateral is replaced with two suitable edges. This suggests that a "nontrivial" snark should have girth at least 5. It has therefore become customary to define a snark as an uncolourable cubic cyclically 4-edge-connected graph with girth at least 5 (see [15], for example).

Fig. 1. Proper k-reductions for $k = 2$ and $k = 3$

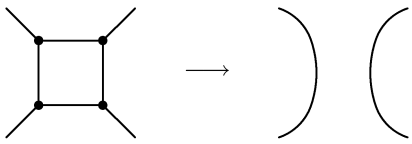

Fig. 2. Proper 4-reduction

Nevertheless, it is possible to view a snark as a "simple variation" of another snark in much more general situations [9]. To see this, consider a snark G which contains an induced subgraph H such that H is not 3-edge-colourable while $G-H$ is. Then $G-H$ does

not contribute to the uncolourability of G, and therefore can be removed. The remaining subgraph H can be converted into a snark H' by adding at most one vertex. Thus G arises from a less trivial snark H' by adding a certain number of unimportant vertices. A digon, a triangle, and a quadrilateral are examples of configurations of vertices which bring nothing essential to the uncolourability of a snark and therefore can be removed in the above sense. The operation which transforms the snark G to the snark H — that is, the operation of removal of unimportant vertices — is a *reduction* of G. Reductions can be classified into *k-reductions* according to how many edges are cut in the process. It is also natural to restrict to *proper* reductions, those where the resulting snark has strictly smaller order than the original one. The removal of a digon, triangle or quadrilateral are thus proper k-reductions for $k = 2, 3, 4$, respectively (see Fig. 1 and Fig. 2).

These considerations suggest that for understanding the substance of nontriviality of snarks it is important to deal with snarks that admit no m-reductions for all positive integers m smaller than a given number k. Such snarks are called *k-irreducible.*

It may seem that by introducing k-irreducible snarks we have obtained infinitely many classes of irreducibility, that is, infinitely many approximations of what a nontrivial snark should be. Surprisingly, this is not true (see [9, Theorem 4.4]):

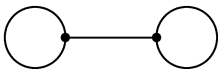

Fig. 3. The dumbbell graph *Db*

For $1 \leqslant k \leqslant 4$, a snark is k-irreducible if and only if it is either cyclically k-edge-connected or the dumbbell graph (see Fig. 3). For $k \in \{5, 6\}$, a snark is k-irreducible if and only if it is *critical* in the following sense: the *suppression* of every edge, indicated in Fig. 4, results in a 3-edge-colourable graph. Equivalently, a snark is critical whenever the removal of any two adjacent vertices produces a 3-edge-colourable graph. For $k \geqslant 7$, a snark is k-irreducible if and only if it is critical and the removal of any two non-adjacent vertices yields a 3-edge-colouring. Such snarks are called *bicritical* or *irreducible* because they are k-irreducible for each k.

Fig. 4. Suppression of an edge e from a graph G

Thus there are only six *irreducibility classes*, with the highest two being formed by the critical and bicritical snarks, respectively. As bicritical snarks admit no proper reductions, it is natural to expect a nontrivial snark to be (at least) bicritical. Observe that the latter requirement fully complies with the classical girth-and-connectivity condition on snarks

for it has been proved [9] that each critical snark is cyclically 4-connected and has girth at least 5.

For reaching the elusive ideal of a nontrivial snark, however, reductions alone are not sufficient. There are still other transformations of snarks that produce simpler snarks from a more complicated one, namely *decompositions*. It has been proved in [9] that there exists a function $\varkappa(k)$ with the following property: *If G is any snark and S is any k-edge-cut in G which yields components H_1 and H_2, both 3-edge-colourable, then it is possible to complete each H_i to a snark G_i by adding at most $\varkappa(k)$ vertices.* In other words, it is possible to *decompose* G into snarks G_1 and G_2. The pair $\{G_1, G_2\}$ is called a *k-decomposition* of G since k edges have been involved. As with reductions, a decomposition is said to be *proper* if both G_1 and G_2 have order smaller than G.

Fig. 5. Snarks G and H, and their dot product $G \cdot H$

In order to be able to handle k-decompositions, the value of $\varkappa(k)$ has to be determined. While k-decompositions with $k \leqslant 3$ are trivial, the only further known values are $\varkappa(4) = 2$ and $\varkappa(5) = 5$. The first interesting case is therefore $k = 4$. It is well known [2, 4, 9] that any 4-decomposition is, essentially, the reverse to the dot product operation. By a *dot product* $G \cdot H$ of two snarks G and H we mean a cubic graph which is constructed as follows. We select two distinct edges e and f in G and two adjacent vertices u and v in H and form $G \cdot H$ from $G - \{e, f\}$ and $H - \{u, v\}$ by joining the resulting vertices of valency 2 in the way depicted in Fig. 5. It is easy to show that the dot product of two snarks is again a snark. A 4-decomposition is thus an operation transforming a snark G into a pair of two (simpler) snarks $\{G_1, G_2\}$, the *factors* of G, such that $G = G_1 \cdot G_2$.

The present paper is devoted to a detailed treatment of 4-decompositions of snarks in terms of dot products and to their interplay with k-reductions.

One can readily verify that the dot product of two k-irreducible snarks where $k \leqslant 3$ is again cyclically k-irreducible. In contrast to this, higher irreducibility classes have a different behaviour. Examples can easily be found to show that the dot product of critical snarks need not be critical, and the same is true for bicritical snarks. The reverse direction — decomposition — is even more interesting (and more difficult). While a 4-decomposition of a cyclically k-connected snark where $2 \leqslant k \leqslant 4$ can result in decreasing the cyclic connectivity and hence the irreducibility class of the constituent factors, for the two highest irreducibility classes this is, essentially, not the case. As we shall see, a 4-decomposition of a bicritical snark yields two smaller bicritical snarks, and a similar but slightly weaker property holds for critical snarks. In the latter case we can even prove the following characterisation result.

Theorem A. *Let G and H be snarks different from the dumbbell graph. Then $G \cdot H$ is critical if and only if H is critical, G is nearly critical, and the pair of edges of G involved in this dot product is essential in G.*

By a *nearly* critical snark we mean one where all the edges are non-suppressible except perhaps those involved in the dot product. The property of being *essential* is a rather technical local property which will be explained later.

In contrast to critical snarks, for bicritical snarks we only have a partial result, nevertheless, one of crucial importance. Its essence is the fact that the class of irreducible snarks is closed under 4-decompositions.

Theorem B. *Let G and H be snarks different from the dumbbell graph. If $G \cdot H$ is bicritical, then both G and H are bicritical. Moreover, the pair of edges of G involved in this dot product is essential in G.*

One naturally asks whether the necessary condition stated in Theorem B is also sufficient. From Theorem A we know that the dot product of irreducible snarks performed by employing an essential pair of edges is certainly critical. Unfortunately, this is not enough: there exist cyclically 4-connected *strictly critical* snarks, snarks that are critical but not bicritical. With the help of a theory based on Theorem A we construct an infinite family of such snarks and show that there exists a strictly critical snark of order n if and only if n is an even integer greater than 30. It should be mentioned that an *ad hoc* construction of strictly critical snarks has been independently given by Steffen and Grünewald [12, 5], but strictly critical snarks were constructed at the same time by the first author of this paper in his Master's Thesis. Our method brings a deeper insight into what makes a snark strictly critical. The construction from [12] is actually covered by our theory. None of these examples, though, excludes the possibility that Theorem B could be reversed.

Theorem B has important consequences. Given an irreducible snark $G \neq Db$ which is not cyclically 5-connected, we can decompose it into a dot product $G = G_1 \cdot G_2$ of two smaller snarks. By the previous theorem, both G_1 and G_2 are irreducible and different from Db. If one of these is again not cyclically 5-connected, we can repeat the process. After a finite number of steps we eventually obtain a collection $H_1, H_2, \ldots, H_r$ of cyclically 5-connected irreducible snarks which cannot be further properly 4-decomposed. Let us call this collection a *composition chain* for G.

Clearly, G can be reconstructed from its composition chain by a successive use of the dot product operation. Unfortunately, the decomposition process, and hence the resulting composition chain, is far from being uniquely determined. The edge-cuts used on our way to a composition chain may intersect in a very complicated fashion, and by choosing one particular 4-edge-cut we may exclude the use of other cuts, including those which do not exist in the original snark but might be created during the decomposition process (note that each 4-decomposition adds two new vertices and one new edge). Moreover, each individual 4-decomposition involves an ordering of the resulting factors, because the two snarks play different roles in the dot product. This may lead to a situation that in different composition chains the same composition factor will play different roles. It

comes therefore as a surprise that regardless of the way in which a given irreducible snark is decomposed, we eventually arrive at essentially the same collection of cyclically 5-connected irreducible snarks. More precisely, any two composition chains contain, up to isomorphism and ordering, exactly the same composition factors. This fact suggests that cyclically 5-connected irreducible snarks can be viewed as basic building blocks of all snarks, and that their role can be compared to the role of prime numbers in the factorisation of integers.

Theorem C. *Every irreducible snark G different from the dumbbell graph can be decomposed into a collection $\{H_1, \ldots, H_n\}$ of cyclically 5-connected irreducible snarks such that G can be reconstructed from them by repeated dot products. Moreover, such a collection is unique up to isomorphism and ordering of the factors.*

We remark that the assumption of a snark G to be irreducible is essential for Theorem C: as we shall see in Section 12, there exist critical snarks that admit non-isomorphic composition chains.

It is natural to ask how the decomposition process of an irreducible snark $G \neq Db$ will be affected if we exclude certain 4-edge-cuts from the decomposition in advance. In this case we will proceed with decomposing as long as permitted edge-cuts are available. By Theorem B, we will again reach a collection $H_1, H_2, \ldots, H_r$ of irreducible snarks which cannot be further properly 4-decomposed by using permitted 4-edge-cuts. For example, given a subgraph K of G, we may require that K must remain intact in each decomposition step. This means that K will eventually become a subgraph of one of the resulting factors H_i. Will then the collection $H_1, H_2, \ldots, H_r$, called a *K-relative composition chain* for G, be still unique? The answer is again surprising.

Theorem D. *Let G be an irreducible snark different from the dumbbell graph, and let K be a fixed subgraph of G of order different from 2. Then G has a K-relative composition chain $\{H_1, H_2, \ldots, H_n\}$ with $K \subseteq H_i$ for some i, and such a chain is unique up to isomorphism and ordering.*

The case where the subgraph K has order 2 is exceptional. Examples in Section 11 show that we may indeed obtain two non-isomorphic K-relative composition chains, depending on the way in which the 4-decompositions are performed.

Our paper is organised as follows. In the next section we collect the basic definitions and give an overview of useful results and techniques to be used later. Section 3 is devoted to a description of colourings of 4-poles that can arise from snarks. Theorems A and B are established in Section 4. Several ideas related to these theorems are further developed in Sections 5 and 6. In particular, we examine the distribution of essential pairs of edges in a snark and construct strictly critical snarks of all possible orders. The last six sections are devoted to factorisation of irreducible snarks. In Sections 7–9 we analyse structural elements of cubic graphs known as *atoms* and study the properties of edge-cuts associated with them. Theorems C and D are proved in Sections 10 and 11, respectively, and the paper closes with a discussion of various aspects of the results proved in this paper.

2 Background

It is convenient to extend the usual definition of a graph by allowing "dangling" and "isolated" edges. These more general objects, often with some additional structure on the set of "free ends" of edges, will be called multipoles. To be more precise, a *multipole* is a pair $M = (V(M), E(M))$ of disjoint finite sets, the *vertex-set* $V(M)$ and the *edge-set* $E(M)$ of M. The size of $V(M)$, denoted by $|M|$, is the the *order* of M. Every edge $e \in E(M)$ has two *ends* and every end of e may or may not be incident with a vertex. If the ends of e are incident with two distinct vertices, then e is a *link*. If both ends are incident with the same vertex, then e is a *loop*. Both loops and links are *proper* edges of a multipole. If one end of e is incident with some vertex but the other not, then e is a *dangling edge*. If no end of e is incident with a vertex, then e is an *isolated edge*. An end of an edge which is incident with no vertex is called a *semiedge*.

All multipoles considered in this paper will be 3-*valent* — that is, every vertex will have valency three — and *ordered* — that is, the set of semiedges of the multipole will be endowed with a linear order. A multipole with k semiedges ($k \geqslant 0$) is called a k-*pole*. Note that a 0-pole is nothing but a cubic *graph*. An ordered k-pole M with semiedges $e_1, e_2, \ldots, e_k$ will be denoted by $M(e_1, e_2, \ldots, e_k)$.

In this paper it is very important to make a clear distinction between identical and isomorphic multipoles as the issue of uniqueness is central to this paper, in particular to Theorems C and D. We therefore assume that the vertices of a graph or a multipole have a fixed labelling and that this labelling is *global*. By this we mean that the label of a vertex is unique not only within a multipole containing it but also within all multipoles which we will be dealing with. If an operation is performed on a multipole, the vertices retain their labels even if moved to another multipole. If a new vertex is added to a multipole, the label of that vertex exists in advance, so technically it does matter which vertex is used for this addition. On the other hand, we will often find it useful to ignore a specific labelling and instead to speak about an isomorphism. What kind of an isomorphism is suitable — that is, what level of abstraction is appropriate — depends on a particular situation. For instance, the global discrimination of vertices is important in the proofs of Lemmas 10.1 and 11.6 and underlies the concept of heredity in Section 9.

As opposed to vertices, no labels are necessary for edges as their identity derives from the identity of their respective end-vertices. The rare case of multiple edges will, if necessary, be handled separately.

There are several operations that can be performed on multipoles. Let M be a multipole and let e and f be two edges of M, not necessarily distinct. Assume that e has a semiedge e' and f has a semiedge f', and that $e' \neq f'$. Then we can perform the *junction* of e' and f' and obtain a new multipole M' as follows:

- If $e \neq f$, we discard e and f from $E(M)$ and replace them by a new edge g whose ends are the other end of e and the other end of f. Thus $E(M') = (E(M) - \{e, f\}) \cup \{g\}$ and g in fact arises from e and f by the identification of e' and f'.

- If $e = f$, then e is an isolated edge. To avoid creating an "isolated loop", we cancel

the edge e and set $E(M') = E(M) - \{e\}$.

The reverse of the junction operation is the *disconnection* of an edge; it produces two new semiedges from any given edge. This operation applies to dangling and isolated edges as well as to links and loops.

Let $M = M(e_1, e_2, \ldots, e_k)$ and $N = N(f_1, f_2, \ldots, f_k)$ be two (ordered) k-poles. Then the *junction* $M * N$ of M and N is the cubic graph which arises from the disjoint union $M \cup N$ by performing the junction $e_i * f_i$ of e_i and f_i for each $i = 1, 2, \ldots, k$. If there are no isolated edges in either M or N, then the set of edges $e_i * f_i$ in $M * N$ forms a k-edge-cut.

The standard notions of graph theory, such as subgraph inclusion extend obviously from graphs to multipoles. In addition to the obvious cases we set $M \subseteq N$ also if the multipole N arises from the multipole M by the junction of some semiedges.

For the sake of clarity we explain how we understand the operations of *removal* of a vertex and that of an edge from a multipole. When removing a vertex, the edges incident with it are not removed. Instead, the ends of edges originally incident with the removed vertex become semiedges. Similarly, when removing an edge, the vertices incident with it are not removed. It follows that to remove a subgraph from a multipole one simply removes all its vertices and edges.

When an edge is removed from a multipole we usually *smooth* any resulting 2-valent vertices as we want to avoid other than 3-valent multipoles. We denote the operation of smoothing of a 2-valent vertex v of the multipole M by $M \sim v$. Further, we define the operation of *suppression* of a link $e = uv$ in a cubic graph G as follows. We remove e from G and subsequently we smooth the 2-valent vertices u and v created by the removal. The resulting graph will be denoted by $G \sim e$ (see Fig. 4).

We proceed to cycles and cuts in multipoles. An easy counting argument shows that there are no acyclic 0-poles and 1-poles, the only acyclic 2-pole is an isolated edge, and the only acyclic 3-pole is a vertex with three dangling edges (a *claw*). There are exactly two acyclic 4-poles, denoted throughout the paper by L and R (see Fig. 6).

Fig. 6.

An edge-cut S in a connected cubic graph G is said to be *cycle-separating* if at least two components of $G - S$ contain cycles. A cubic graph is called *cyclically k-connected* if it has no cycle-separating m-edge-cut for every $m < k$. The largest integer k such that G is cyclically k-connected, provided that it exists, is called the *cyclic connectivity* of G. There are only three cubic graphs (namely K_4, $K_{3,3}$ and θ_2 — the graph consisting of two vertices connected by three parallel edges) for which cyclic connectivity is not defined in the above sense. For these we set the cyclic connectivity to be equal to their cycle rank

(see [8]). Note that the cyclic connectivity of any cubic graph is bounded above by its *girth*, the length of a shortest cycle.

Any minimum cycle-separating edge-cut S in a cubic graph is clearly independent. Indeed, if S contained two adjacent edges (say e and f), let h be the third edge adjacent to both e and f. Then $(S-\{e,f\})\cup\{h\}$ is a cycle-separating edge-cut, too, contradicting the minimality of S. On the other hand, an independent edge-cut separates induced subgraphs with minimum valency at least 2 which means that they must be cyclic. Therefore the study of cycle-separating edge-cuts in cubic graphs is in fact the study of independent edge-cuts.

A 3-*edge-colouring*, or simply a *colouring* of a multipole is a mapping which assigns *colours* 1, 2 and 3 to the edges of the multipole so that adjacent edges (more precisely, adjacent ends) receive distinct colours. A multipole is called *colourable* if it admits a colouring, otherwise it is called *uncolourable*. An uncolourable cubic graph is called a *snark*. We thus leave the notion of a snark as broad as possible.

The following lemma (usually stated for edge-cuts in cubic graphs) is well-known.

Lemma 2.1 (Parity Lemma)**.** *In a k-pole that has been coloured with three colours* 1, 2 *and* 3*, let k_i be the number of semiedges coloured with colour i. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

This result implies, in particular, that a 1-pole is never colourable and that in a 3-pole every colouring assigns three different colours to its three dangling edges.

We often transform a multipole into a snark by adding a few vertices and edges. The following definition makes this idea precise. For $k \neq 1$, we say that a k-pole M *extends* to a snark, if there exists a colourable multipole N such that $M * N$ is a snark. Such $M * N$ will be called a snark *extension* of M. In the special case where $k = 1$ we say that a 1-pole M *extends* to a snark by definition, since there is no colourable 1-pole. A snark extension with minimum order will be called a snark *completion*.

Let G be a snark which can be expressed as a junction $M * N$ of two k-poles M and N, $k \geqslant 0$. If one of M and N, say M, is uncolourable, then M can be extended to a snark $\tilde{M} \supseteq M$ of order $|\tilde{M}| \leqslant |G|$ by adding at most 1 vertex (in the worst case we can take $\tilde{M} = G$). We call $\tilde{M}$ a *k-reduction* of G. This means that the "uncolourable core" of G and $\tilde{M}$ (that is, the multipole M) is the same, but $\tilde{M}$ may be smaller.

A k-reduction $\tilde{M}$ of G is *proper* if $|\tilde{M}| < |G|$. A snark is *k-irreducible* if it has no proper m-reduction for each $m < k$. A snark is *irreducible* if it is k-irreducible for every $k > 0$, that is, if it admits no proper reductions at all. Observe that a k-irreducible snark is also r-irreducible for every $r \leqslant k$; in particular, it is 1-irreducible and hence connected.

A set of vertices or edges of a snark is said to be *removable* if its removal leaves an uncolourable multipole; otherwise it is called *non-removable*. A link of a snark is *suppressible* if its suppression leaves an uncolourable graph; otherwise it is *non-suppressible*. (Note: Our terminology aims to reflect which operations may be performed on a snark without affecting its uncolourability.)

From the Parity Lemma it immediately follows that a set consisting of a single vertex or a single edge is always removable from a snark. Therefore non-removable sets consist of

at least two vertices or edges. Those with exactly two elements are therefore particularly interesting. (Throughout the paper, the term *pair* will automatically be meant to contain two *distinct* objects; degenerate pairs containing the same element twice are excluded.)

Suppressible links and removable pairs of adjacent vertices are closely related. In fact, the following holds.

Proposition 2.2. [9] *A link is suppressible from a snark G if and only if the pair of its end-vertices is removable from G.*

In view of this proposition, we will use the above stated terms as synonyms, always choosing the one that appears to be more suitable for the particular purpose.

We define a snark to be *critical*, if all pairs of its distinct adjacent vertices are non-removable. By Proposition 2.2, this is equivalent to the condition that all its links are non-suppressible. Similarly, we say that a snark is *cocritical*, if all pairs of its distinct non-adjacent vertices are non-removable. If a snark is both critical and cocritical, then we say that it is *bicritical*. The following theorem characterises various degrees of irreducibility in terms of non-removability.

Theorem 2.3. [9] *Let G be a snark. Then the following statements hold true.*

(a) *If $1 \leqslant k \leqslant 4$, then G is k-irreducible if and only if it is either cyclically k-connected or the dumbbell graph.*
(b) *If $k \in \{5, 6\}$, then G is k-irreducible if and only if it is critical.*
(c) *If $k \geqslant 7$, then G is k-irreducible if and only if it is bicritical.*

In particular, the previous theorem shows that irreducible snarks coincide with bicritical ones. These two terms will therefore be used interchangeably.

The standard notion of a snark (cf. [4], for example) requires it to have girth at least 5 and to be cyclically 4-connected. The following proposition shows that critical (and hence also irreducible) snarks are snarks in this traditional sense.

Proposition 2.4. [9] *A critical snark other than the dumbbell graph is cyclically 4-connected and has girth at least 5. Both values are best possible.*

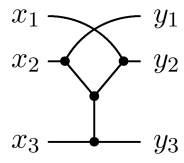

Fig. 7. The multipole Y

There is a well-known infinite family of snarks constructed by Isaacs [6] called *flower snarks* and denoted by I_n. They can be constructed as follows. Let Y be the 6-pole shown in Fig. 7; it is obtained from $K_{3,3}$ by the removal of two non-adjacent vertices. Then I_n is the cubic graph that arises from the disjoint union of n copies Y_i of Y by

identifying the semiedge y_j in Y_i with the semiedge x_j in Y_{i+1} where $j = 1, 2, 3$ and the indices $i \in \{1, 2, \ldots, n\}$ are taken modulo n. For odd $n \geqslant 3$ the graph I_n is a snark, and for $n \geqslant 7$ it is cyclically 6-connected. The flower snarks I_3 and I_5 are displayed in Fig. 8. Moreover, the following is true.

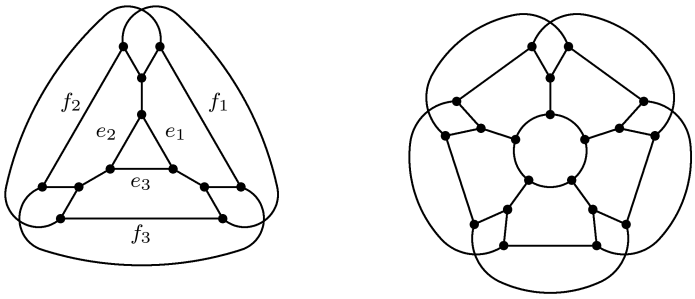

Fig. 8. The flower snarks I_3 and I_5

Proposition 2.5. [9, 11] *The flower snark I_n is irreducible for each odd $n \geqslant 5$.*

Our paper focuses on irreducible snarks of cyclic connectivity 4. The fundamental operation for construction of such snarks is the operation of a *dot product*. It can be defined as follows. Let G and H be cubic graphs. Take two distinct edges e and f in G and disconnect them into semiedges e_1, e_2 and f_1, f_2, respectively. Denote the resulting multipole by G_L. In H, take a link x with end-vertices u and v and remove x together

Fig. 9.

with its end-vertices. The resulting 4-pole H_R has four semiedges x_1^u, x_2^u, and x_1^v, x_2^v, respectively, the former two being originally incident with u in $H - x$, and the latter two

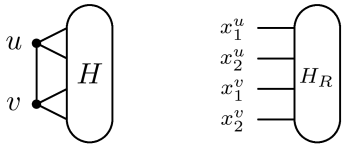

Fig. 10.

being originally incident with v in $H - x$. By joining e_i to x_i^u and f_i to x_i^v $(i = 1, 2)$ we obtain a cubic graph $G_L * H_R$, commonly denoted by $G \cdot H$ and called a *dot product* of G

and H. In terms of multipoles, if $G = G_L * R$ and $H = L * H_R$, then $G \cdot H = G_L * H_R$. We will employ this useful multipole notation of a dot product throughout the paper.

Of course, the symbol $G \cdot H$ does not determine the dot product uniquely, but in each particular case the precise meaning will be clear from the context. We refer to the graphs G and H as to the *left* and *right factor* of $G \cdot H$, respectively. We also say that the edges e, f, and x $(= uv)$ and the vertices u and v are *involved* in the dot product. Note that the edges e and f are only required to be distinct; that is, they can be adjacent. Our definition of the dot product is therefore less restrictive than the usual one.

If there is no loop at either u or v in H and there is no multiple adjacency between these two vertices, then the four newly established edges between G and H form a 4-edge-cut which will be called the *bond* of $G \cdot H$. The edges of the bond naturally split into two *couples*, the first consisting of the edges arising from the junction of the semiedges e_i and x_i^u $(i = 1, 2)$ and the other one consisting of the remaining two edges.

It is well known and easy to see that the dot product of two snarks is a snark; moreover, if both factors are cyclically 4-connected and have girth at least 5, and if the edges e and f from the left factor are non-adjacent, then the dot product is also cyclically 4-connected and has girth at least 5 [2, 4, 6]. Note that the dot product operation has the unique left and right identity element, the dumbbell graph Db, but the operation is not commutative in general.

The dot product operation can be extended to multipoles in the obvious way (see Fig. 11). Thus, if M is an m-pole and N is an n-pole, we can select two *distinct* edges of M and one link of N which will be involved in the dot product. By appropriate transformations we obtain an $(m+4)$-pole M_L and an $(n+4)$-pole N_R such that $M_L * R = M$ and $L * N_R = N$. The dot product $M \cdot N$ of M and N is then the $(m+n)$-pole obtained by the partial junction of M_L and N_R.

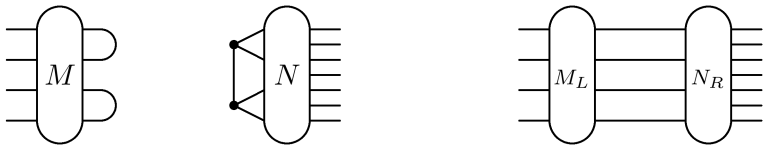

Fig. 11. Dot product of multipoles

Note that the semiedges of M_L can naturally be partitioned into two groups called *connectors*. These connectors are formed by the original semiedges of M and the four new semiedges of M_L, respectively. Therefore we can say that M_L is an $(m, 4)$-pole and similarly, N_R is a $(4, n)$-pole. A multipole with precisely two connectors, such as M_L and N_R, for example, will be called a *dipole*.

3 Colour-open 4-poles

In order to analyse the dot product operation in a greater detail we need to look at colourings that can be induced on the semiedges of a 4-pole. Given a k-pole M =

$M(e_1, e_2, \ldots, e_k)$ we define the *colouring set* of M to be the set

$$\mathrm{Col}(M) = \left\{ \big(\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_k)\big);\ \varphi \text{ is a colouring of } M \right\}.$$

As the colourings "inside" a multipole can usually be neglected, we define two multipoles M and N to be *colour-equivalent* if $\mathrm{Col}(M) = \mathrm{Col}(N)$.

Any colouring of a colourable multipole can be changed to a different colouring by simply permuting the colours in use. In order to describe all admissible colourings of a multipole it is therefore sufficient to know which semiedges are coloured with the same colour and which are coloured with distinct colours. By doing this we actually define the type of a colouring. Technically speaking, the *type* of a colouring φ of a multipole M is the lexicographically smallest sequence of colours assigned to the semiedges of M which can be obtained from φ by permuting the colours. Thus, by the Parity Lemma (2.1), each colouring of a 4-pole has one of the following types: 1111, 1122, 1212, and 1221.

Any colourable 4-pole admits at least two different types of colourings. Indeed, we can start with any colouring and take an arbitrary *Kempe chain* — an alternating chain which begins and ends with a semiedge — and interchange the colours on it to obtain a colouring of another type. Colourable 4-poles thus can have two, three, or four different types of colourings. Those attaining the minimum of precisely two types are of special importance for the study of snarks as will become clear from Proposition 3.4 below; we call them *colour-open* 4-poles (in contrast to colour-closed multipoles discussed in [9]).

Colour-open 4-poles occur in two varieties — isochromatic and heterochromatic 4-poles.

- A 4-pole M will be called *isochromatic* if its semiedges can be partitioned into two pairs such that in every colouring of M the semiedges within each pair are coloured by the same colour. (Example: multipole R).

- A 4-pole M will be called *heterochromatic* of its semiedges can be partitioned into two pairs such that in every colouring of M the semiedges within each pair are coloured by distinct colours. (Example: multipole L).

The pairs of semiedges of an isochromatic or a heterochromatic 4-pole mentioned above will be called *couples*. No confusion should arise between the couples of semiedges in a multipole and the couples of edges in a bond of a dot product since (as we shall see later) they correspond to each other.

Observe that each colour-open 4-pole has one of $\binom{4}{2} = 6$ possible combinations of colouring types. By checking all these combinations we obtain the following two results.

Proposition 3.1. *Every colour-open* 4*-pole is either isochromatic or heterochromatic, but not both. Moreover, it is isochromatic if and only if it admits a colouring of type* 1111*.*

Proposition 3.2. *Every colour-open* 4*-pole can be extended to a snark by adding at most two vertices, and such an extension is unique up to isomorphism. A heterochromatic multipole is extended by joining the semiedges in each couple, that is, by adding no new vertex. An isochromatic multipole is extended by attaching the semiedges in each couple to a new vertex, and by connecting these two vertices with a new edge.*

The extensions described in the preceding proposition are, in fact, completions. Thus every colour-open 4-pole has a completion to a snark which is unique up to isomorphism.

Definition 3.3. *For a colour-open 4-pole M its isomorphically unique completion will be denoted by $\tilde{M}$.*

The next result shows that the completion operator $M \mapsto \tilde{M}$ does not extend to a wider class of 4-poles.

Proposition 3.4. *A colourable 4-pole extends to a snark if and only if it is colour-open.*

Proof. The sufficient condition follows from Proposition 3.2. We prove the necessity. Assume that a colourable 4-pole M extends to a snark. Then there is a colourable 4-pole N such that $M * N$ is a snark. Each of M and N admits at least two different types of colourings. However, they cannot have a colouring type in common since a suitable permutation of colours would yield a colouring of $M * N$, which is a contradiction. As there are only four admissible types altogether, each of M and N must admit precisely two of them. In particular, M must be colour-open. □

Having characterised colour-open 4-poles we can now proceed deeper into the structure of snarks which can be expressed as a junction of two such 4-poles.

Proposition 3.5. *If M and N are colourable 4-poles such that $M*N$ is a snark, then one of them is isochromatic and the other one is heterochromatic. Moreover, the semiedges belonging to the same couple in M are joined with the semiedges belonging to the same couple in N.*

Proof. By Proposition 3.4, each of $M = M(e_1, e_2, e_3, e_4)$ and $N = N(f_1, f_2, f_3, f_4)$ is colour-open and hence, by Proposition 3.1, either isochromatic or heterochromatic. Since there are only four admissible colouring types, and the multipoles cannot have a colouring type in common, exactly one of them, say M, allows the type 1111. Proposition 3.1 now implies that M is isochromatic and N is heterochromatic.

Along with 1111, M contains one of the remaining three colouring types. Assume that M admits the type 1212 (for 1122 and 1221 the consideration is similar). Then N has types 1221 and 1122, and we see that the couples of M are $\{e_1, e_3\}$ and $\{e_2, e_4\}$ while those of N are $\{f_1, f_3\}$ and $\{f_2, f_4\}$. Thus the couples are joined correspondingly. □

As a corollary of the previous proposition we get:

Proposition 3.6. *Let K be a snark which can be expressed as a dot product $K = G \cdot H$ of snarks G and H. If G_L is colourable, then it is colour-equivalent to L and hence heterochromatic. If H_R is colourable, then it is colour-equivalent to R and hence isochromatic.*

Proof. Since G_L is colourable and $G = G_L * R$, Proposition 3.4 implies that G_L is colour-open. It cannot be isochromatic, for otherwise Proposition 3.1 would yield a colouring of type 1111 and G would be colourable. Hence, it is heterochromatic. By a direct verification we deduce that the only possibility for the distribution of colouring types on

the semiedges of G_L not contradicting the uncolourability of G is the one equivalent to L. Moreover, the couples of semiedges of G_L as a heterochromatic multipole are precisely the dot product couples of $K = G \cdot H = G_L * H_R$.

The proof of the second part is similar. □

As we have just seen, the dot product couples and the couples of colour-open multipoles coincide. This coincidence is important for the proof of the next theorem which shows that the dot product operation can sometimes be reversed in a unique way.

Theorem 3.7. (cf. [2, 4, 9]) *Every 4-edge-cut S in a critical snark G separates G into two colour-open multipoles M and N, one isochromatic and one heterochromatic. Moreover, G can be expressed as a dot product of their respective completions $\tilde{M}$ and $\tilde{N}$ in such a way that S is the bond. This expression is unique up to isomorphism.*

Proof. As above, let M and N be the 4-poles resulting from the disconnection of S. Obviously, both of them have at least two vertices and must be colourable. If, say, M were uncolourable, it could be extended to a snark without adding a single vertex, and this snark would then be a proper 4-reduction of G. Therefore, both M and N are colour-open. Since $G = M * N$, from Proposition 3.5 we deduce that one of these 4-poles (say M) is heterochromatic and the other one (say N) is isochromatic, with the couples being mutually joined. It follows that $\tilde{N}$ arises from N by joining the semiedges of each couple to a new vertex and adding an edge between the new pair of vertices, whereas $\tilde{M}$ is obtained from M by joining the semiedges of each couple to each other. Since the couples in M and N match, we see that $G = \tilde{M} \cdot \tilde{N}$ and that S is the bond of this dot product. □

We have just shown that, given a critical snark G and a 4-edge-cut S in G, there is a unique way of decomposing G along S into a dot product of two snarks with S being the bond. These two snarks will be called the *S-factors* of G, or the *direct factors* of G, if no cut has been explicitly specified. We also allow the trivial factorisation of G in the form $G = G \cdot Db = Db \cdot G$, although in this case the direct factors need not be necessarily determined by a 4-edge-cut.

The uniqueness of $\tilde{M}$ and $\tilde{N}$ stated in Theorem 3.7 depends on the chosen cut and therefore is only local, in contrast to the global uniqueness of Theorem C. Theorem 3.7 constitutes the first major step towards Theorem C, nevertheless it does not imply the global uniqueness in general. As we shall see in Section 12, there exist critical snarks which admit two essentially different ways of factorisation although each factorisation step produced a unique pair of factors.

4 Critical and bicritical snarks

In this section we establish conditions under which a dot product of two snarks becomes critical or bicritical. Our aim is to prove the first two main results of this paper, Theorem A and Theorem B.

Theorem 3.7 implies that all critical snarks of cyclic connectivity 4 can be constructed from smaller snarks by using a repeated dot product. It is therefore possible to characterise all critical snarks of cyclic connectivity 4 by finding necessary and sufficient conditions for a dot product of two snarks to be critical. Analogously, to characterise all irreducible snarks of cyclic connectivity 4 it is sufficient to find necessary and sufficient conditions for a dot product of two snarks to be irreducible. In order to make such characterisations possible, we first develop a simple idea of replacing a colour-open 4-pole by a smaller colour-equivalent 4-pole. In fact, we have already used this idea in the proof of Proposition 3.6.

Let M_1 and M_2 be colour-equivalent m-poles, and let N be an arbitrary (m, n)-dipole. Consider the multipoles $M_1 * N$ and $M_2 * N$. In these partial junctions the m-connector of N is joined to the multipole M_1 or M_2, respectively. The semiedges of the n-connector of N are not involved in these partial junctions and are therefore the semiedges of both $M_1 * N$ and $M_2 * N$. Obviously, the multipoles $M_1 * N$ and $M_2 * N$ are colour-equivalent. This situation can be conveniently seen as a substitution of the sub-multipole M_1 in $M_1 * N$ by a colour-equivalent multipole M_2. In other words, we have the following result.

Lemma 4.1 (Substitution Lemma)**.** *Substitution of a sub-multipole in a multipole M by a colour-equivalent multipole results in a multipole that is colour-equivalent to M.*

Actually, we can view the dot product as an instance of substitution: If G and H are snarks such that the multipoles G_L and H_R obtained by cutting the bond of $G \cdot H$ are colourable, then Proposition 3.6 implies that G_L and H_R are colour-equivalent to the multipoles L and R, respectively. Thus $G \cdot H$ can be regarded both as a substitution of R in G by H_R, and as a substitution of L in H by G_L. With the help of Theorem 3.7 this point of view can be reversed as the following remark shows.

Remark 4.2. Snark completions $\tilde{M}$ and $\tilde{N}$ in Theorem 3.7 can be interpreted in terms of substitution. The snark $\tilde{M}$ arises from G by substituting N with that one of L and R which is colour-equivalent to N, while $\tilde{N}$ arises by substituting M by the other one, the couples being matched. Thus each 4-decomposition of a critical snark into a dot product of snarks can be regarded as a pair of colour-equivalent substitutions.

The following result uses the Substitution Lemma.

Proposition 4.3. *Let K be a snark which can be expressed as a dot product $K = G \cdot H$ of snarks G and H such that both G_L and H_R are colourable. Then a pair of distinct vertices of G_L (respectively, of H_R) is non-removable from $G \cdot H$ if and only if it is non-removable from G (respectively, from H).*

Proof. Let $\{u, v\}$ be a pair of distinct vertices of G_L. By removing u and v from G and from $G \cdot H$ we obtain multipoles M and $M \cdot H$, respectively. The pair $\{u, v\}$ is non-removable from G if and only if M is colourable, and it is non-removable from $G \cdot H$ if and only if $M \cdot H$ is colourable. By Proposition 3.6, the assumption of colourability of H_R implies that H_R is colour-equivalent to R. One thus obtains $M \cdot H$ from M by a substitution of R with H_R. The Substitution Lemma now implies that $M \cdot H$ and M are colour-equivalent, as our statement claims.

The proof of the second part is similar. □

We continue with a detailed analysis of suppressible edges in a dot product $G \cdot H$ of two snarks G and H. Proposition 4.3 implies that if an edge $e = uv$ already exists in G or in H, then it is suppressible from $G \cdot H$ if and only if it is suppressible from the factor containing it, provided that both G_L and H_R are colourable. However, $G \cdot H$ contains four new edges originally present in neither G nor H — namely the edges of the bond. To handle these edges we introduce the following concept.

Definition 4.4. *A pair of distinct edges $\{e, f\}$ of a snark G is essential if it is non-removable and for every 2-valent vertex v of $G - \{e, f\}$ the graph $(G - \{e, f\}) \sim v$ is colourable.*

This definition can be restated as follows. For an ordered m-pole M without isolated edges, $m \geqslant 1$, let $M_{(i)}$ denote the $(m-1)$-pole obtained from M by removing its i-th dangling edge and by suppressing the resulting 2-valent vertex. Furthermore, put $M_{(0)} = M$. Now, let M be the 4-pole obtained from a snark G by disconnecting a pair of edges. Then a necessary and sufficient condition for the pair in question to be essential is: *any of the five multipoles $M_{(i)}$, $i \in \{0, 1, 2, 3, 4\}$, is colourable.*

Observe that in any snark G a non-removable pair of edges is independent. Indeed, if the edges e and f shared a common vertex v, then the 4-pole M constructed from G by disconnecting e and f would contain the 3-pole M' obtained from G (or from M) by removing v. By the Parity Lemma (2.1), M' is uncolourable, so the same must be true for $M \supseteq M'$. As this is in contradiction with the definition of non-removability, we deduce that the edges in an essential pair are non-adjacent.

Definition 4.4 does not confine itself to pairs of links, so $G - \{e, f\}$ can contain fewer than four 2-valent vertices. For example, the only pair of non-adjacent edges in the dumbbell graph is essential although both edges of this pair are loops.

The following lemma examines the dot product of an arbitrary snark with a critical snark.

Proposition 4.5. *Let $G \cdot H$ be a dot product of snarks where H is a critical snark different from the dumbbell graph and the pair of edges $\{e, f\}$ of G involved in the dot product is essential. Then the following hold.*

(a) *Every suppressible link of G different from e and f is a suppressible edge of $G \cdot H$.*
(b) *Every suppressible link of $G \cdot H$ is a suppressible edge of G.*

In other words, the suppressible links of $G \cdot H$ are those inherited from G.

Proof. First of all, observe that both G_L and H_R are colourable. The former follows from the assumption that $\{e, f\}$ is an essential pair of edges of G while the latter is a consequence of the fact that H is critical.

To prove (a), let $x \neq e, f$ be a suppressible edge of G which is not a loop. Then x is a proper edge of G_L and hence it is an edge of $G \cdot H$. The conclusion now follows from Proposition 4.3. (The edges e and f have to be excluded since they are destroyed by the dot product operation and hence do not exist in $G \cdot H$.)

For the froof of (b), let a_i $(i = 1, 2, 3, 4)$ denote the edge of $G \cdot H$ obtained by the junction of the i-th semiedge of G_L with that of H_R. Furthermore let u and v be the end-vertices of the edge of H involved in the dot product $G \cdot H$.

Consider a suppressible link x of $G \cdot H$. Since H is critical, it follows from Proposition 4.3 that x cannot be a proper edge of H_R. Thus x is either an edge of the bond or an edge of G_L.

Assume that $x = a_i$ for some $i \in \{1, 2, 3, 4\}$. As H is critical and different from Db, Proposition 2.4 implies that it is cyclically 4-connected; in particular, it contains no parallel edges and loops. Thus there are four pairwise distinct edges in H adjacent to either u or v other than uv. Denote these edges by x_i $(i = 1, 2, 3, 4)$ so that their numbering agrees with the numbering of the edges a_i. In this situation the edges a_i cannot be loops in $G \cdot H$.

Consider the graph $H \sim x_i$. The suppression of x_i destroys exactly one of the vertices u and v. Without loss of generality assume that the destroyed vertex is u. Since H is critical, $H \sim x_i$ is colourable, and hence the removal of the other vertex (that is, v) produces a colourable 3-pole $(H \sim x_i) - v$. Of course, the semiedges of this 3-pole must be coloured by three distinct colours. On the other hand, the pair $\{e, f\}$ is essential in G. Hence the 3-pole $(G_L)_{(i)}$ is colourable, too, and its semiedges will, again, be coloured by different colours. By applying a suitable permutation of colours we obtain a colouring of the junction $(G_L)_{(i)} * ((H \sim x_i) - v)$. However, this graph is nothing but $(G \cdot H) \sim a_i$ which means that $a_i = x$ is a non-suppressible edge of $G \cdot H$. We have arrived at a contradiction.

There is only one possibility left, namely that x is a proper edge of G_L. But in this case x is an edge of G different from e and f. By Proposition 4.3, it is a suppressible edge of G. □

Now we are ready to prove Theorem A.

Theorem 4.6. *Let G and H be snarks, $H \neq Db$. Further, let $\{e, f\}$ be the pair of edges of G involved in the dot product $G \cdot H$. Then $G \cdot H$ is critical if and only if H is critical, the pair $\{e, f\}$ is essential in G, and every link of G different from e and f is non-suppressible from G.*

Roughly speaking, the result says that the right factor of a critical dot product must be critical, but the left factor only has to be "nearly" critical. The theorem also explains the background of the term *essential*: such a pair of edges is essential for a dot product to be critical.

Proof. ($\Leftarrow$) Let G and H satisfy the assumptions of the theorem. If $G \cdot H$ had a suppressible edge, by Proposition 4.5 it would have to be a suppressible link of G_L. Since there are no such edges, $G \cdot H$ is critical.

($\Rightarrow$) Let $G \cdot H$ be a critical snark, and let u and v be the end-vertices of the edge of H involved in $G \cdot H$. As above, denote by a_i $(i = 1, 2, 3, 4)$ the edge of $G \cdot H$ obtained by the junction of the i-th semiedge of G_L with that of H_R.

Clearly, $|G_L| = |G| \geqslant 1$. Furthermore, the assumption that $H \neq Db$ implies that $|H_R| \geqslant 1$. By Theorem 3.7, the multipoles G_L and H_R are colourable and we may use Proposition 4.3 to conclude that all links of G different from e and f are non-suppressible from G. This is exactly what we need to know about G.

We show that H is critical. First of all, Proposition 4.3 implies that all links of H_R are non-suppressible from H. Moreover, the pair $\{u, v\}$ of adjacent vertices is non-removable from H because H_R is colourable. Let us look closer at the edges of H incident with u or v. As $H \neq Db$, at most one of these two vertices is incident with a loop. The existence of a loop at just one of these vertices would lead to a cycle-separating 2-edge-cut in $G \cdot H$. But since $G \cdot H$ is critical, it is cyclically 4-connected or the dumbbell graph, as Proposition 2.4 claims. The latter possibility would imply $G = H = Db$, which contradicts the assumption. Therefore, there are no loops at u and v in H. Moreover, there is no edge parallel to uv in H for otherwise the colouring of H_R would directly extend to a colouring of H. Thus, in addition to uv, there are exactly four pairwise distinct edges in H incident with u or v. Denote these edges by x_i $(i = 1, 2, 3, 4)$ so that their numbering agrees with that of the edges a_i.

To show that H is critical it remains to prove that all the graphs $H \sim x_i$ are colourable. Since there are no loops at both u and v in H and the edge uv has no parallel counterpart, the end-vertices of each a_i are one in G_L and the other in H_R. As $G \cdot H$ is critical, $(G \cdot H) \sim a_i$ is colourable. By disconecting the remaining three edges of the original dot product 4-edge cut in $(G \cdot H) \sim a_i$ we obtain two 3-poles: $(G_L)_{(i)}$ on the side of G, and another one on the side of H. The three semiedges of the latter 3-pole are, by the Parity Lemma, always coloured by three distinct colours. Hence by joining them to a new vertex we obtain a colourable graph. It is obvious from the construction that this graph is exactly $H \sim x_i$. Thus H is critical.

The only thing left is to verify that the pair $\{e, f\}$ is essential in G. We have already seen that each $(G \cdot H) \sim a_i$ is a colourable graph. Therefore the multipole $(G_L)_{(i)}$ is colourable for $i = 1, 2, 3, 4$. As $(G_L)_{(0)} = G_L$ is colourable, too, the pair $\{e, f\}$ is essential, as required. This completes the proof. □

The following straightforward corollary shows that in a critical dot product we can substitute the right factor (which is necessarily critical) by any other critical snark different from the dumbbell graph.

Corollary 4.7. *Let $G \cdot H$ be a critical snark arising from snarks G and H, $H \neq Db$, and let $H' \neq Db$ be any other critical snark. Then $G \cdot H'$ is critical provided that the pair of edges of G involved in $G \cdot H'$ is the same as that involved in $G \cdot H$.*

We proceed to Theorem B.

Theorem 4.8. *Let G and H be snarks. If $G \cdot H$ is irreducible, then both G and H are irreducible. Moreover, the pair of edges of G involved in this dot product is essential in G.*

Proof. Assume that $G \cdot H$ is irreducible. From Proposition 4.3 we already know that a pair of distinct vertices of G is removable from G if and only if it is removable from $G \cdot H$. As $G \cdot H$ is irreducible, it follows that so is G.

Since $G \cdot H$ is critical, from Theorem 4.6 we deduce that H is critical and the pair of edges of G involved in this dot product is essential in G. Thus it remains to show that H is cocritical. In order to do so, we need to verify that an arbitrary pair $\{x, y\}$ of distinct non-adjacent vertices of H is non-removable from H. We distinguish two cases with respect to the mutual position of $\{x, y\}$ and the pair $\{u, v\}$ of vertices involved in the dot product $G \cdot H$.

Case 1. The pairs $\{x, y\}$ and $\{u, v\}$ are disjoint. This means that $\{x, y\}$ is contained in H_R. Again, we may use Proposition 4.3 to conclude that the pair $\{x, y\}$ is non-removable from H.

Case 2. The pairs $\{x, y\}$ and $\{u, v\}$ have exactly one vertex in common. Without loss of generality we can assume that $x = u$ and $y \neq v$. Observe that the vertex u is incident in H with two of the semiedges of H_R. Denote these two semiedges arbitrarily by f_1 and f_2, and the other two semiedges, also arbitrarily, by f_3 and f_4. Furthermore, denote the semiedges of G_L by e_1, e_2, e_3, and e_4 consistently with the ordering of the semiedges in H_R, that is, in such a way that $G \cdot H$ is obtained by the junction of e_i with f_i, $1 \leqslant i \leqslant 4$. In turn, denote the four resulting edges by a_i.

Now let w be the end-vertex of the dangling edge corresponding to e_1 in G_L and take the multipole $M = (G \cdot H) - \{w, y\}$. Since $G \cdot H$ is irreducible and $w \neq y$, we see that M is colourable. We claim that in any colouring of M, the edges a_3 and a_4 receive distinct colours. To see this, observe that M can be expressed as a partial junction of $M = (G_L - w)$ and $(H_R - y)$ where e_i is joined to f_i for $i = 2, 3, 4$, and the other semiedges of $G_L - w$ and $H_R - y$ remain free. Now, if the edges a_3 and a_4 were coloured with the same colour, the colouring of $G_L - w$ could be extended to a colouring of $G - w$ by performing the junction of e_3 and e_4. Since $G - w$ is a 3-pole, its semiedges must be coloured with three distinct colours and such a colouring can further be extended to a colouring of G, which is impossible.

From the fact that the edges a_3 and a_4 must in M be coloured with distinct colours we conclude that the colours of f_3 and f_4 in $H_R - y \subseteq M$ must also be distinct. Hence, $H_R - y$ has a colouring that can be extended to a colouring of $H - \{u, y\}$ by attaching the vertex v to the semiedges f_3 and f_4. This means that the pair of vertices $\{u, y\}$ is non-removable from H.

Summing up, we have shown that any pair of distinct non-adjacent vertices of H is non-removable from H; thus H is cocritical and hence irreducible. The proof is complete. □

5 Essential pairs of edges

Theorem 4.6 and Theorem 4.8 indicate that essential pairs of edges play a crucial role in the study of snarks of cyclic connectivity 4. We now explore their distribution in snarks along with the effect of the dot product on them. We start with two corollaries of the Substitution Lemma 4.1.

Lemma 5.1. *Let G be a snark and let N be an m-pole ($m \geqslant 1$) without isolated edges. Assume that the pair of edges of G involved in the dot product $G \cdot N$ is essential in G.*

Then for each $i \in \{0, 1, \ldots, m\}$ the multipole $(G \cdot N)_{(i)}$ is colour-equivalent to $N_{(i)}$.

Proof. Let $\{u, v\}$ be the pair of distinct vertices of N involved in the dot product $G \cdot N$. For $i = 0$ we clearly have that $(G \cdot N)_{(i)} = G \cdot N_{(i)}$. The same also holds for $i = 1, 2, \ldots, m$ provided that the end-vertex of the i-th dangling edge of N is different from both u and v. Indeed, the multipole $G \cdot N_{(i)}$ arises from $N_{(i)}$ by substituting its sub-multipole L, determined by the vertices u and v, with G_L. Since these two multipoles are colour-equivalent by Proposition 3.6, our claim follows from the Substitution Lemma 4.1.

We are thus left with the case where $i \neq 0$ and the end-vertex of the i-th dangling edge of N is one of u or v, say u. Note that $G \cdot N = G_L * N_R$. In this situation the i-th semiedge of N_R is the semiedge of an isolated edge in N_R. Let N' be the $(3, m-1)$-dipole obtained by removing this edge from N_R. Then $(G \cdot N)_{(i)} = (G_L)_{(i)} * N'$.

Since the pair of edges of G involved in the dot product $G \cdot N$ is essential, it follows that $(G_L)_{(i)}$ is colourable. The semiedges of this 3-pole must be coloured by three distinct colours. This means that $(G_L)_{(i)}$ is colour-equivalent to the *claw* C, that is, to three dangling edges incident with one vertex. Therefore, by the Substitution Lemma, $(G_L)_{(i)} * N'$ is colour-equivalent to $C * N' = N_{(i)}$, as required. □

Lemma 5.2. *Let H be a snark and let M be an m-pole ($m \geqslant 1$) without isolated edges. Assume that the edge of H involved in the dot product $M \cdot H$ is non-suppressible from H. Then the following statements hold:*

(a) *The multipoles $(M \cdot H)_{(0)}$ and $M_{(0)}$ are colour-equivalent.*
(b) *For $i = 1, 2, \ldots, m$, if the i-th dangling edge of M is none of the edges involved in the dot product $M \cdot H$, then multipole $(M \cdot H)_{(i)}$ is colour-equivalent to $M_{(i)}$.*

Proof. Under the conditions stated in parts (a) and (b) of the lemma we have that $(M \cdot H)_{(i)} = M_{(i)} \cdot H$. However, the multipole $M_{(i)} \cdot H$ arises from $M_{(i)}$ by substituting its sub-multipole R with the colour-equivalent multipole H_R. Our claim now follows from the Substitution Lemma. □

The following two theorems are the main results of this section.

Theorem 5.3. *Let G and H be snarks. Assume that the pair of edges involved in the dot product $G \cdot H$ is essential in G and that the edge of H involved in this dot product is non-suppressible from H. Then the following statements hold:*

(a) *A pair of distinct edges of G neither of which is involved in the dot product $G \cdot H$ is essential in $G \cdot H$ if and only if it is essential in G.*
(b) *A pair of distinct edges of H neither of which is involved in the dot product $G \cdot H$ is essential in $G \cdot H$ if and only if it is essential in H.*

Proof. Let $\{e, f\}$ be the pair of edges of G and let $\{u, v\}$ be the pair of vertices of H that are involved in the dot product $G \cdot H$.

To prove (a) consider a pair $\{x, y\}$ of edges of G neither of which is involved in the dot product $G \cdot H$. Thus $\{x, y\}$ is disjoint from $\{e, f\}$. Let M denote the 4-pole obtained

from G by disconnecting both x and y. By disconnecting the same edges in $G \cdot H$ we obtain $M \cdot H$. From Lemma 5.2 we obtain that M and $M \cdot H$ are colour-equivalent, and so will be $M_{(i)}$ and $(M \cdot H)_{(i)}$ for each $i = 0, 1, 2, 3, 4$. Therefore $\{x, y\}$ is essential in G if and only if it is essential in $G \cdot H$.

The proof of part (b) uses Lemma 5.1 and is otherwise similar to the proof of part (a). □

Theorem 5.4. *Let G and H be snarks, $H \neq Db$, and let $G \cdot H$ be a critical snark. Then the following statements hold:*

(a) *A pair of edges in the bond of $G \cdot H$ belonging to the same couple is removable from $G \cdot H$, and hence not essential.*
(b) *A pair of edges in the bond of $G \cdot H$ belonging to different couples is essential in $G \cdot H$.*

Proof. Let $\{u, v\}$ be the pair of vertices involved in this dot product. Let us express $G \cdot H$ as the junction $G_L * H_R$ of the ordered multipoles $G_L(e_1, e_2, e_3, e_4)$ and $H_R(f_1, f_2, f_3, f_4)$ in such a way that G is obtained from G_L by joining e_1 to e_2 and e_3 to e_4, and H is obtained from H_R by adding two adjacent vertices u and v and attaching f_1 and f_2 to u, and f_3 and f_4 to v. For $i \in \{1, 2, 3, 4\}$ let a_i denote the edge arising from the junction of e_i to f_i in $G \cdot H$. Then $\{a_1, a_2\}$ is the first couple of the bond and $\{a_3, a_4\}$ is the second one. The edges a_i are pairwise distinct and have one end-vertex in G_L and the other one in H_R because $G \cdot H$ is cyclically 4-connected and $H \neq Db$.

We first prove (a). Without loss of generality consider the pair $\{a_1, a_2\}$. By disconnecting both a_1 and a_2 in $G_L * H_R$ we obtain a 4-pole which is a partial junction of $G_L(e_1, e_2, e_3, e_4)$ and $H_R(f_1, f_2, f_3, f_4)$ where e_3 is joined to f_3 and e_4 to f_4, the other four semiedges remaining free. Denote this 4-pole by S (see Fig. 12). Note that S cannot be colourable, since, by Proposition 3.6, the edges a_3 and a_4 would have to be coloured with the same colour and at the same time with two distinct colours. It follows that the pair $\{a_1, a_2\}$ is removable from $G \cdot H$.

Fig. 12.

We proceed to part (b). Without loss of generality consider the pair $\{a_1, a_3\}$. By disconnecting both a_1 and a_3 in $G_L * H_R$ we obtain a 4-pole which is a partial junction of $G_L(e_1, e_2, e_3, e_4)$ and $H_R(f_1, f_2, f_3, f_4)$ where e_2 is joined to f_2 and e_4 to f_4 and the other four semiedges remain free. Denote this 4-pole by T (see Fig. 12). We need to prove that the multipoles $T_{(i)}$ $(i = 0, 1, 2, 3, 4)$ are all colourable.

Clearly, $|G_L| = |G| \geqslant 1$ and $|H_R| \geqslant 1$. Moreover, by Theorem 3.7, both G_L and H_R are colourable. Proposition 3.6 further implies that G_L has a colouring of type 1212 and H_R has a colouring of type 1111. These colourings can easily be combined into a colouring of T; hence $T_{(0)} = T$ is colourable.

Now let $i \neq 0$. The semiedges of T are of two kinds: those belonging to G_L and those belonging to H_R. First consider the semiedges of T belonging to G_L. Without loss of generality, let $T_{(i)}$ be obtained from T by the removal of the dangling edge with semiedge e_1 and by the subsequent suppression of the resulting 2-valent vertex. Then $T_{(i)}$ is a partial junction of $(G_L)_{(1)}(e_2, e_3, e_4)$ and $H_R(f_1, f_2, f_3, f_4)$ where e_2 is joined to f_2 and e_4 to f_4, the other three semiedges remaining free. Since $G \cdot H$ is critical, Theorem 4.6 implies that the pair of edges of G involved in this dot product is essential; hence $(G_L)_{(1)}$ is colourable. The Parity Lemma (2.1) forces the semiedges of $(G_L)_{(1)}$ to be coloured differently, therefore the colours of e_2 and e_4 are distinct, say 1 and 2. In view of Proposition 3.6, H_R has a colouring of type 1122. It follows that the colourings of $(G_L)_{(1)}$ and H_R can be extended to a colouring of $T_{(i)}$.

Now consider the semiedges of T belonging to H_R. Without loss of generality, let $T_{(i)}$ arise from T by the removal of the dangling edge with semiedge f_1 and by the subsequent suppression of the resulting 2-valent vertex. Denote this vertex by w. Then $T_{(i)}$ is a partial junction of $G_L(e_1, e_2, e_3, e_4)$ and $(H_R)_{(1)}(f_2, f_3, f_4)$ where e_2 is joined to f_2 and e_4 to f_4, the other three semiedges remaining free.

The multipole $(H_R)_{(1)}$ arises from the graph $H \sim uw$ by removing v. Since H is critical by Theorem 4.6, $H \sim uw$ is colourable, so the same must be true for $(H_R)_{(1)}$. By the Parity Lemma, the semiedges of $(H_R)_{(1)}$ must be coloured by three distinct colours. Let f_2 be coloured by 2 and f_4 by 1. By Proposition 3.6, G_L has a colouring of type 1221. It follows that the colourings of G_L and $(H_R)_{(1)}$ can be extended to a colouring of $T_{(i)}$. The proof is complete. □

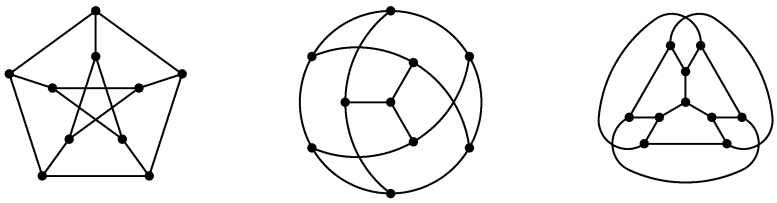

Fig. 13. Different drawings of the Petersen graph *Ps*

Example 5.5. Let us examine essential pairs of edges in Isaacs' flower snarks. We claim that each pair of non-adjacent edges of the flower snark I_n, $n \geqslant 5$ odd, is essential. To prove this we employ induction with respect to n. The validity of the statement for $n = 5$ and $n = 7$ has been verified by using a computer. Let $n \geqslant 9$ and let N_k denote the 6-pole arising from the union of k copies Y_i of the multipole Y (Fig. 7) by identifying the semiedge y_j in Y_i with the semiedge x_j in Y_{i+1} for $j = 1, 2, 3$ and $i = 1, 2, \ldots, k-1$.

Taking into account the colouring properties of Y it can easily be checked that N_{2m} is colour-equivalent to N_2 for every $m \geqslant 1$.

Select a pair of non-adjacent edges in I_n and form a 4-pole M from I_n by disconnecting both of them. We have to verify that each of the multipoles $M_{(i)}$, $i = 0, 1, 2, 3, 4$, is colourable. It is easy to see that, regardless of the choice of such a pair, the multipole $M_{(i)}$ contains a copy of N_4. By replacing this copy of N_4 with a copy of N_2 we obtain a similar multipole arising from I_{n-2}. The latter multipole is colourable by the induction hypothesis. Since N_4 is colour-equivalent to N_2, the Substitution Lemma implies that $M_{(i)}$ is colour-equivalent to the corresponding multipole obtained from I_{n-2}. Hence, $M_{(i)}$ is colourable, too. This proves that any pair of non-adjacent edges in a flower snark is essential.

It is worth of mentioning that both the Petersen graph (Fig. 13) and the double-star snark Ds (Fig. 14) have the property that every pair of non-adjacent edges is essential. □

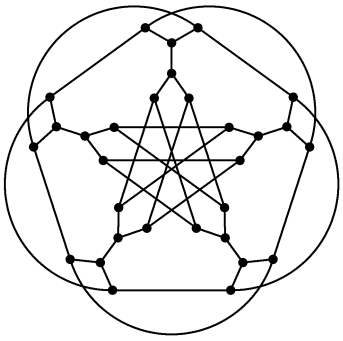

Fig. 14. The double-star snark Ds

Our description of essential pairs of edges in critical snarks of cyclic connectivity 4 stated in Theorems 5.3 and 5.4 is incomplete since it only covers pairs of edges in the same factor or in the bond of a dot product. Additional essential pairs are provided by the next theorem.

Theorem 5.6. *Any pair of edges of a critical snark at distance* 1 *is essential.*

Proof. Let G be a critical snark and let $\{e, f\}$ be a pair of edges of G such that there is an edge g connecting an end-vertex u of e to and end-vertex v of f. Let x be the third edge at u and let y be the third edge at v.

Let us form the dot product $H = G \cdot Ps$ which involves the pair $\{e, f\}$ on the side of G. Since both G and Ps are critical, Theorem 4.6 yields that H is critical if and only if the pair $\{e, f\}$ is essential. It is therefore sufficient to show that H is critical.

Let a_i, $i = 1, 2, 3, 4$, be the edges of the bond of H labelled in such a way that a_2 is incident with u and a_3 is incident with v; thus $\{a_1, a_2\}$ and $\{a_3, a_4\}$ are the couples of the bond. Since the edges e and f have distance 1, the set $R = \{a_1, x, y, a_4\}$ is a cycle-separating 4-edge-cut, too. (This observation anticipates the situation studied in Section 8 where similar pairs of cuts are termed atomic and quasiatomic cuts, respectively.)

By Theorem 3.7, the latter cut gives rise to a decomposition of H into a dot product of two smaller snarks. It is easy to see that in this case H is expressed as $Ps \cdot G$ where the pair vertices of G involved in the dot product is just $\{u, v\}$. On the side of Ps the pair of edges involved in this dot product has distance 1, and it is easy to see that such a pair is essential (cf. end of Example 5.5). It follows that the snark $H = Ps \cdot G = G \cdot Ps$ is critical, and consequently that the pair $\{e, f\}$ is essential in G. □

Recall that every essential pair of edges of a snark is non-removable. Although the technical definition of an essential pair of edges requires far more, it is not clear whether the resulting concept is indeed stronger. It is therefore desirable to provide an example of a critical snark which has a non-removable pair of edges that is not essential. Such a snark indeed exists. It is constructed in Section 6 and shown in Fig. 16. The required non-removable pair of edges that is not essential consists of the edges x and y indicated in the figure. The fact that the pair $\{x, y\}$ is not essential has been checked by using a computer.

The snark from Fig. 16, however, is not irreducible. In fact, we have not been able to find any irreducible snark containing a non-removable pair of edges that would not be essential. If such an irreducible snark exists, then it certainly has more than 30 vertices (verified by checking the catalogue of snarks [1]). This suggests to the following problem:

Problem 5.7. *Is any non-removable pair of edges of an irreducible snark essential?*

6 Strictly critical snarks

Results of Sections 4 and 5 offer a fairly general method of constructing cyclically 4-connected *strictly critical snarks*, snarks that are critical but not bicritical. Strictly critical snarks have previously been constructed by Steffen [12] and by Grünwald and Steffen [5] using a different approach. Our construction is a substantial generalisation of Steffen's method and covers all examples supplied by [12]. Moreover, it provides a tool for generating interesting examples related to Theorems A-D some of which will be given later in this paper.

Our point of departure is the observation that Theorem 4.6 allows the existence of a critical snark $G = H \cdot K$ where only K is critical, but not H. For instance, assume that we have found a non-critical snark H which contains an essential pair of edges $\{e, f\}$ such that each suppressible edge of H (and there must be one) is one of e and f. If we perform the dot product of H as the left factor with an arbitrary critical snark different from the dumbbell graph using the pair $\{e, f\}$, we obtain a snark which is critical (because it fulfils the sufficient condition of Theorem 4.6), but is not bicritical (because it fails to fulfil the necessary condition of Theorem 4.8). All that remains is to find such a snark H.

To do this, we start with a non-critical snark H_0, even one which is not cyclically 4-connected. It is very likely that H_0 will contain many suppressible edges. We supplement them by some other edges and distribute all these edges into disjoint essential pairs, if possible. If such a distribution in H_0 has been possible, we take one of these pairs and use

it for the dot product of H_0, as the left factor, with an arbitrary critical snark different from the dumbbell graph. Let H_1 be the resulting snark. By Proposition 4.5, H_1 has the same suppressible edges as H_0 had, with the exception of those involved in the dot product. Moreover, by Theorem 5.3, all essential pairs of edges, except for those involved in the dot product, remain essential. This means that with H_1 we are in the same position as we have previously been with H_0, but now there are fewer undesirable edges, because the dot product has absorbed those two which are involved in it. If we continue in this way, we finally obtain a snark $H_n = H$ with the required property, and by an additional dot product we obtain a strictly critical snark G.

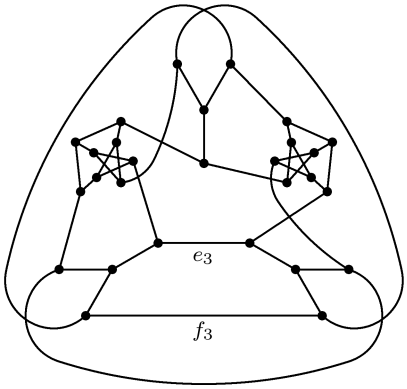

Fig. 15. The snark Z

For the just described procedure we employ the flower snark I_3 as the starting graph H_0. It has three pairwise adjacent suppressible edges e_1, e_2 and e_3 constituting its central triangle. We supplement each of e_i by the edge f_i indicated in Fig. 8. It can be shown

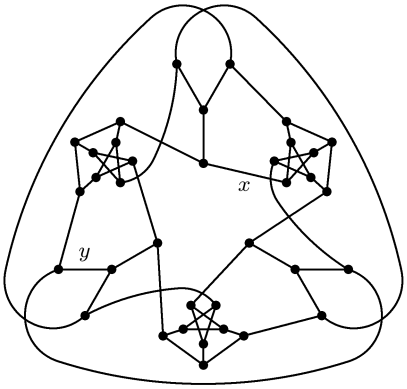

Fig. 16. The strictly critical snark Sc

that the pairs $\{e_1, f_1\}$, $\{e_2, f_2\}$, $\{e_3, f_3\}$ are essential in I_3. Then, starting with I_3, we perform three dot products, each time taking the Petersen graph as the right factor and using one of the selected pairs of edges in the left factor. We thus construct the snarks

$H_0 = I_3$, $H_1 = H_0 \cdot Ps$, $H_2 = H_1 \cdot Ps$, and $G = H_2 \cdot Ps$. As the only suppressible edge of H_2 is e_3, G is a strictly critical snark. We denote H_2 by Z (Fig. 15) and G by Sc (Fig. 16); the order of Sc is 36.

It can easily be seen from the way in which Sc was constructed that any pair formed from the three vertices of Sc originally constituting the central triangle in I_3 is removable from Sc. In fact, there are no other removable pairs of vertices in Sc.

Instead of the Petersen graph as the right factor we could have used any other critical snark different from the dumbbell graph. Since by Theorem 2.5 there are infinitely many irreducible snarks, it follows that there are infinitely many cyclically 4-connected strictly critical snarks. With a little bit more care we can determine all possible orders of strictly critical snarks.

Theorem 6.1. *There exists a strictly critical snark of each even order $n \geqslant 32$. There are no strictly critical snarks of order $n \leqslant 30$.*

Before proving this theorem we need the following:

Proposition 6.2. *There exist critical snarks of orders 2, 10 and each even order $n \geqslant 18$ except $n = 24$. There are no critical snarks of other orders.*

Proof. The dumbbell graph Db (Fig. 3), the Petersen graph Ps (Fig. 13), the Blanuša snarks B_1 and B_2 (Fig. 17), the flower snark I_5 (Fig. 8), and the Goldberg-Loupekine snarks GL_1 and GL_2 (Fig. 18) are critical snarks of orders 2, 10, 18, 20, and 22, respectively. Note that all these snarks are even irreducible.

Fig. 17. The Blanuša snarks B_1 and B_2

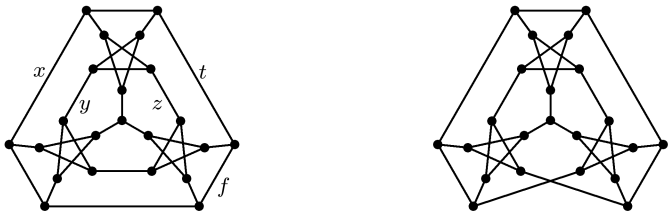

Fig. 18. The Goldberg-Loupekine snarks GL_1 and GL_2

There are also critical snarks of orders 28, 30, and 32, for example the flower snark I_7, the double-star snark Ds (Fig. 14), and S_{32} (Fig. 19), respectively. These snarks are

irreducible, too. To indicate the order in the notation, we denote the first two of them by S_{28} and S_{30}, respectively, and both of the Blanuša snarks by S_{18}.

Now, we perform a k-fold dot product of the Petersen graph as the left factor with S_i ($i \in \{18, 28, 30, 32\}$) as the right factor to obtain snarks

$$Ps^k \cdot S_i = Ps \cdot \Big(\cdots \big(Ps \cdot (Ps \cdot S_i) \big) \Big)$$

of orders $8k + i$, respectively. Since, by Example 5.5, any pair of non-adjacent edges of the Petersen graph is essential, the snarks $Ps^k \cdot S_i$ are critical by Theorem 4.6. It follows that there is a critical snark of order n for every even $n \geqslant 26$.

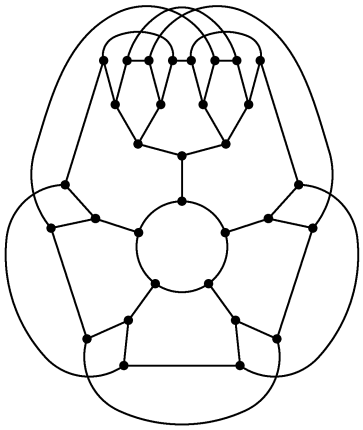

Fig. 19. The snark S_{32}

There are no critical snarks of orders 4, 6, 8, 12, 14, 16, and 24; see [1, 9] for a discussion of this topic. □

Proof of Theorem 6.1. The nonexistence of strictly critical snarks of orders smaller than 32 has been verified by Brinkmann and Steffen [1] by exhaustive computer search. We now show that for each even $n \geqslant 32$ there exists at least one strictly critical snark of order n.

A strictly critical snark of order 32 has been constructed by Steffen in [12]. By employing our method such a snark can be constructed as follows. Take the Goldberg-Loupekine snark GL_1 shown in Fig. 18, subdivide the edges denoted x and y with one 2-valent vertex each, and join the vertices with a new edge e. Since $\{x, y\}$ is a removable pair of edges (as easily seen from Goldberg's construction [4]), the resulting graph GL_1' is a snark with e being a suppressible edge. In fact, e is the only such edge because GL_1 is irreducible. Moreover, e can be supplemented by the edge f indicated in Fig. 18 in the original GL_1 to obtain an essential pair of edges. Thus in our construction of a strictly critical snark of the form $G = H \cdot K$ we can take $H = GL_1'$ and $K = Ps$. The result is a strictly critical snark of order 32. Note that another such snark arises from GL_2 as a dot product of GL_2' (defined analogously) and Ps.

If we replace Ps in the dot product $GL_1' \cdot Ps$ by any other critical snark different from Db, we get a strictly critical snark as long as the same essential pair of edges $\{e, f\}$ of GL_1' is used for the dot product. Since there exists a critical snark of each even order $n \geqslant 18$ except $n = 24$ (Proposition 6.2), we deduce that there exists a strictly critical snark of each even order $n \geqslant 40$ except possibly $n = 46$.

We have already constructed a strictly critical snark of order 36 in the form $Z \cdot Ps$ where Z is the snark of order 28 shown in Fig. 15. By replacing Ps with I_5 we obtain a strictly critical snark of order $n = 46$. Thus it remains to construct strictly critical snarks of orders 34 and 38.

A strictly critical snark of order 34 can be constructed as follows. We start again with GL_1, but this time we add two edges: the edge e across x and y (as above), and an edge g across the edges z and t which are indicated in Fig. 18. Since e and g are the only suppressible edges of the resulting snark GL_1'' and they form an essential pair, we obtain a strictly critical snark of order 34 by a suitable dot product of GL_1'' with Ps.

Finally, to construct a strictly critical snark of order 38 take the flower snark I_3, select a vertex at distance 2 from its central triangle (see Fig. 8), and expand this vertex into a triangle. The resulting graph is a snark of order 14 whose suppressible edges are exactly those lying on the triangles. Since each pair consisting of edges from different triangles can be shown to be essential, we can distribute the six suppressible edges into three disjoint essential pairs. By performing three dot products, each time using one of the essential pairs in the left factor and the Petersen graph as the right factor, we obtain a strictly critical snark of the required order. □

Strictly critical snarks are closely related not only to Theorem A but also to Theorem B. According to the latter theorem, every bicritical snark of cyclic connectivity 4 is a dot product of two smaller bicritical snarks such that the pair of edges of the left factor involved in the dot product is essential. It is natural to ask whether Theorem B can be reversed and used to construct bicritical snarks, that is, whether any dot product of two bicritical snarks involving an essential pair of edges will necessarily be bicritical. If this is not true, then by Theorem A any counterexample will be a strictly critical snark, moreover, one which decomposes into a dot product of two bicritical snarks. Recall that the very basic idea behind strictly critical snarks constructed in the present section is that they are formed by a dot product of a non-critical snark with a critical snark. Thus the potential counterexamples to the converse of Theorem B will have to be strictly critical snarks of a fundamentally different nature and therefore probably difficult to find.

In addition, the following problem is open.

Problem 6.3. *Do there exist cyclically* 6*-connected strictly critical snarks?*

7 Factorisation chains and atoms

This section starts our preparations for the proofs of Theorems C and D and introduces their main technical ingredients. First of all we revisit Theorem 3.7 and note that with the help of Theorem B (4.8) it can be strengthened as follows:

Theorem 7.1. *Every 4-edge-cut S in an irreducible snark G separates G into two colour-open multipoles M and N, one isochromatic and one heterochromatic. Their respective completions $\tilde{M}$ and $\tilde{N}$ are irreducible snarks, and G can be expressed as a dot product of $\tilde{M}$ and $\tilde{N}$ in such a way that S is the bond. Moreover, this expression is unique up to isomorphism.*

In other words, Theorem 7.1 asserts that each 4-edge-cut S in an irreducible snark G gives rise to a decomposition of G into a dot product of two other irreducible snarks, the S-factors of G. It is follows from Propositions 3.2 and 3.5 that the factor $\tilde{M}$ uniquely determines the factor $\tilde{N}$, and vice versa. More formally, if G is expressed as $H \cdot K_1 = H \cdot K_2$, then K_1 coincides with K_2 up to the the identity of the added vertices and edges; if $H_1 \cdot K = H_2 \cdot K$, the same holds for H_1 and H_2. Thus, for an irreducible snark $G = H \cdot K$ these cancellation rules justify the definition of a *quotient snark* as $G/H = K$ and $G/K = H$. In particular, $G = H \cdot (G/H)$ and $G = (G/K) \cdot K$.

Note that our notation ignores the difference between the "right" and the "left" quotient of a snark although such quotients could be introduced. In fact, this ambiguity will later prove to be very convenient.

Consider an arbitrary irreducible snark G different from Db. If G has a cycle-separating 4-edge-cut, we can decompose G into a dot product of two smaller snarks. Since direct factors of G are again irreducible, we can repeat the process with any 4-edge-cut in either direct factor and continue in this manner as long as we like. At each stage we obtain a sequence of snarks called a *factorisation chain*. The following definition extends this concept to all snarks.

Let G be a snark.

- The single-term sequence (G) is a factorisation chain of G.
- If $(G_1, G_2, \ldots, G_m)$ is a factorisation chain of G, and one of its members G_j can be expressed as a dot product $G_j = K_1 \cdot K_2$ of snarks K_1 and K_2, then $(G_1, \ldots, G_{j-1}, K_1, K_2, G_{j+1}, \ldots, G_m)$ is again a factorisation chain of G. The latter chain is an *elementary refinement* of the former one.

The number m of members in a factorisation chain $\mathcal{F} = (G_1, G_2, \ldots, G_m)$ will be called the *length* of $\mathcal{F}$ and will be denoted by $|\mathcal{F}|$.

Although each factorisation chain is inherently ordered, it is sometimes useful to ignore this ordering and regard it as a set (or even a multiset, when necessary). In such a case we will write $\{G_1, G_2, \ldots, G_m\}$ instead of $(G_1, G_2, \ldots, G_m)$. Graphs which occur as members of factorisation chains of a snark G will be called *subsnarks* of G. The fact that a snark H is a subsnark of G will be denoted by $H \leqslant G$. A subsnark different from G will be called a *proper* subsnark. Because of the trivial factorisation, a factorisation chain may contain the dumbbell graph as a member arbitrarily many times. In order to avoid such situations we define a factorisation chain to be *clean* if it does not contain the dumbbell graph as a member, or if it is just (Db). A clean factorisation chain with no clean refinements will be called a *composition chain* and its members will be *composition factors*. A composition

chain is thus the final product of successive refinement of any factorisation chain. By Theorem B, composition factors of an irreducible snark different from Db are cyclically 5-connected irreducible snarks, and therefore have no proper subsnarks other than Db (which, of course, is a subsnark of any snark).

Given two factorisation chains $\mathcal{F}$ and $\mathcal{G}$ we set $\mathcal{F} \succcurlyeq \mathcal{G}$ and say that $\mathcal{F}$ is a *refinement* of $\mathcal{G}$ if there exists a sequence of factorisation chains $\mathcal{F} = \mathcal{G}_r, \mathcal{G}_{r-1}, \ldots, \mathcal{G}_0 = \mathcal{G}$ $(r \geqslant 0)$ such that $\mathcal{G}_i$ is an elementary refinement of $\mathcal{G}_{i-1}$ for $i = 1, 2, \ldots, r$. The relation "$\succcurlyeq$" endows the set of all clean factorisation chains of a snark G with a partial order which has a single minimal element (G) and, as a rule, many maximal elements, the composition chains.

The structure of composition chains of a snark can be very complicated in general. Although Theorem 7.1 claims that in an irreducible snark each particular 4-decomposition produces uniquely determined direct factors, this local uniqueness hints nothing about global properties of composition chains. For example, at this moment we do not even know whether any two composition chains of an irreducible snark have the same length. This question is relevant since there are strictly critical snarks having composition chains with different lengths (see Section 12). Our ultimate aim, though, is to prove a much stronger result: any two composition chains of an irreducible snark can be obtained from each other by reordering the elements and replacing any member by an isomorphic copy. Such factorisation chains will be called *equivalent*.

The fundamental device which we utilise for establishing the relationship between different composition chains of a snark is the concept of an atom, a building block of the connectivity structure of a cubic graph. For an induced subgraph F of a cubic graph G let ∂F denote the set of all edges of G with exactly one end in F; whenever a reference to G is necessary, the symbol $\partial_G F$ will be used. Clearly, ∂F is an edge-cut provided that F is nonempty and different from G. If ∂F is a mininum cycle-separating edge-cut, we will call F a *cyclic fragment* of G (in [8] the term *cyclic part* was used). A cyclic fragment minimal under inclusion is an *atom*.

Note that an atom is not a cubic graph but can be easily converted to a cubic multipole: it is sufficient to attach the appropriate number of dangling edges to all vertices with valency smaller than 3. When this operation is performed on a graph H with maximum valency 3, the resulting multipole will be denoted by $H^{\#}$. Of course, $H^{\#} = H \cup \partial H$.

If F is a cyclic fragment of an irreducible snark G of cyclic connectivity 4, then $G - F^{\#}$ is a cyclic fragment, too. According to Theorem 7.1, the multipoles $F^{\#}$ and $G - F$ have isomorphically unique completions to irreducible snarks whose dot product is G. To simplify the notation, in similar situations we will always denote the snark completion of $F^{\#}$ by $\tilde{F}$. With this notation, the direct factors of G resulting from the choice of F are $\tilde{F}$ and $G/\tilde{F}$.

The distribution of atoms provides a useful insight into the internal structure of a cubic graph. In a general setting, the distribution of atoms can be intricate with regard to both the mutual position of two atoms and to the position of an atom and a minimum cycle-separating edge-cut. Things become much simpler when cyclic connectivity is strictly smaller than girth.

Theorem 7.2. [8] *Let G be a connected cubic graph whose cyclic connectivity is strictly smaller than its girth. Then any atom of G is disjoint from any minimum cycle-separating edge-cut, and any two distinct atoms are disjoint.*

By Proposition 2.4, critical and hence irreducible snarks of cyclic connectivity 4 satisfy the assumption of Theorem 7.2. Thus we have:

Corollary 7.3. *In an irreducible snark of cyclic connectivity 4, any two distinct atoms are disjoint and any atom is disjoint from any cycle-separating 4-edge-cut.*

In contrast to atoms, two minimum cycle-separating edge-cuts need not be disjoint at all. Therefore a closer look at the mutual position of such cuts will be necessary.

8 Edge-cuts associated with atoms

Let A be an atom in a cubic graph G of cyclic connectivity k. Then ∂A is the cycle-separating k-edge-cut which separates A from the rest of G. This cut will be called the *atomic* cut associated with A. Besides the atomic cut, there may be other related edge-cuts associated with A. Since the edges in ∂A are independent, k of their end-vertices are in $G - A$. Let us call them $v_1, \ldots, v_k$. It may happen that two of these vertices, say v_1 and v_2, are adjacent and that the edge-cut separating the subgraph H induced by $A \cup \{v_1, v_2\}$ is again a minimum cycle-separating edge-cut in G. This latter cut will be called a *quasiatomic* cut and the subgraph H will be called a *quasiatom* associated with A. The pair $\{v_1, v_2\}$ of vertices lying "between" the atomic and a quasiatomic cut will be called a *quasiatomic pair* of vertices. Any atom A thus gives rise to one atomic and several quasiatomic cuts, in general at most $k/2$, but possibly none.

For irreducible snarks of cyclic connectivity 4 the situation is more specific.

Proposition 8.1. *In an irreducible snark of cyclic connectivity 4 each atom gives rise to at most one quasiatom and to at most one quasiatomic cut.*

Proof. Assume to the contrary that an atom A in a snark G of cyclic connectivity 4 has two associated quasiatomic cuts S and T with quasiatomic pairs of vertices $\{v_1, v_2\}$ and $\{w_1, w_2\}$, respectively. Then S (as well as T) separates $\{v_1, v_2\}$ from $\{w_1, w_2\}$. By verifying a few simple cases one can easily see that the isochromatic S-factor always contains a cycle of size at most 4, contradicting the fact that irreducible snarks have girth at least 5 (Proposition 2.4). □

In view of Proposition 8.1, we can unambiguously denote the quasiatomic cut and the quasiatom associated with an atom A of an irreducible snark G by $\partial' A$ and A', respectively; in particular, $\partial A' = \partial' A$.

Next we show that in the factorisation process atomic and quasiatomic cuts play essentially the same role.

Proposition 8.2. *Let G be an irreducible snark G of cyclic connectivity 4 and let A be an atom of G. Then $\tilde{A}' \cong \tilde{A}$ and $G/\tilde{A}' \cong G/\tilde{A}$.*

Proof. Assume that G contains the quasiatomic cut $\partial' A$ associated with A. Then G can be expressed as a dot product of $\tilde{A}'$ and $G/\tilde{A}'$. In this dot product, $\tilde{A}'$ cannot act as the right factor. To see this, we identify $(A')^{\#}$ with the multipole obtained from $A^{\#}$ by a partial junction with the 4-pole L employing a pair of non-adjacent semiedges of L. Denote the semiedges of $A^{\#}$ by a_1, a_2, a_3, a_4 and those of L by l_1, l_2, l_3, l_4, where l_1 is adjacent to l_2 and l_3 is adjacent to l_4 (cf. Fig. 6 and Fig. 20). Without loss of generality we may assume that the partial junction of $A^{\#}$ and L joins a_1 to l_2, and a_2 to l_3, so that the semiedges of $(A')^{\#}$ are l_1, l_4, a_3, and a_4. Now, if $\tilde{A}'$ was the right factor of the dot product, then l_1 and one of the remaining three semideges would have to be incident with a common vertex. However, l_4 is excluded since $\tilde{M}$ would contain a triangle, and both a_3 and a_4 are excluded because $\tilde{M}$ would then contain a quadrilateral, contradicting Proposition 2.4. It follows that $\tilde{A}'$ acts as a left factor, and hence can be obtained from $(A')^{\#}$ by joining the semiedges in each couple of $(A')^{\#}$. However, it is now easy to see that $\tilde{A}'$ can also be obtained directly from $A^{\#}$ by adding two adjacent vertices. By Proposition 3.2, the latter can be done uniquely up to isomorphism. Therefore $\tilde{A}' \cong \tilde{A}$, and consequently $\tilde{G}/\tilde{A}' \cong G/\tilde{A}$. □

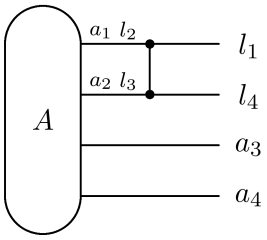

Fig. 20. The quasiatomic cut associated with A

It follows from Proposition 8.2 that the factorisation chain $\{\tilde{A}, G/\tilde{A}\}$ resulting from the use of the atomic cut ∂A is equivalent to the chain $\{\tilde{A}', G/\tilde{A}'\}$ which arises from the quasiatomic cut $\partial' A$. Besides the identity of the newly added vertices and edges the main difference between these two chains consists in the position of the original quasiatomic pair of vertices. While in the former chain the quasiatomic pair is "moved" to $G/\tilde{A}$ (and two new vertices are used for the other factor $\tilde{A}$), in the latter chain the quasiatomic pair stays with A (and new vertices are used for the other factor $G/\tilde{A}'$). In other words, the isomorphisms $\tilde{A}' \cong \tilde{A}$ and $G/\tilde{A}' \cong G/\tilde{A}$ substitute the quasiatomic pair for the pair of new vertices and vice versa. The same fact causes the exchange of roles between the left and right factor of the respective dot products.

9 Heredity of atoms and cuts

In this section we deal with the situation when an atom A and a cycle-separating 4-edge-cut S not associated with A are given in an irreducible snark G. The atom A and the cut S offer two possibilities to decompose G into a dot product of two smaller irreducible

snarks. Our aim is to show that in either case the object not used for the decomposition is inherited into one of the resulting factors.

We need two lemmas.

Lemma 9.1. *Let G be an irreducible snark of cyclic connectivity 4 and let A be an atom in G. Then for any cycle-separating 4-edge-cut S in G we have*

$$|S \cap \partial A| \neq 3.$$

Proof. Assume, to the contrary, that a cycle-separating edge-cut S with $|S \cap \partial A| = 3$ does exist. Let $\partial A = \{a_1, a_2, a_3, a_4\}$ and $S = \{a_1, a_2, a_3, e\}$, where $e \neq a_i$ for $i = 1, 2, 3, 4$. By Corollary 7.3, A contains no edge of S; in particular, e does not belong to A. Therefore G can be split into three induced subgraphs, H, K and A, such that $\partial A \cap \partial H = \{a_1, a_2, a_3\}$, $\partial A \cap \partial K = \{a_4\}$, and $\partial H \cap \partial K = \{e\}$; see Fig. 21.

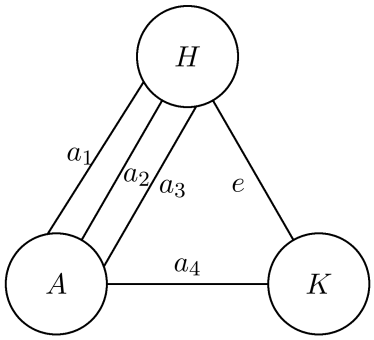

Fig. 21. S-cut with $|S \cap \partial A| = 3$

However, $K^{\#}$ is a 2-pole distinct from an isolated edge, so it must be cyclic. As $A^{\#}$ is cyclic, too, it follows that $\{a_4, e\}$ is a cycle-separating 2-edge-cut in G, contradicting our assumption about G. This establishes the lemma. □

Lemma 9.2. *Let G be an irreducible snark of cyclic connectivity 4 and let A be an atom in G. Then for any cycle-separating 4-edge-cut S in G with $|S \cap \partial A| = 2$ the edges in $S \cap \partial A$ do not form a couple of S.*

Recall that any cycle-separating 4-edge-cut S can serve as the bond of a dot product and that its couples correspond to the couples of the colour-open 4-poles arising from the disconnection of S.

Proof. By way of contradiction, assume that G contains a cycle-separating edge-cut S which intersects the atomic cut ∂A in two edges forming a couple of S. Let $\partial A = \{a_1, a_2, a_3, a_4\}$ and $S = \{a_1, a_2, e_1, e_2\}$ where $a_i \neq e_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2$, and $e_1 \neq e_2$, and where $\{a_1, a_2\}$ forms a couple in S.

Since A is an atom, Corollary 7.3 implies that $A \cap S = \emptyset$, so the edges e_1 and e_2 cannot belong to A. It follows that G can be split into three induced subgraphs, H, K and A, in

such a way that $\partial A \cap \partial H = \{a_1, a_2\}$, $\partial A \cap \partial K = \{a_3, a_4\}$, and $\partial H \cap \partial K = \{e_1, e_2\}$; see Fig. 22.

By disconnecting the cut S we obtain two 4-poles, one of which is $H^{\#} = M$. Let N be the other one. By Proposition 3.5, one of M and N is heterochromatic and the other one is isochromatic. Moreover, by Theorem 7.1, the snark $G = M * N$ can be obtained as a dot product of the snark completions $\tilde{M}$ and $\tilde{N}$. At this moment, however, it is not clear which of $\tilde{M}$ and $\tilde{N}$ serves as the left S-factor and which as the right one.

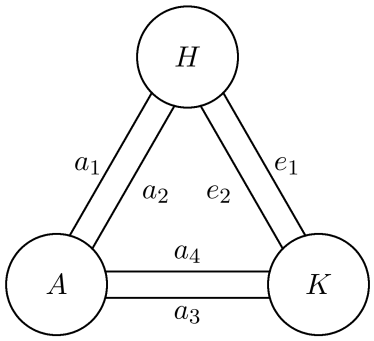

Fig. 22. S-cut with $|S \cap \partial A| = 2$

Obviously, if N is isochromatic, then $G = \tilde{M} \cdot \tilde{N}$ and if N is heterochromatic, then $G = \tilde{N} \cdot \tilde{M}$. In either case $\tilde{N}$ must be irreducible and different from Db (because the edge-cut S is cycle-separating). We now discuss these two cases in detail.

Case 1. First suppose that N is heterochromatic. By Proposition 3.2, the snark $\tilde{N}$ is obtained from N by joining the semiedges of each couple of N. It follows that $\{a_3, a_4\}$ is an edge-cut in $\tilde{N}$ that separates A with one edge added and K with one edge added. As this cut is clearly cycle-separating, we get a contradiction with Proposition 2.4.

Case 2. Now suppose that N is isochromatic. By Proposition 3.2, the snark $\tilde{N}$ is obtained from N by attaching the semiedges in each couple to a new vertex and by connecting these vertices with a new edge. Denote this new edge by f. Now it is clear that the edges in $\{a_3, a_4, f\}$ form an independent 3-edge-cut in $\tilde{N}$. Again, this contradicts Proposition 2.4. □

Proposition 9.3. *Let G be an irreducible snark of cyclic connectivity 4 and let A be an atom of G. Further let S be a cycle-separating 4-edge-cut in G not associated with A. Then one of the S-factors of G has cyclic connectivity 4 and contains A as an atom.*

Proof. Let H and K be the S-factors of G. By Corollary 7.3, A is disjoint from S and hence a subgraph of one of H and K, say H. We show that A is, in fact, an induced subgraph of H. To see this, first observe that $|S \cap \partial_G A| \leqslant 2$. Indeed, $|S \cap \partial_G A| < 4$ for S is not the atomic cut associated with A, and $|S \cap \partial_G A| \neq 3$ due to Lemma 9.1. Now, if A were not an induced subgraph of H, then the completion which creates H from a component of $G - S$ would have to add a new edge between two vertices of A in H. This

could only happen when H was the left factor of this dot product. As $|S \cap \partial_G A| \leqslant 2$, we would get that $|S \cap \partial_G A| = 2$, and the edges of $S \cap \partial_G A$ would necessarily form a couple in S, contrary to Lemma 9.2. Therefore A is indeed an induced subgraph of H.

We want to prove that $\partial_H A$ is a cycle-separating cut in H, or equivalently, that $H - A$ contains a cycle. Suppose, to the contrary, that $H - A$ is acyclic. Since A is an induced subgraph of H, $H - A$ has no isolated edges, and hence $|H - A| \geqslant 2$. Thus $H - A$ is isomorphic to L with possibly permuted semiedges. At this point we have to distinguish two cases depending on whether H is the left or the right factor of the dot product.

First assume that H is the right factor. The only way H can be involved in this dot product subject to the restriction that A has to "survive" the operation and become an induced subgraph of H is that the vertices of $H - A$ are involved in the dot product. However, the bond of $G = K \cdot H$, the cut S, then coincides with $\partial_G A$, contradicting our assumption.

Now let H be the left factor. If A has to "survive" the dot product and become an induced subgraph of H, then two edges of $\partial_H A$ non-adjacent in H must be involved in the dot product. In this case the the bond of $G = H \cdot K$ will be the quasiatomic cut associated with A, which contradicts our assumption again.

Thus $\partial_H A$ is a cycle-separating 4-edge-cut in H. Since $H = A^{\#} * (H - A)$ is an irreducible snark, it is cyclically 4-connected. It follows that the cyclic connectivity of H is 4 and that A is a cyclic fragment of H. To prove that A is in fact an atom of H, we need to verify that A is minimal under inclusion. However, this is obvious: if there existed a cyclic fragment F in H such that $F \subsetneqq A$, then F would also be a cyclic fragment in G, contradicting the fact that A is an atom of G. Thus there is no such F and hence A is an atom in H. This completes the proof. □

Given an atom A in an irreducible snark G of cyclic connectivity 4 and a cycle-separating 4-edge-cut S not associated with A, one of the S-factors inherits A as an atom. Our next aim is to show that if the cut ∂A is used instead, then S will be inherited in some way into one of the ∂A-factors. The latter situation, however, requires a careful preparation because the correspondence is less straightforward.

Recall that by disconnecting the atomic cut ∂A we obtain two 4-poles $A^{\#}$ and $G - A$ such that $A^{\#} * (G - A) = G$. At the same time, G can be expressed as a dot product of $\tilde{A}$ and $G/\tilde{A}$, the respective completions of the latter 4-poles. We now establish a natural correspondence between the edges of $G - A$ and those of $G/\tilde{A}$ by defining a mapping p_A: $E(G - A) \to E(G/\tilde{A})$, a *projection*, as follows:

Each edge e of $G - A$ with both end-vertices in $G - A$ is *inherited* into $G/\tilde{A}$ directly, and we set $p_A(e) = e$ in this case.

For the edges of $\partial_G A$ the definition differs according to whether $A^{\#}$ is isochromatic or heterochromatic. Let $\partial_G A = \{a_1, a_2, a_3, a_4\}$. By disconnecting the edges a_i in G we obtain the 4-poles $A^{\#}$ and $G - A$ with dangling edges denoted correspondingly by a_i^A in $A^{\#}$ and a_i^{G-A} in $G - A$ $(i = 1, 2, 3, 4)$.

- Let $A^{\#}$ be heterochromatic. Then $\tilde{A}$ is the left and $G/\tilde{A}$ the right factor of the dot product. By Proposition 3.2, $G/\tilde{A}$ arises from $G - A$ by adding two new vertices

joined by a new edge. Therefore we set $p_A(a_i) = a_i^{G-A}$.

- Let $A^\#$ be isochromatic. Then $\tilde{A}$ is the right factor and $G/\tilde{A}$ is the left one. By Proposition 3.2, $G/\tilde{A}$ is obtained from M by joining the semiedges in each couple. Without loss of generality, if the couples are $\{a_1^{G-A}, a_2^{G-A}\}$ and $\{a_3^{G-A}, a_4^{G-A}\}$ and the newly created edges are e and f (for the first and the second couple, respectively), we define $p_A(a_i) = e$ for $i = 1, 2$ and $p_A(a_i) = f$ for $i = 3, 4$.

Let S be an arbitrary cycle-separating 4-edge-cut in G. By Corollary 7.3, S is disjoint from the atom A. Hence $S \subseteq G - A$, and so the projection $p_A(e)$ is defined for all edges $e \in S$. We proceed to show that if S is not associated with A, then $p_A(S)$ is a cycle-separating 4-edge-cut in $G/\tilde{A}$. The cut $p_A(S)$ will be referred to as the cut of $G/\tilde{A}$ *corresponding* to the cut S of G.

Proposition 9.4. *Let G be an irreducible snark of cyclic connectivity 4 and let A be an atom of G. Further let S be a cycle-separating 4-edge-cut in G not associated with A. Then $p_A(S)$ is a cycle-separating 4-edge-cut in $G/\tilde{A}$.*

Proof. It is sufficient to show that $p_A(S)$ is an independent 4-edge-cut in $G/\tilde{A}$. We will separately treat the cases when the 4-pole $A^\#$ is isochromatic or heterochromatic.

Case 1. If $A^\#$ is heterochromatic, then p_A is injective on its domain of definition; hence $|p_A(S)| = 4$. Moreover, the edges of G with both end-vertices in $G - A$ are directly inherited into $G/\tilde{A}$ together with their end-vertices. Any edge of $\partial_G A$ has exactly one vertex in $G - A$. As the edges in S are independent, the only way for the edges in $p_A(S)$ to become adjacent is that one of the couples of edges in $\partial_G A$ is present in S. This is because p_A maps such a pair of originally non-adjacent edges to a pair of adjacent edges. To show that this does not occur assume the contrary. Then $|S \cap \partial_G A| \geqslant 2$ and a couple of $\partial_G A$ is contained in $S \cap \partial_G A$. However, by Lemma 9.2, the size of this intersection cannot be 2 and, by Lemma 9.1, it cannot be 3. It follows that $|S \cap \partial_G A| = 4$ in which case $S = \partial_G A$, contradicting our assumption. Therefore $p_A(S)$ is an independent set of edges. In fact, $p_A(S)$ is an edge-cut in $G/\tilde{A}$ separating components which are formed from the components of G separated by S by substituting the 4-pole $A^\#$ (which is connected) with the 4-pole L (which is also connected) in the appropriate way (cf. Remark 4.2).

Case 2. We now assume that $A^\#$ is isochromatic. In this case the situation is more complicated because the mapping p_A is not injective anymore.

First of all, we have that $|p_A(S)| = 4$. Otherwise S would have to contain a couple of edges of $\partial_G A$, and by similar arguments as in Case 1 we could derive a contradiction by showing that S is the atomic cut associated with A.

Next we show that $p_A(S)$ is an edge-cut in $G/\tilde{A}$. Since S is disjoint from A, $G/\tilde{A}$ arises from G by substituting $A^\#$ (which is connected) with the 4-pole R (which is disconnected). The components of $(G/\tilde{A}) - p_A(S)$ thus arise from those of $G - S$ by the same substitution (cf. Remark 4.2). Denote the component of $G - S$ containing $A^\#$ by K. Then $|K| > |A|$, since $S \neq \partial_G A$, so K will not vanish by replacing $A^\#$ with R. The resulting 4-pole either remains connected or splits into more components, as R is disconnected. In either case, $(G/\tilde{A}) - p_A(S)$ is disconnected which means that $p_A(S)$ is an edge-cut in $G/\tilde{A}$.

It remains to show that the edges in $p_A(S)$ are independent. In the general case, the edges of $G - A$ are directly inherited into $G/\tilde{A}$ together with their original end-vertices. Hence non-adjacent edges are mapped onto non-adjacent edges; so $p_A(S)$ is independent. There is however one exception: an edge x independent from a bond edge $a_i \in \partial_G A$ in G can become adjacent to the corresponding edge $p_A(a_i)$ in $G/\tilde{A}$.

To see this, assume that the edges of $\partial_G A = \{a_1, a_2, a_3, a_4\}$ are ordered in such a way that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ form the couples. Let e and f be the new edges of $G/\tilde{A}$ created by joining a_1 to a_2 and a_3 to a_4, respectively. Thus $p_A(a_1) = p_A(a_2) = e$ and $p_A(a_3) = p_A(a_4) = f$. Let v_1, v_2, v_3, and v_4 be the respective end-vertices in $G - A$ of the edges a_i. Thus e joins v_1 to v_2 and f joins v_3 to v_4 in $G/\tilde{A}$. Of course, these four vertices are pairwise distinct, since $\partial_G A$ was an independent edge-cut in G (cf. Fig. 23).

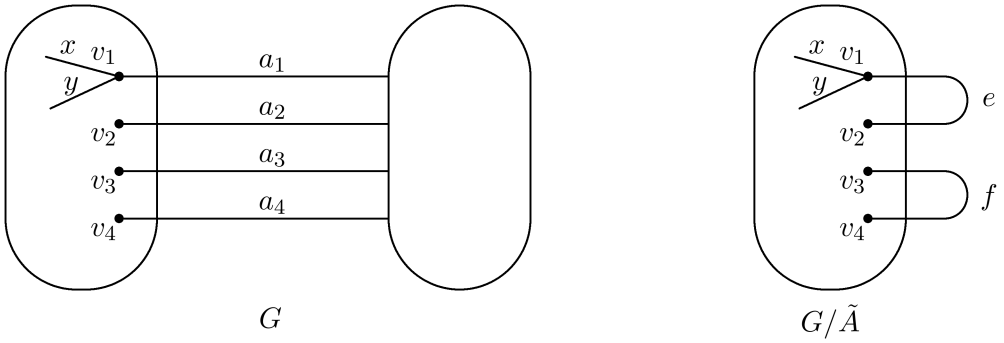

Fig. 23. Case 2 of the proof of Proposition 9.4

Pick an edge x of $G/\tilde{A}$ (and of G) incident with some v_i, say v_1, other than e or f. Obviously, x is not incident with v_2 (neither in G nor in $G/\tilde{A}$), otherwise the edges x and e would form a digon in $G/\tilde{A}$ which is impossible. Then x and a_2 are non-adjacent in G while $p_A(x) = x$ and $p_A(a_2) = e$ are adjacent in $G/\tilde{A}$.

To finish the proof, it remains to show that, without loss of generality, S does not contain the edges a_2 and x at the same time. Assume it does. Besides a_1 and x there is the third edge y incident with v_1 in G. As S is independent, necessarily $y \notin S$. Furthermore we see that $y \notin p_A(S)$ because x and y are inherited from G to $G/\tilde{A}$. Nevertheless, $x = p_A(x)$ and $e = p_A(a_2)$ are adjacent in $G/\tilde{A}$. Since $p_A(S)$ is an edge-cut in $G/\tilde{A}$, so is $(p_A(S) - \{e, x\}) \cup \{y\}$. However, this is a 3-edge-cut in an irreducible snark $G/\tilde{A}$, so it is not cycle-separating, and hence it must separate some vertex v from the rest of $G/\tilde{A}$.

Since S is independent, the two edges in $S - \{a_2, x\}$ are non-adjacent in G. In contrast, the edges in $p_A(S - \{a_2, x\}) = p_A(S) - \{e, x\}$ are adjacent in $G/\tilde{A}$, their common vertex being v. Should both edges of $S - \{a_2, x\}$ be directly inherited from G to $G/\tilde{A}$, their independence would be preserved. Therefore S contains one of a_1, a_3, or a_4. However, $a_1 \notin S$ because $|p_A(S)| = 4$, so S contains a_3 or a_4, and v is necessarily one of v_3 and v_4. This means that S is the quasiatomic cut in G associated with A, contradicting our assumptions.

In either case $p_A(S)$ is an independent 4-edge-cut in $G/\tilde{A}$. □

The above results enable us to observe the position of a fixed atom during its "life" in refining factorisation chains. The relevant information about the "life-cycle" of an atom can be drawn from Proposition 9.3. If we are given an irreducible snark G of cyclic connectivity 4 and a fixed atom A of G, A is always inherited as an atom into one of the resulting subsnarks regardless of the way in which G is being factored. Eventually one of the cuts associated with A is used for factorisation and at this moment the "life" of our atom A ends. By decomposing G along any of these cuts — by Proposition 8.2, no matter which one — either A is not inherited into the corresponding component as an induced graph (namely, when $A^\#$ is heterochromatic), or it is not a cyclic fragment (when $A^\#$ is isochromatic), or the corresponding component is cyclically 5-connected and the meaning of an atom turns out to be completely different.

10 Factorisation

In this section we apply our knowledge of the interplay between atoms, cuts and 4-decompositions to proving Theorem C. The main proof will be preceded by two lemmas, a "switching lemma" and a "splitting lemma". The first of these explores the same situation as the one treated in the previous section: in an irreducible snark G an atom A and a cycle-separating 4-edge-cut S not associated with A are given. Depending on the order taken, the cuts ∂A and S offer two ways of factorising G into a 3-element chain. If ∂A is taken first, then by Proposition 9.4 the cut S is inherited into one of the direct factors as $p_A(S)$. In the next step we can use $p_A(S)$ and form a factorisation chain $\mathcal{A}$. If S is used first, then by Proposition 9.3 the atom A is inherited into an appropriate factor again as an atom. This direct factor can be further decomposed by employing the atomic cut ∂A, thereby producing another factorisation chain $\mathcal{S}$. The switching lemma compares these two chains.

Lemma 10.1. *Let G be an irreducible snark of cyclic connectivity 4, let A be an atom in G, and let S be a cycle-separating 4-edge-cut of G not associated with A. Let $\mathcal{A}$ be the factorisation chain of G obtained by first using ∂A and then $p_A(S)$, and let $\mathcal{S}$ be the factorisation chain obtained by first using S and then ∂A. Then $\mathcal{A}$ and $\mathcal{S}$ are equivalent. In fact, the corresponding subsnarks only differ in the identity of the newly added vertices and edges.*

Proof. Let P and Q be the S-factors of G. By Proposition 9.3, one of them, say P, contains A as an atom. Since the role of A in P may differ from that in its original place in G, during the whole proof we must carefully distinguish between operations performed in P and those performed in G.

First note that the 4-poles $A_P^\# = A \cup \partial_P A \subseteq P$ and $A_G^\# = A \cup \partial_G A \subseteq G$ are isomorphic and, by Proposition 3.2, have isomorphically unique completions to snarks $\tilde{A}_P$ and $\tilde{A}_G$, respectively. Thus $\tilde{A}_P$ and $\tilde{A}_G$ are isomorphic, the difference between them being limited to the identity of the newly added vertices and edges.

Let us decompose G along $\partial_G A$ to obtain the factorisation chain $\{\tilde{A}_G, G/\tilde{A}_G\}$. By Proposition 9.4, $G/\tilde{A}_G$ contains the 4-edge-cut $p_A(S)$ corresponding to S. Using the latter

cut $G/\tilde{A}_G$ decomposes into a pair of direct factors, say H and K. Thus $\mathcal{A} = \{\tilde{A}_G, H, K\}$. On the other hand, the factorisation chain $\mathcal{S}$ arises from G by first decomposing G along S into a dot product of P and Q with $A \subseteq P$, and then by decomposing P along $\partial_P A$ into a dot product of $\tilde{A}_P$ and $P/\tilde{A}_P$. Thus $\mathcal{S} = \{\tilde{A}_P, P/\tilde{A}_P, Q\}$. To finish the proof it remains to match the subsnarks H and K to $P/\tilde{A}_P$ and Q. We therefore focus on the multipoles resulting from the individual 4-decompositions.

Due to Theorem 7.1, the snarks P and Q resulting from the decomposition of G along S are isomorphically unique completions of certain 4-poles M and N, respectively; of course, this uniqueness is determined up to the identity of the newly added vertices and edges. As Proposition 9.3 claims, one of these multipoles, say M, contains $A_P^{\#}$. So $A_P^{\#} \subseteq M \subseteq P$.

By disconnecting the edge-cut $p_A(S)$ in $G/\tilde{A}_G$ we again obtain two multipoles. It is easy to see that one of them is exactly N while the other one, denoted by M_A, can be obtained from M by substituting its sub-multipole $A_P^{\#}$ with either L or R, depending on which of them is colour equivalent to $A_P^{\#}$ (cf. Remark 4.2). Recall, however, that the $p_A(S)$-factors of $G/\tilde{A}_G$ are H and K. Without loss of generality we may assume that H is a completion of M_A while K is a completion of N.

Note that the way in which M, M_A and N are completed to snarks is uniquely determined by the colouring sets of the respective multipoles (Proposition 3.2). This immediately implies that K and Q, both being completions of N, can only differ in the identity of the newly added vertices and edges.

Since M_A arises from M by a colour-equivalent substitution of $A_P^{\#}$ with one of L and R, M_A is colour-equivalent to M by the Substitution Lemma. It follows that both M_A and M will be completed to snarks by using the same 4-pole $I \in \{L, R\}$. As the completion of M is P and the completion of M_A is H, the snark H arises from P by substituting $A_P^{\#}$ with I. Note that in G the substitution of $A_G^{\#}$ with I produces $G/\tilde{A}_G$, so the same substitution in P gives rise to $P/\tilde{A}_P$. By comparison, $P/\tilde{A}_P$ coincides with H up to the identity of newly added vertices and edges. The result follows. □

The result which we have just proved can be interpreted as follows. Let G be an irreducible snark to be factorised into cyclically 5-connected snarks, and let A be a fixed atom of G. We already know that sooner or later one of the cuts associated with A will be used for decomposition within a certain subsnark H of G. Consider the edge-cut S employed in the step just before this happens, and assume that S lives in a subsnark $J \geqslant H$. Lemma 10.1 now implies that by interchanging the order of S and ∂A (or, equivalently, $\partial' A$) within J, no potential composition chain of G will be lost from a further factorisation. As we shall see next, the immediate precedence of S to ∂A is not necessary: we can decompose G along ∂A in the very first step without any significant effect on the final result. This allows us to completely reorganise the whole factorisation process in such a way that the given composition chain of G can be "split" into a composition chain of $\tilde{A}$ and a composition chain of $G/\tilde{A}$.

To formalise these ideas, we need two more definitions. Let G be a snark expressed as a dot product $H \cdot K$. Then for any factorisation chain $\mathcal{H} = (H_1, H_2, \ldots, H_m)$ of H and any factorisation chain $\mathcal{K} = (K_1, K_2, \ldots, K_n)$ of K the sequence $(H_1, \ldots, H_m, K_1, \ldots, K_n)$ is

a factorisation chain of G. We denote the latter chain by $\mathcal{H} \cup \mathcal{K}$, the *union* being ordered whenever necessary.

Let $\mathcal{F} \preccurlyeq \mathcal{G}$ be factorisation chains of a snark G and let H be a member of $\mathcal{F}$. Define the *restriction* $\mathcal{G}|_H$ of $\mathcal{G}$ to H to be the subsequence of $\mathcal{G}$ consisting of all members of $\mathcal{G}$ that are subsnarks of H. An easy inductive argument shows that $\mathcal{G}|_H$ forms a connected interval in $\mathcal{G}$ and constitutes a factorisation chain of H.

Now we are ready for the splitting lemma.

Lemma 10.2. *Let G be an irreducible snark of cyclic connectivity 4 and let A be an atom of G. Then for any composition chain $\mathcal{C}$ of G there is a composition chain $\mathcal{C}_1$ of $\tilde{A}$ and a composition chain $\mathcal{C}_2$ of $G/\tilde{A}$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof. Let $\mathcal{C}$ be an arbitrary composition chain of G, and let A be a fixed atom of G. Proposition 9.3 implies that there exists a factorisation chain $\mathcal{M} \preccurlyeq \mathcal{C}$ such that either $\tilde{A}$ or $\tilde{A}' \cong \tilde{A}$ is a member of $\mathcal{M}$. Assume that $\mathcal{M}$ is the shortest such chain, and define $h = h_{\mathcal{C}}(\tilde{A})$, the *height* of $\tilde{A}$ in $\mathcal{C}$, to be $|\mathcal{M}| - 1$.

To establish the result we employ induction on h. If $h = 1$, then without loss of generality $\mathcal{M} = \{\tilde{A}, G/\tilde{A}\}$, and any refinement of $\mathcal{M}$ either refines $\tilde{A}$ or $G/\tilde{A}$. Therefore it is sufficient to take $\mathcal{C}_1$ and $\mathcal{C}_2$ to be the restrictions of $\mathcal{C}$ to $\tilde{A}$ and to $G/\tilde{A}$, respectively, and the required conclusion follows.

For the induction step assume that $h \geqslant 2$ and that the statement is true for all composition chains of G with height of $\tilde{A}$ smaller than h. Before proceeding further let us note that if the statement holds for a certain composition chain, then it also holds for all composition chains equivalent to it. Therefore, to accomplish the induction step, it is sufficient to construct a composition chain $\mathcal{U}$ of G equivalent to $\mathcal{C}$ such that the height of $\tilde{A}$ in $\mathcal{U}$ is smaller than h.

Again consider the shortest chain $\mathcal{M} \preccurlyeq \mathcal{C}$ which contains $\tilde{A}$ or $\tilde{A}'$ as a member. We may clearly assume $\tilde{A}$ to be a member of $\mathcal{M}$. Obviously, $\mathcal{M}$ is an elementary refinement of a chain $\mathcal{H} \preccurlyeq \mathcal{C}$ such that A is contained as an atom in a member H of $\mathcal{H}$. As $|\mathcal{H}| \geqslant 2$, H is a proper subsnark of G and therefore itself arises from a subsnark J (possibly $J = G$) by decomposition along a cycle-separating 4-edge-cut S. Let $\mathcal{J} \preccurlyeq \mathcal{C}$ be the longest chain which contains J. Clearly, $\mathcal{J} \preccurlyeq \mathcal{H} \preccurlyeq \mathcal{M}$. We now distinguish two cases according to whether $\mathcal{H}$ is or is not an elementary refinement of $\mathcal{J}$.

Case 1. If $\mathcal{H}$ is an elementary refinement of $\mathcal{J}$, then $\mathcal{H}|_J = \{H, J/H\}$ and $\mathcal{M}|_J = \{\tilde{A}, H/\tilde{A}, J/H\}$. The latter chain thus arises from J by first using the cut S and then using one of the cuts associated with A. By Proposition 8.2, we may assume the latter cut to be ∂A (more precisely, $\partial_H A$). As in Proposition 10.1, we can reverse the order of these cuts and decompose J beginning with ∂A (more precisely, $\partial_J A$) to obtain the chain $\{\tilde{A}, J/\tilde{A}\}$, and continuing with the cut $p_A(S)$ in $J/\tilde{A}$ to produce a chain $\{\tilde{A}, H_1, J_1\}$ equivalent to $\{\tilde{A}, H/\tilde{A}, J/H\}$, with $H_1 \cong H/\tilde{A}$ and $J_1 \cong J/H$. We now implement the modified factorisation of J into the factorisation of G in the following manner. We construct $\mathcal{H}_1$ from $\mathcal{H}$ by replacing $\mathcal{H}|_J$ with $\{\tilde{A}, J/\tilde{A}\}$, and similarly we form $\mathcal{M}_1$ from $\mathcal{M}$ by replacing $\mathcal{M}|_J$ with $\{\tilde{A}, H_1, J_1\}$. Clearly, both $\mathcal{H}_1$ and $\mathcal{M}_1$ are factorisation chains of G, moreover, $\mathcal{J} \preccurlyeq \mathcal{H}_1 \preccurlyeq \mathcal{M}_1$ and $\mathcal{M}_1$ is equivalent to $\mathcal{M}$. Since $\mathcal{C}$ is a refinement of $\mathcal{M}$,

the chain $\mathcal{M}_1$ has an equivalent refinement $\mathcal{B}$ (which obviously must be a composition chain of G). However, $\tilde{A}$ is a member of $\mathcal{H}_1$ and $\mathcal{H}_1 \preccurlyeq \mathcal{B}$ but $\mathcal{H}_1$ is shorter than $\mathcal{M}$. Therefore $h_{\mathcal{B}}(\tilde{A}) < h$, as required.

Case 2. Assume that $\mathcal{H}$ is not an elementary refinement of $\mathcal{J}$. Then $\mathcal{H}$ must be an elementary refinement of a chain $\mathcal{L} \preccurlyeq \mathcal{C}$ such that $\mathcal{J} \preccurlyeq \mathcal{L} \preccurlyeq \mathcal{H}$ and both $\mathcal{L}$ and $\mathcal{H}$ contain H. Note that $\mathcal{H}$ arises from $\mathcal{L}$ by using a cycle-separating 4-edge-cut outside H whereas $\mathcal{M}$ arises from $\mathcal{H}$ by using the cut $\partial_H A$ inside H. In particular, by restricting $\mathcal{L}, \mathcal{H}$ and $\mathcal{M}$ to J we obtain the chains $\{H\} \cup \mathcal{L}|_{J/H} \preccurlyeq \{H\} \cup \mathcal{H}|_{J/H} \preccurlyeq \{\tilde{A}, H/\tilde{A}\} \cup \mathcal{H}|_{J/H}$ where $\mathcal{L}|_{J/H} \preccurlyeq \mathcal{H}|_{J/H}$. Since $\mathcal{H}$ and $\mathcal{M}$ arise from $\mathcal{L}$ by using cuts in different factors of $\mathcal{L}$, we may interchange the order of these cuts. In other words, we can construct a factorisation chain $\mathcal{H}_2$ from $\mathcal{H}$ by replacing $\mathcal{H}|_J$ with $\{\tilde{A}, H/\tilde{A}\} \cup \mathcal{L}|_{J/H}$ and a factorisation chain $\mathcal{M}_2$ from $\mathcal{H}_2$ by replacing $\mathcal{H}_2|_{J/H} = \mathcal{L}|_{J/H}$ with $\mathcal{H}|_{J/H}$ so that $\mathcal{L} \preccurlyeq \mathcal{H}_2 \preccurlyeq \mathcal{M}_2$ and $\mathcal{M}_2$ is equivalent to $\mathcal{M}$. In fact, the latter two chains only differ by the identity of vertices and edges added in the last two steps. Now, $\mathcal{M}_2$ has a refinement $\mathcal{D}$ equivalent to $\mathcal{C}$, but $h_{\mathcal{D}}(\tilde{A}) < h$, because $\tilde{A}$ belongs to $\mathcal{H}_2$, and $\mathcal{H}_2$ is shorter than $\mathcal{M}$.

This completes the induction step, and the lemma is proved. □

Now we are ready to prove Theorem C.

Theorem 10.3. *Any irreducible snark $G \neq Db$ can be decomposed into a collection $\{G_1, \ldots, G_n\}$ of cyclically 5-connected irreducible snarks in such a way that G can be reconstructed from them by repeated dot products. Moreover, if $\{H_1, \ldots, H_m\}$ is another such decomposition of G, then $n = m$ and there is a permutation π of $\{1, 2, \ldots, n\}$ such that $H_i \cong G_{\pi(i)}$ for all $i = 1, 2, \ldots, n$.*

In other words, any two composition chains of an irreducible snark are equivalent.

Proof. We proceed by induction on the order of G. The result clearly holds for cyclically 5-connected snarks, so we let G be an irreducible snark of cyclic connectivity 4, and assume the theorem to be true for all irreducible snarks of order smaller than $|G|$. Let $\mathcal{C}$ and $\mathcal{D}$ be any two composition chains of G. Clearly, G contains at least one atom, say A. By Lemma 10.2, there exist composition chains $\mathcal{C}_1$ and $\mathcal{D}_1$ of $\tilde{A}$ and $\mathcal{C}_2$ and $\mathcal{D}_2$ of $G/\tilde{A}$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{D}$ equivalent to $\mathcal{D}_1 \cup \mathcal{D}_2$. Since both $|\tilde{A}| < |G|$ and $|G/\tilde{A}| < |G|$, the induction hypothesis implies that $\mathcal{C}_1$ is equivalent to $\mathcal{D}_1$, and $\mathcal{C}_2$ is equivalent to $\mathcal{D}_2$. Thus the same is true for $\mathcal{C}$ and $\mathcal{D}$. □

It may be instructive to analyse the nature of the permutation π in Theorem 10.3 by identifying elementary isomorphisms and permutations of composition factors that make up an equivalence of two composition chains. A careful analysis of the proof reveals two types of elementary isomorphisms. First, there are isomorphisms related to the identity of the newly added vertices and edges; these are inherent to all multipole extensions. A different type of an isomorphism was described in Lemma 8.2 and the subsequent discussion. It comes from the choice between an atomic and a quasiatomic cut. As regards the ordering of factors, this is determined by the two kinds of decisions. If we use a quasiatomic cut instead of an atomic one, the factors swap. The order of factors

may also change if we apply Lemma 10.1. However, in the factorisation process these actions are interwoven, and eventually the particular effect of a single action can hardly be recognised.

11 Relative factorisation

Factorisations considered in previous sections were "absolute" in the sense that any 4-edge-cut available at the current stage of the factorisation process could be used in the next factorisation step. In this section we study factorisations subjected to one very natural restriction: a given subgraph of a snark has to "survive" all factorisation steps.

To be more formal, let $K \subseteq G$ be a fixed subgraph of a snark G, and let H be a subsnark of G. A cycle-separating 4-edge-cut S of H will be called *K-consistent* if either $V(K) \cap V(H) = \emptyset$, or else $K \subseteq H$ and one of the S-factors of H inherits K as a subgraph. A factorisation chain of G obtained by a successive use of K-consistent 4-edge-cuts will be called *K-relative*. Its members are *K-relative subsnarks* of G.

We would like to emphasise that these definitions do not require K to be induced, therefore two vertices of K not adjacent in G can become adjacent in a K-relative subsnark of G.

As with the usual factorisation, the ultimate aim of a K-relative factorisation is a K-relative factorisation chain with no clean K-relative refinements. Such a chain will be called a *K-relative composition chain* of G. Note that K-relative composition factors need not be cyclically 5-connected — they may contain cycle-separating 4-edge-cuts as long as none of them is K-consistent. However, any K-relative composition chain contains at most one factor of cyclic connectivity 4.

For $|K| \leqslant 1$ the condition of K-consistency means no real restriction. In this case any two K-relative composition chains of an irreducible snark are equivalent by Theorem 10.3. For subgraphs of order 2 the situation dramatically changes. By specifying a subgraph K of order two we may enable the existence of non-equivalent K-composition chains in a given irreducible snark.

Example 11.1. Given cubic graphs G and H, let $G + H$ be a cubic graph constructed as follows. Select two adjacent vertices u and v in G and two adjacent vertices p and q in H. Remove u, v and the edge uv from G to produce a 4-pole with semiedges $e_1^u, e_2^u, e_1^v, e_2^v$, where the superscripts indicate the original end-vertex of the respective semiedge. Similarly, remove p, q and the edge pq from H to obtain a 4-pole with semiedges $f_1^p, f_2^p, f_1^q, f_2^q$. Finally, take two new vertices x and y joined by a new edge xy and connect e_1^u to f_1^p, e_1^v to f_1^q, e_2^u and f_2^p to x, and e_2^v and f_2^q to y. As an example, consider the graph $B_1 + Ps$ shown in Fig. 24. Note that $H + G$ is isomorphic to $G + H$ provided that the choice of vertices and the ordering of semiedges remain the same.

We claim that $G + H$ is a snark whenever G and H are. To see this, it is sufficient to realise that $G + H$ can be interpreted as a dot product of G and H for an appropriate choice of edges in G and adjacent vertices in H. In fact, $G + H$ can be viewed both as $G \cdot H$ and as $H \cdot G$.

Consider the family of snarks $C_n = B_1 + I_n$ where B_1 is the Blanuša snark (Fig. 17) and I_n is the Isaacs flower snark (Fig. 8) for odd $n \geqslant 5$. The snark $B_1 + Ps$ can be included as the initial member of this family exactly in the same way as the Petersen graph can be viewed the smallest of the Isaacs flower snarks (see Example 5.5). We claim that all these snarks are irreducible. Indeed, $B_1 + Ps$ is irreducible by Theorems 4.6, 6.1(a), and the facts stated in Example 5.5. Several further members (up to C_{13}) have been verified by a computer. For the rest of the family, one can employ an induction argument similar in structure to the one used in Example 5.5.

The snark C_n has two atoms A_1 and A_2 related by the following equalities: $\partial A_1 = S_1 = \partial' A_2$ and $\partial A_2 = S_2 = \partial' A_1$ (see Fig. 24 which for simplicity shows $B_1 + Ps$). Furthermore, $A_1 = A_2'$ and $A_2 = A_1'$, and both S_1 and S_2 are associated with the same quasiatomic pair of vertices $\{x, y\}$ created by the operation $B_1 + I_n$.

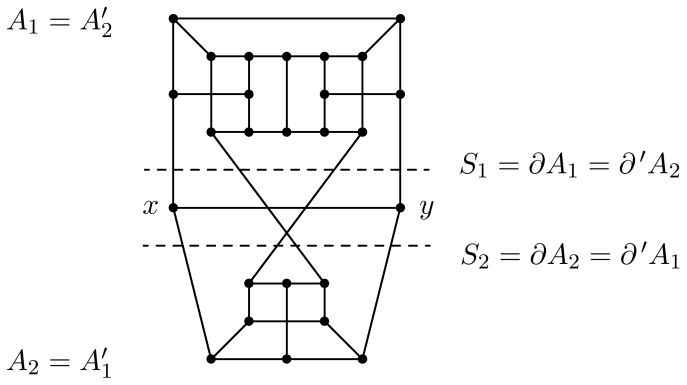

Fig. 24. The snark $B_1 + Ps$

Since B_1 is a dot product of two copies of the Petersen graph, the usual unrestricted composition chain of C_n consists of two copies of Ps and a copy of I_n.

Let us choose $K \subseteq C_n$ to be any subgraph with vertex-set $\{x, y\}$; it does not matter whether K does or does not include the edge xy. We now explore the K-relative composition chains of C_n. We may start factorisation with any of the cuts S_1 and S_2, as both of them are K-consistent. Since $S_1 = \partial A_1$ and $S_2 = \partial' A_1$, Proposition 8.2 implies that in either case we obtain the same direct factors, namely B_1 and I_n. The vertices x and y will be inherited into I_n if S_1 has been used in the first step, otherwise they will be inherited into B_1. In the former case we carry on with K-consistent factorisation of B_1, thus obtaining the usual three composition factors of $B_1 + I_n$. In the latter case, however, there are no further clean K-relative refinements. The vertices $\{x, y\}$ must always be inherited into the same factor, but all cycle-separating 4-edge-cuts of B_1 separate them.

Therefore, depending on the choice of particular factorisation steps we can really arrive at essentially different K-relative composition chains of $B_1 + I_n$. □

Surprisingly, the case where $|K| = 2$ is the only exception. The rest of this section is devoted to proving this claim, that is to say, to establishing Theorem D. The proof

basically follows the same lines as that of Theorem C and relies on K-consistent versions of the lemmas used for Theorem C. One important additional device, however, is needed first. We show that for a K-relative factorisation of an irreducible snark edges of K are actually irrelevant and we may therefore think of K as having no edges. In such a situation we simply identify K with its vertex-set $V(K)$.

Proposition 11.2. *Let G be an irreducible snark with a distinguished subgraph $K \subseteq G$, and let H be a subsnark of G. Then a cycle-separating 4-edge-cut of H is K-consistent if and only if it is $V(K)$-consistent.*

Proof. A K-consistent edge-cut is obviously $V(K)$-consistent. For the converse, let us take a $V(K)$-consistent cycle-separating 4-edge-cut S of H and show that it is also K-consistent. If $V(K) \cap V(H) = \emptyset$, there is nothing to prove. We therefore assume that $K \subseteq H$ and that S is not K-consistent. Since S is $V(K)$-consistent, all the vertices of K will be inherited into the same S-factor B of G. On the other hand, S is not K-consistent, so there must be two vertices u and v in K which are adjacent in H but are not adjacent in B. However, the only edges of H which do not continue to exist in any of its S-factors are those of S. Thus $uv \in S$, and consequently u and v belong to different S-factors of G. This contradiction establishes the lemma. □

Corollary 11.3. *A factorisation chain of an irreducible snark G is K-relative if and only if one of its members contains $V(K)$.*

The previous proposition and its corollary allow us to speak of K-consistent edge-cuts and K-relative factorisation chains while actually dealing only with vertices of K.

We now proceed to the K-relative versions of Proposition 8.2, Proposition 9.3, and Lemma 10.2, respectively.

Proposition 11.4. *Let G be an irreducible snark of cyclic connectivity 4 with a distinguished subgraph K of order at least 3. Let A be an atom of G such that both ∂A and $\partial' A$ are K-consistent. Then for any K-relative factorisation chain $\mathcal{F} \succcurlyeq \{\tilde{A}', G/\tilde{A}'\}$ there exists an equivalent K-relative factorisation chain $\mathcal{G}$ such that $\mathcal{G} \succcurlyeq \{A, G/\tilde{A}\}$.*

Proof. From Proposition 8.2 and the discussion following it we know that $\tilde{A}' \cong \tilde{A}$ and $G/\tilde{A}' \cong G/\tilde{A}$. Moreover, the former isomorphism simply substitutes the quasiatomic pair $\{v_1, v_2\}$ in $\tilde{A}'$ with the new pair of vertices in $\tilde{A}$ while the latter isomorphism does the exact reverse between $G/\tilde{A}'$ and $G/\tilde{A}$. Since $\mathcal{F} \succcurlyeq \{\tilde{A}', G/\tilde{A}'\}$, we can express $\mathcal{F}$ as $\mathcal{F}|_{\tilde{A}'} \cup \mathcal{F}|_{G/\tilde{A}'}$, and it is obvious that the restricted chains are K-relative. By applying the above isomorphisms to members of $\mathcal{F}|_{\tilde{A}'}$ and $\mathcal{F}|_{G/\tilde{A}'}$ we obtain factorisation chains $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\tilde{A}$ and $G/\tilde{A}$, respectively, and set $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. Clearly, $\mathcal{G}$ is a factorisation chain of G. It remains to show that $\mathcal{G}$ is K-relative.

To see this we first observe that K has no vertex in common with the quasiatomic pair $\{v_1, v_2\}$ associated with A. Suppose to the contrary that K contains, say, v_1. As $|K| \geqslant 3$, there is a third vertex u in K different from both v_1 and v_2. Clearly, either ∂A or $\partial' A$ separates u from v_1. However, this is a contradiction since both these cuts are assumed to be K-consistent.

It follows that $K \cap \{v_1, v_2\} = \emptyset$ implying that exactly one of the inclusions $K \subseteq A$ and $K \subseteq G - A^{\#}$ holds. If $K \subseteq A$, then $K \subseteq \tilde{A}$ and $K \subseteq \tilde{A}'$. Since $K \cap \{v_1, v_2\} = \emptyset$, the isomorphism $\tilde{A}' \cong \tilde{A}$ maps a K-consistent cycle-separating 4-edge-cut of a subsnark of $\tilde{A}'$ to a K-consistent cycle-separating 4-edge-cut of the corresponding subsnark of $\tilde{A}$. Hence $\mathcal{G}_1$ is K-relative. On the other hand, any cycle-separating 4-edge-cut in a subsnark H of $G/\tilde{A}$ is automatically K-consistent because $V(K) \cap V(H = \emptyset$. Therefore $\mathcal{G}_2$ is K-relative, too, and hence so is the whole $\mathcal{G}$.

If $K \subseteq G - A^{\#}$, the roles of $\tilde{A}'$ and $G/\tilde{A}'$ are interchanged, and a similar reasoning leads to the conclusion that $\mathcal{G}$ is K-consistent again. This completes the proof. □

Proposition 11.5. *Let G be an irreducible snark of cyclic connectivity 4 with a distinguished subgraph K, and let A be an atom of G with K-consistent atomic cut. Further let S be a K-consistent cycle-separating 4-edge-cut in G not associated with A. Then one of the S-factors of G has cyclic connectivity 4, contains A as an atom, and the inherited cut ∂A remains K-consistent.*

Proof. Let H be the S-factor which inherits A from G. In view of Lemma 9.3, H has cyclic connectivity 4 and A is an atom of H. It remains to show that the cut $\partial_H A$ is K-consistent. Assume to the contrary that it is not. Then $K \subseteq H$ and there is a pair of vertices of K separated by $\partial_H A$. Exactly one of these vertices belongs to A. Since A has the same vertices in H as it has in G, the cut $\partial_G A$ separates the same pair of vertices in G as well. However, this contradicts the assumption that $\partial_G A$ is K-consistent. □

Lemma 11.6. *Let G be an irreducible snark of cyclic connectivity 4 with a distinguished subgraph K of order at least 3, and let A be an atom of G with K-consistent atomic cut. Then for any K-relative composition chain $\mathcal{C}$ of G there exists a K-relative composition chain $\mathcal{C}_1$ of $\tilde{A}$ and a K-relative composition chain $\mathcal{C}_2$ of $G/\tilde{A}$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof. Let $\mathcal{C}$ be an arbitrary K-relative composition chain of G, and let A be a fixed atom of G. By Proposition 11.5, there exists a factorisation chain $\mathcal{M} \preccurlyeq \mathcal{C}$ such that either $\tilde{A}$ or $\tilde{A}' \cong \tilde{A}$ is a member of $\mathcal{M}$. Assuming that $\mathcal{M}$ is the shortest such chain, we define the *height* of $\tilde{A}$ in $\mathcal{C}$, denoted by $h = h_{\mathcal{C}}(\tilde{A})$, to be $|\mathcal{M}| - 1$, and proceed by induction on h.

If $h = 1$, then without loss of generality $\mathcal{M} = \{\tilde{A}, G/\tilde{A}\}$, and we can take $\mathcal{C}_1 = \mathcal{C}|_{\tilde{A}}$ and $\mathcal{C}_2 = \mathcal{C}|_{G/\tilde{A}}$. Since $\mathcal{C}$ is K-relative, so are the restricted chains. Thus the statement holds in this case.

For the induction step it is enough to prove that, whenever $h \geqslant 2$, there exists a K-relative composition chain $\mathcal{U}$ equivalent to $\mathcal{C}$ such that $h_{\mathcal{U}}(\tilde{A}) < h$. So let $h \geqslant 2$, and consider the shortest chain $\mathcal{M} \preccurlyeq \mathcal{C}$ containing $\tilde{A}$ or $\tilde{A}'$. We may again assume $\mathcal{M}$ to contain $\tilde{A}$. Clearly, $\mathcal{M}$ is an elementary refinement of a chain $\mathcal{H} \preccurlyeq \mathcal{C}$ a member H of which contains A as an atom. Since $|\mathcal{H}| \geqslant 2$, H arises from a subsnark J by decomposition along a K-consistent cycle-separating edge-cut S. Let $\mathcal{J} \preccurlyeq \mathcal{C}$ be the longest K-relative factorisation chain containing J; hence, $\mathcal{J} \preccurlyeq \mathcal{H} \preccurlyeq \mathcal{M}$. We now consider two cases depending on whether $\mathcal{H}$ is or is not an elementary refinement of $\mathcal{J}$.

Case 1. If $\mathcal{H}$ is an elementary refinement of $\mathcal{J}$, then $\mathcal{H}|_J = \{H, J/H\}$ and $\mathcal{M}|_J = \{\tilde{A}, H/\tilde{A}, J/H\}$. The latter chain, denoted by $\mathcal{S}$, arises from J by first using the cut S and then by using one of the cuts associated with A. As Proposition 11.4 shows, we may assume the latter cut to be ∂A (more precisely, $\partial_H A$). Due to Lemma 10.1, we can reverse the order of the cuts and decompose J by first taking ∂A (more precisely, $\partial_K A$) to obtain $\{\tilde{A}, J/\tilde{A}\}$, and then by employing $p_A(S)$ in $J/\tilde{A}$ to produce a chain $\mathcal{A} = \{\tilde{A}, H_1, J_1\}$ equivalent to $\mathcal{S}$, with $H_1 \cong H/\tilde{A}$ and $J_1 \cong J/H$. Since $\mathcal{S}$ is a restriction of a K-relative factorisation chain, it is itself K-relative. It follows that either $V(K) \cap V(J) = \emptyset$, or one of members of $\mathcal{S}$ contains all the vertices of K. Can we claim that $\mathcal{A}$ is K-relative, too? If $V(K) \cap V(J) = \emptyset$, the answer is trivially *yes*. So let us assume that K is contained in a member of $\mathcal{S}$. Regarded as a subgraph of G, the distinguished subgraph K contains no new vertices resulting from factorisation. However, Lemma 10.1 guarantees that $\mathcal{S}$ and $\mathcal{A}$ only differ in the ordering of their members and in the identity of newly added vertices and edges. Thus there must be a member of $\mathcal{A}$ which also contains all the vertices of K. By Corollary 11.3 this means that $\mathcal{A}$ is K-relative. From this point on we can continue as in the proof of Lemma 10.2: in $\mathcal{M}$ we replace $\mathcal{M}|_J$ with $\mathcal{A}$ to form a chain $\mathcal{M}_1$ which can be refined to a composition chain $\mathcal{B}$ equivalent to $\mathcal{C}$, but with $h_{\mathcal{B}}(\tilde{A}) < h$.

Case 2. This case can be handled by a straightforward modification of arguments employed in the corresponding place of the proof of Lemma 10.2. We leave the details to the reader. □

We are ready to prove Theorem D.

Theorem 11.7. *Let G be an irreducible snark with a distinguished subgraph $K \subseteq G$ of order different from 2. If $(G_1, \ldots, G_n)$ and $(H_1, \ldots, H_m)$ are any two K-relative composition chains of G, then $n = m$ and there exists a permutation π of $\{1, 2, \ldots, n\}$ such that $H_i \cong G_{\pi(i)}$ for all $i = 1, 2, \ldots, n$.*

Proof. Let K be any subgraph of G such that $|K| \neq 2$. If $|K| = 1$, then the result follows from Theorem C. Therefore we may assume that $|K| \geqslant 3$, and proceed by induction on the order of G.

The conclusion obviously holds for cyclically 5-connected snarks and for snarks with no K-consistent cycle-separating 4-edge-cuts. For the induction step let G be an irreducible snark with at least one K-consistent cycle-separating 4-edge-cut, and assume the theorem to be true for all irreducible snarks of smaller order. We claim that G contains at least one atom with a K-consistent atomic cut. Indeed, if S is any K-consistent cycle-separating 4-edge-cut in G, then one of the resulting cyclic fragments contains all vertices of K. The other fragment contains an atom, and this atom, say A, is separated from K by S. Hence $K \cap A = \emptyset$, and the cut ∂A is K-consistent in G.

Now let $\mathcal{C}$ and $\mathcal{D}$ be any two K-relative composition chains of G. By Lemma 11.6, there exist K-relative composition chains $\mathcal{C}_1$ and $\mathcal{D}_1$ of $\tilde{A}$ and $\mathcal{C}_2$ and $\mathcal{D}_2$ of $G/\tilde{A}$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{D}$ equivalent to $\mathcal{D}_1 \cup \mathcal{D}_2$. Since both $|\tilde{A}| < |G|$ and $|G/\tilde{A}| < |G|$, the induction hypothesis implies that $\mathcal{C}_1$ is equivalent to $\mathcal{D}_1$ and $\mathcal{C}_2$ is equivalent to $\mathcal{D}_2$. Thus $\mathcal{C}$ is equivalent to $\mathcal{D}$. □

As we already know, Theorem D (11.7) fails when the preassigned subgraph K has order 2. In spite of the fact that the proof of Theorem D is similar to that of Theorem C, a significant difference can be found in Proposition 11.4. This lemma, in contrast to its counterpart, Proposition 8.2, guarantees the equivalence of atomic and quasiatomic cuts only when the subgraph K has order at least 3. Moreover, Example 11.1 shows that Proposition 11.4 does not immediately extend to the 2-vertex case. The same example, however, seems to suggest that to save Proposition 11.4 for factorisations with respect to 2-vertex subgraphs it might be sufficient to avoid quasiatomic pairs of vertices. Unfortunately, this is false. The reason is that a pair of vertices which is not quasiatomic with respect to any atom in the original snark G can become quasiatomic in a subsnark of G as the factorisation proceeds. To recognise whether this happens or not only from the structure of G may not be easy. However, if this does not happen for any choice of refinement steps, then all relative composition chains with respect to such a 2-vertex subgraph will indeed be equivalent. Whether this property can be replaced by a "simple" condition which refers only to G rather than to the collection of all its factorisation chains is not known.

12 Conclusions

We conclude this paper with several remarks concerning Theorem C (10.3). In particular, we examine the question of whether the result is best possible.

Canonical factorisation. Theorem 10.3 asserts that all composition chains of an irreducible snark are pairwise equivalent. Some of them, however, can be regarded as canonical. To see this, let G be an irreducible snark of cyclic connectivity 4 and let $A_1, A_2, \ldots, A_s$, $s \geqslant 2$, be the complete list of atoms of G. We know that the factors resulting from the decomposition of G along any of the atomic cuts ∂A_i are $\tilde{A}_i$ and $G/\tilde{A}_i$, and that $\tilde{A}_i$ arises from the 4-pole $A_i^{\#}$ either by a junction with L while $G/\tilde{A}_i$ arises by a junction with R, or vice versa. Since $\tilde{A}_i$ is uniquely determined by A_i, and the couples of L, R and the bond ∂A_i correspond to each other, the way of attaching L or R to $G - A_i$ is also uniquely determined by A_i. As any two distinct atoms A_i and A_j are disjoint, the same holds for the edge-ends of ∂A_i and ∂A_j incident with vertices of A_i and A_j, respectively. It follows that the necessary junctions will be the same irrespectively of the order taken. In other words, we can disconnect all atomic cuts at once and perform the junctions on both sides of the cuts unambiguously, or we can just do it in an arbitrary order. As a result we obtain a factorisation chain $\mathcal{F}_1$ containing the snarks $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_s$ in some order and possibly some additional snarks $G_1, G_2, \ldots, G_t$, $t \geqslant 0$, which arise from vertices of G not belonging to atoms and from some of the newly added vertices and edges. At any rate, G determines $\mathcal{F}_1$ uniquely up to isomorphism and ordering of the factors. If $\mathcal{F}_1$ is not a composition chain, we inductively repeat the process with each member of $\mathcal{F}_1$ until we reach a composition chain of G.

Note that the existence of this composition chain only relies on results of Sections 3 and 7 and thus avoids the use of Theorem C. On the other hand, the mere existence of a

canonical composition chain is not sufficient for establishing Theorem C.

Role of atoms. Atoms play a crucial role in the proof of Theorem C. Are however atoms the only reason why Theorem C holds? To put it differently, is the fact that all composition chains of an irreducible snark are equivalent solely caused by the uniqueness of atoms? Yet in other words, is the collection of all atoms in all subsnarks encountered on the way from an irreducible snark to a particular composition chain the same for all composition chains? The answer is, surprisingly, negative. To see this, consider the snark $B_{1,2}$ depicted in Fig. 25. Clearly, $B_{1,2}$ is a dot product of three copies of the Petersen graph. The first two of these copies are joined in the same way as in B_1 whereas the second and third copies are joined in the same way as in B_2. Therefore $B_{1,2}$ is a kind of crossbreed between B_1 and B_2. It follows from Theorems 4.6 and 6.1 applied to Example 5.5 that $B_{1,2}$ is irreducible.

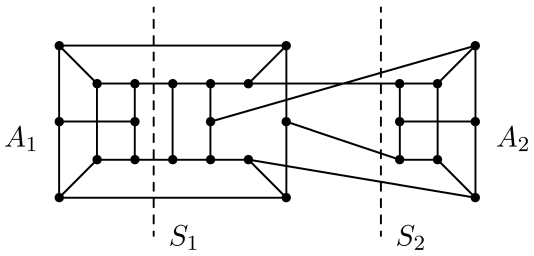

Fig. 25. The snark $B_{1,2}$

There are two significant cycle-separating 4-edge-cuts in $B_{1,2}$, namely S_1 and S_2 in the notation of Fig. 25. Thus we may factorise $B_{1,2}$ in two different ways depending on the order in which S_1 and S_2 are used. Starting with S_1 we obtain the chains $\{B_{1,2}\} \preccurlyeq \{Ps, B_2\} \preccurlyeq \{Ps, Ps, Ps\}$. On the other hand, when S_2 is applied first, the chains will be $\{B_{1,2}\} \preccurlyeq \{B_1, Ps\} \preccurlyeq \{Ps, Ps, Ps\}$. Out of all these snarks — that is $B_{1,2}$, B_1, B_2, and Ps — it is only B_2 which contains an atom of order 10. Thus the two composition chains induce different collections of atoms.

Elusive subsnarks. Each elementary refinement of a factorisation chain creates two new vertices and five new edges not present in the original snark. It is actually possible that certain subsnarks arising during the factorisation process exclusively comprise new vertices and new edges. We call such subsnarks *elusive* since they are not immediately "visible" in the composite snark.

As an example consider the irreducible snark Sz shown in Fig. 26 left, known as the Szekeres snark (see [14, 15]). Its composition chain consists of six copies of the Petersen graph. Five of them are evident from the figure, the sixth one is elusive and is responsible for the overall structure of the graph.

At the first glance elusive subsnarks seem to be important for determining the internal structure of the composite snark. Surprisingly, however, the property of being elusive is

not invariant over the composition chains of an irreducible snark, and therefore elusive subsnarks do not provide any structural information about an irreducible snark in general.

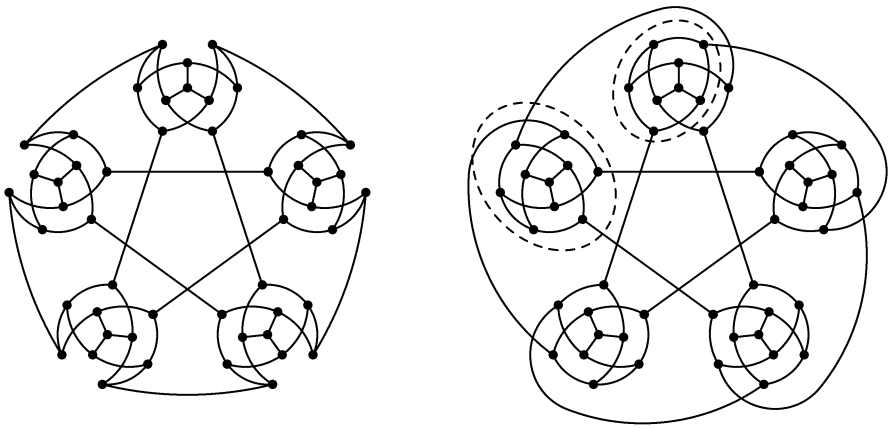

Fig. 26. The Szekeres snark Sz and the Watkins snark Wa

To see this, examine a related irreducible snark Wa (Fig. 26 right) discovered by Watkins [14] (see also [15]). This snark is obtained by a dot product of six copies of the Petersen graph, however, performed in a way different from the Szekeres snark; the distinction is similar to that between the Blanuša snarks B_1 and B_2 — Fig. 17. Although each composition chain of the Watkins snark consists of six copies of the Petersen graph, there are ten cycle-separating 4-edge-cuts in the composite snark, therefore different decisions can be taken in particular factorisation steps. For example, there are five atomic and five quasiatomic cuts (see Fig. 26 where dashed lines indicate one atomic and one quasiatomic cut). If the quasiatomic cuts are used for factorisation in each of the five cases, the sixth (central) copy of the Petersen graph will be elusive. Nonetheless, if the atomic cuts are used for factorisation in each of the five cases, the sixth (central) copy of the Petersen graph will completely consist of the original vertices while the five copies of the Petersen graph whose cores are evident in the figure will gain two new vertices each. The resulting composition chain thus contains no elusive factor. It follows that the composition subsnark elusive in one composition chain can completely lose this property in another composition chain.

Composition chains of critical snarks. Another important problem related to Theorem C is the question whether its result is best possible. To give an answer we look at snarks that are just one step below irreducible snarks in our hierarchy of k-irreducible snarks, that is, at (strictly) critical snarks.

To start with, it is useful to realise that composition factors of a critical snark need not be cyclically 5-connected. For instance, the strictly critical snark Sc from Fig. 16 can be factorised into a collection of three copies of the Petersen graph and a copy of I_3 whose cyclic connectivity is only 3. Thus for composition factors of k-irreducible snarks with $k < 7$ the cyclic connectivity, girth, and the irreducibility class may decrease.

In spite of the fact that Sc has strange composition factors, it still has only one equivalence class of composition chains. To give an example of a critical snark with two non-equivalent composition chains let us consider the snark T_{44} of order 44 depicted in Fig. 27. Its only cycle-separating 4-edge-cut S connects two isomorphic uncolourable 4-poles each of which can be obtained from the Goldberg-Loupekine snark GL_1 by disconnecting the edges denoted x and y in Fig. 18. Note that T_{44} is not critical, because each edge not belonging to S is suppressible. In contrast, the edges of S are non-suppressible. The 62 edges complementing S can be partitioned into 31 pairwise disjoint pairs, each consisting of an edge of $GL_1 - \{x, y\}$ and its counterpart in the other copy of GL_1. Since all these pairs are essential, T_{44} can be converted into a strictly critical snark by the technique described in Section 6. After performing a repeated dot product of T_{44} on the left with 31 copies of Ps on the right, each time using one of the essential pairs of edges in the left factor, we obtain a strictly critical snark W on $44 + 31 \times 8 = 292$ vertices.

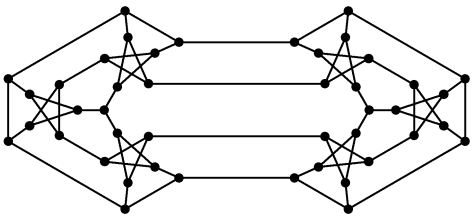

Fig. 27. The snark T_{44}

Let us inspect composition chains of W. Since W contains T_{44} as a subsnark, it admits a composition chain which includes a composition chain of T_{44}. Given the structure of W, such a composition chain will consist of 31 copies of Ps and a composition chain of T_{44}. For the latter composition chain we have three possibilities corresponding to three non-equivalent ways of expressing T_{44} as a dot product of two S-factors. Two of them produce composition chains consisting of cyclically 5-connected factors, namely $\{GL_1, GL_1'\}$ and $\{GL_2, GL_2'\}$. These two chains are obviously non-equivalent, and so are the corresponding composition chains of W. This shows that Theorem C does not extend to reducible snarks. The third possibility for T_{44} gives rise to an S-factor with cyclic connectivity 4 which can be further decomposed. The resulting composition chain of T_{44} will have three, not just two, factors and, of course, will not be equivalent to any of the previous two chains. It follows that strictly critical snarks can have composition chains of different lengths. With this point of view the result of Theorem C appears even stronger.

Decompositions along 5-cuts. The dot product has a less well-known 5-connected analogue first mentioned by Cameron et al. in [2]. Given two cubic graphs G and H let us form 5-poles $G(e_0, e_1, \ldots, e_4)$ and $H(f_0, f_1, \ldots, f_4)$ by removing a 5-cycle from G and a 5-cycle from H, and by deriving the ordering of the resulting semiedges from the cyclic ordering of the corresponding cycles. Define $G\#H$ to be the cubic graph obtained by joining each e_i to f_{2i}, the indices being reduced modulo 5. It can be shown that if both

G and H are snarks, so is $G\#H$. Moreover, $G\#Ps \cong Ps\#G \cong G$ which means that the Petersen graph is both the right and the left identity of this operation.

With this in mind, it is natural to attempt building a decomposition theory based on this and perhaps some similar operations. An additional reason for such a project would be the fact that 5-decompositions have already been investigated in detail and the following 5-Decomposition Theorem is known [2]:

Let G be a snark with a 5-edge-cut whose disconnection leaves 5-poles M and N. Then either one of M and N is not colourable, or both M and N can be extended to snarks G_1 and G_2 by adding at most five vertices to each.

In addition to this result it was shown in [9] that critical snarks admitting no proper 5-decomposition in the above sense are exactly those which are cyclically 5-connected and have the property that every cycle-separating 5-edge-cut separates a 5-cycle from the rest of the graph.

An analysis of the proofs reveals that the snark completion of each G_i either adds one, three or five vertices (a claw plus an isolated edge, a 2-path with dangling edges, and a 5-cycle with dangling edges, respectively). Unfortunately, only the last of these possibilities corresponds to a well-defined 5-composition operation as the reverse of a 5-decomposition, namely to the #-product described above. Examples suggest that the remaining two types of completion cannot be set into a framework of unrestricted operations (see [9, p. 273]) although such operations have been reported in [3]. This drawback would certainly make a possible theory of 5-decomposition of snarks more complicated, although undoubtedly interesting.

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