# On a Rado Type Problem for Homogeneous Second Order Linear Recurrences

Hayri Ardal* Zdeněk Dvořák† Veselin Jungić‡ Tomáš Kaiser§

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**Abstract**

In this paper we introduce a Ramsey type function $S(r;a,b,c)$ as the maximum $s$ such that for any $r$-coloring of $\mathbb{N}$ there is a monochromatic sequence $x_1, x_2, \ldots, x_s$ satisfying a homogeneous second order linear recurrence $ax_i + bx_{i+1} + cx_{i+2} = 0$, $1 \leqslant i \leqslant s-2$. We investigate $S(2;a,b,c)$ and evaluate its values for a wide class of triples $(a,b,c)$.

## 1 Introduction

In this paper we are interested in the following question: If the set of positive integers $\mathbb{N}$ is finitely colored, is it possible to find a monochromatic sequence of a certain length that satisfies a given second order homogeneous recurrence? A reader that is even remotely familiar with Ramsey Theory would quickly note that Van der Waerden's theorem affirmatively answers this question for the recurrence $x_i - 2x_{i+1} + x_{i+2} = 0$, any finite coloring of $\mathbb{N}$, and any finite sequence length. But what about other second order homogeneous recurrences?

In 1997 Harborth and Maasberg [4] considered the recurrence $x_i + x_{i+1} = ax_{i+2}$ and obtained a puzzling sequence of results that have inspired a large portion of the work presented in this paper:

*Department of Mathematics, Simon Fraser University, Burnaby, B.C., V5A 2R6, Canada. E-mail: hardal@sfu.ca.

†Department of Mathematics, Simon Fraser University, Burnaby, B.C., V5A 2R6, Canada. E-mail: rakdver@kam.mff.cuni.cz.

‡Department of Mathematics, Simon Fraser University, Burnaby, B.C., V5A 2R6, Canada. E-mail: vjungic@sfu.ca.

§Department of Mathematics and Institute for Theoretical Computer Science, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic. E-mail: kaisert@kma.zcu.cz. Supported by project 1M0545 and Research Plan MSM 4977751301 of the Czech Ministry of Education.

i. If $a = 1$ then any finite coloring of positive integers yields a 4-term monochromatic sequence that satisfies the recurrence.

ii. If $a = 2$ then any finite coloring of positive integers yields arbitrarily long monochromatic sequences that satisfy the recurrence.

iii. If $a = 4$ then any 2-coloring of $[1, 71]$ will produce a monochromatic 4-term sequence that satisfies the recurrence.

iv. For any odd prime $a$ there is a 2-coloring of the set positive integers with no monochromatic 4-term sequence that satisfies the recurrence.

We were intrigued with the question what we can learn about monochromatic sequences that satisfy the recurrence $x_i + x_{i+1} = 2^k x_{i+2}$, $k \geqslant 3$ or the recurrence $x_i + x_{i+1} = 2k x_{i+2}$, $k \geqslant 3$.

The problem of finding monochromatic sequences that satisfy homogeneous recurrences belongs to the rich and exciting segment of Ramsey Theory that has its roots in the celebrated Ph.D. thesis of Richard Rado. Here we mention two results of Rado [8] that are used in developing ideas presented in this paper.

**Theorem 1.** *Let $L$ be a linear homogeneous equation with integer coefficients. Assume that $L$ has at least three and not all coefficients of the same sign. Then any 2-coloring of $\mathbb{N}$ admits a monochromatic solution to $L$.*

Let $r$ be a positive integer. A linear equation or a system of linear equations $L$ is $r$-regular if every $r$-coloring of positive integers admits a monochromatic solution to $L$. Hence Theorem 1 states that a linear homogeneous equation in more than two variables and with integer coefficients, both positive and negative, is at least 2-regular. Fox and Radoičić [2] showed that the equation $x_1 + 2x_2 - 4x_3 = 0$ is not 3-regular, so Rado's result is best possible. Moreover, a recent result by Alekseev and Tsimerman [1] affirmatively settled Rado's conjecture that for any $r \geqslant 3$ there is a homogeneous linear equation that is not $r$-regular.

We say that a linear equation or a system of linear equations $L$ is regular if it is $r$-regular for all $r \in \mathbb{N}$.

**Theorem 2.** *For a linear homogeneous system $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$, where $\mathbf{A}$ is an $m \times n$ matrix with integer entries, to be regular it is necessary and sufficient that the matrix $\mathbf{A}$ satisfies the columns condition, i.e., that there is a partition $S_1 \cup \ldots \cup S_k$ of the set of columns of the matrix $\mathbf{A}$ such that elements of $S_1$ add up to $\mathbf{0}$ and that, for any $j \in \{2, \ldots k\}$, the sum of all elements of $S_j$ is a rational linear combination of the elements from $\cup_{i=1}^{j-1} S_i$.*

A version of Rado's proof of Theorem 1 in English can be found in [7]. A version of the proof of Theorem 2 and more information about $r$-regularity, regular systems, the columns condition, and related problems is possible to find, for example, in [6].

In [3] and [4] Harborth and Maasberg considered the following problem.

**Problem 3.** [3] Let $a, b, c, s$ be integers such that $a, c \neq 0$ and $s \geqslant 3$. Find the largest $r \in \mathbb{N}$, if it exists, such that every $r$-coloring of $\mathbb{N}$ yields a monochromatic $s$-term sequence $x_1, x_2, \ldots, x_s$ that satisfies the homogeneous second order recurrence $ax_n + bx_{n+1} + cx_{n+2} = 0$. If such an $r$ exists, it is called the *degree of partition regularity* of the given recurrence for $s$-term sequences and denoted by $k_0(s; a, b, c)$.

We note that the problem of finding the degree of partition regularity of the given recurrence for $s$-term sequences is equivalent to the problem of finding the largest $r$ for which the linear homogeneous system

$$\begin{array}{ccccccccc} ax_1 & + & bx_2 & + & cx_3 & & & = & 0 \\ \vdots & & & & & & & & \\ & & ax_{s-2} & + & bx_{s-1} & + & cx_s & = & 0 \end{array}$$

is $r$-regular. We write $k_0(s; a, b, c) = 0$ if the corresponding system has no solution and $k_0(s; a, b, c) = \infty$ if the corresponding system is regular.

**Observation 4.** *The following is true for all $a, b, c \in \mathbb{Z}$ and $s \geqslant 3$:*

*i.* $k_0(s; a, b, c) = k_0(s; c, b, a)$

*ii.* $k_0(s; a, b, c) = k_0(s; na, nb, nc)$*, for any nonzero integer $n$.*

*iii. For any $s \geqslant 3$, $k_0(s + 1; a, b, c) \leqslant k_0(s; a, b, c)$.*

Harborth and Maasberg proved in [3] the following fact.

**Theorem 5.** $k_0(s; a, b, c) = \infty$ *if and only if one of the following is true:*

*i. $s = 3$ and one of $a + b + c, a + b, a + c, b + c$ is equal to zero.*

*ii. $s = 4$ and $a + b + c = 0$ or $a = b = -c$ or $a = -b = -c$.*

*iii. $s \geqslant 5$ and $a + b + c = 0$*

The results by Harborth and Maasberg mentioned at the beginning of this section now can be stated in the following form:

i. $k_0(4; 1, 1, -1) = \infty$.

ii. $k_0(s; 1, 1, -2) = \infty$, for any $s \geqslant 3$.

iii. $k_0(4; 1, 1, -4) = 2$.

iv. $k_0(4; 1, 1, -p) = 1$, for all odd primes $p$.

In an attempt to further examine the function $k_0(s;a,b,c)$ and related problems, we introduce, for $r,a,b,c \in \mathbb{N}$, a new Ramsey type function

$$S(r;a,b,c) = \max\{s \geqslant 0 : k_0(s;a,b,c) \geqslant r\}.$$

Thus $S(r;a,b,c)$ is the maximum $s \geqslant 0$ such that for any $r$-coloring of $\mathbb{N}$ there is a monochromatic sequence $x_1, x_2, \ldots, x_s$ satisfying the recurrence $ax_i + bx_{i+1} + cx_{i+2} = 0$, $1 \leqslant i \leqslant s-2$. We write $S(r;a,b,c) = \infty$ if the set $\{s \geqslant 0 : k_0(s;a,b,c) \geqslant r\}$ is not bounded. For example, $S(r;1,-2,1) = \infty$.

It is the purpose of this paper to investigate $S(2;a,b,c)$ and to evaluate its values for a wide class of triples $(a,b,c)$. The paper is organized in the following way. In Section 2 we give some basic properties of the function $S(2;a,b,c)$ and we discuss the case when there is a prime $p$ which divides exactly two elements of $\{a,b,c\}$ to the same power. In Section 3 we consider the case when there is a prime $p$ that divides exactly one of the coefficients $a$, $b$, and $c$. Our results in this Section show that the value of $S(2;a,b,c)$ depends on the order of a certain element, that is determined by the coefficients $a$, $b$, and $c$, in the multiplicative group $\mathbb{Z}_p^*$. In Section 4 we introduce a computer-based method for finding values of $S(2;a,b,c)$. We finish with a few observations and open problems.

To an impatient reader who wonders what happens with the recurrence $x_i + x_{i+1} = 2^k x_{i+2}$ we suggest to take a quick peek at Corollary 17.

The following notation will be used in the remainder of this paper. For $x \in \mathbb{N}$ and $t \in \mathbb{N}\backslash\{1\}$, if $x = t^u(tv+w)$, for some integers $u,v,w \in \mathbb{Z}$ with $u,v \geqslant 0$ and $1 \leqslant w \leqslant t-1$, then we will write $x = (u,v,w)_t$. For a prime $p$, if $l \in \mathbb{Z}$ is such that $p \nmid l$, $o_p(l)$ denotes the order of $l$ in the multiplicative group $\mathbb{Z}_p^*$. For $n,x,y \in \mathbb{Z}$, by $x \equiv_n y$, we mean $x \equiv y \ (mod\ n)$. And lastly, for $n \in \mathbb{Z}$, let $(n)_2$ be the remainder when $n$ is divided by 2.

## 2 The function $S(2;a,b,c)$

In the rest of this paper we will write $S(a,b,c)$, or just $S$, to denote the function $S(2;a,b,c)$.

We start with a few simple facts.

**Theorem 6.** *The following is true for any $a,b,c \in \mathbb{Z}$.*

*i.* $S(a,b,c) = S(c,b,a)$.

*ii.* $S(a,b,c) = S(na,nb,nc)$ *for any nonzero integer $n$.*

*iii.* $S(a,b,c) \geqslant 3$ *if $a$, $b$, and $c$ are nonzero integers not all of the same sign.*

*iv.* *If $a+b+c=0$ then $S(a,b,c) = \infty$.*

*Proof.* Statements (i) and (ii) follow from Observation 4, statement (iii) follows from Theorem 1, and (iv) follows from Theorem 2. $\square$

Since it is enough to consider the case when $gcd(a, b, c) = 1$, we will focus our attention to the following two cases:

1. There is a prime $p$ that divides exactly two elements of the set $\{a, b, c\}$.
2. There is a prime $p$ that divides exactly one element of the set $\{a, b, c\}$.

For a wide class of triples, the size of the middle coefficient determines an upper bound on the values of $S$.

**Theorem 7.** *Let $a, b, c \in \mathbb{N}$ with $c \leqslant b$. Then $S(a, b, -c) \leqslant 4$.*

*Proof.* Let $\alpha = \dfrac{a+b}{c}$. For each non-negative integer $i$, let $B_i = [\alpha^i, \alpha^{i+1}) \cap \mathbb{N}$. Let $\chi$ be a 2-coloring of $\mathbb{N}$ defined by $\chi(x) = (i)_2$ if $x \in B_i$, $i \geqslant 0$. We will show that under $\chi$ there is no 5-term monochromatic sequence satisfying the recurrence $ax_i + bx_{i+1} = cx_{i+2}$.

Assume for a contradiction that the sequence $x_1, x_2, x_3, x_4, x_5$ is $\chi$-mono- chromatic and it satisfies the recurrence $ax_i + bx_{i+1} = cx_{i+2}$. Since $b \geqslant c$ and $a > 0$, we have $x_2 < x_3 < x_4 < x_5$. If $x_2, x_3 \in B_i$ for some $i$, then

$$\alpha^{i+1} \leqslant x_4 = \frac{a}{c}x_2 + \frac{b}{c}x_3 < \alpha^{i+2},$$

which is impossible since this would imply $\chi(x_4) \neq \chi(x_3)$. Similarly, there is no $i$ such that $x_3, x_4 \in B_i$. Since $\chi(x_2) = \chi(x_3)$ and $x_2 < x_3$, there exist $i, j \in \mathbb{N}$, with $j - i$ positive and even, such that $x_2 \in B_i$ and $x_3 \in B_j$. But then, we have

$$\alpha^j \leqslant x_3 < x_4 < \alpha x_3 < \alpha^{j+2},$$

i.e., $x_4 \in B_j \cup B_{j+1}$. Hence $x_4$ has to be in $B_j$, which gives the desired contradiction. □

A special case of Theorem 7, together with an earlier mentioned result by Harborth and Maasberg, covers the Fibonacci recurrence $x_i + x_{i+1} - x_{i+2} = 0$.

**Corollary 8.** $S(1, 1, -1) = 4$. *In fact, $S(r; 1, 1, -1) = 4$ for all $r \geqslant 1$.*

Next we consider recurrences of the form $x_i - bx_{i+1} + x_{i+2} = 0$, $b \geqslant 1$. We note that there is no 4-term sequence of positive integers that satisfies the recurrence $x_i - x_{i+1} + x_{i+2} = 0$. Thus $S(1, -1, 1) = 3$. By Theorem 2, $S(1, -2, 1) = \infty$. The remaining cases are given by the following theorem.

**Theorem 9.** $S(1, -b, 1) = 3$ *for all $b \geqslant 3$.*

*Proof.* Since $b$ is positive, by Theorem 1, $S(1, -b, 1) \geqslant 3$.

First, assume that $b$ is odd and define a 2-coloring $\chi$ as

$$\chi(x) = \begin{cases} 0 & \text{if } x \equiv_b 1, 2, \ldots, \frac{b-1}{2}, \\ 1 & \text{if } x \equiv_b \frac{b+1}{2}, \ldots, b-1, \\ \chi(x/b) & \text{if } x \equiv_b 0. \end{cases}$$

Let the sequence $x_1, x_2, x_3, x_4$ be monochromatic and let it satisfy the recurrence $x_i - bx_{i+1} + x_{i+2} = 0$, with $x_4$ minimal possible. Then $x_1 + x_3 \equiv_b 0$ and $x_2 + x_4 \equiv_b 0$. This is possible only if $x_1 \equiv_b x_2 \equiv_b x_3 \equiv_b x_4 \equiv_b 0$. Let $y_i = x_i/b$, $1 \leqslant i \leqslant 4$. It follows that $\chi(y_i) = \chi(x_i)$, for all $i \in \{1,2,3,4\}$, and $y_i - by_{i+1} + y_{i+2} = 0$, $i \in \{1,2\}$, with $y_4 < x_4$. This contradicts our assumption that $x_4$ is minimal.

Now assume that $b = 2b'$ for some $b' \geqslant 2$ and define a 2-coloring $\chi$ as

$$\chi(x) = \begin{cases} 0 & \text{if } x \equiv_b 1, 2, \ldots, b'-1, \\ 1 & \text{if } x \equiv_b b'+1, \ldots, b-1, \\ \chi(x/b') & \text{if } x \equiv_{b'} 0. \end{cases}$$

The remainder of the proof is similar to the proof of the odd case. □

In [3] Harborth and Maasberg proved that if $gcd(a,b,c) = 1$ and if there is a prime $p$ which divides exactly two elements of $\{a,b,c\}$ to the same power, i.e., there are positive integers $k$, $A$, $B$, and $C$, $p \nmid ABC$, such that $\{a,b,c\} = \{Ap^k, Bp^k, C\}$, then $k_0(4;a,b,c) \leqslant 2$. We strengthen their result in the following way.

**Theorem 10.** *Let $a,b,c$ be integers such that $\gcd(a,b,c) = 1$. If there is a prime $p$ which divides exactly two of the coefficients to the same power, then $S(a,b,c) \leqslant 3$.*

*Proof.* Suppose that $p$ is a prime which divides exactly two elements of the set $\{a,b,c\}$ to the same power $k$, say $a = Ap^k$ and $b = Bp^k$, $p \nmid AB$, and $p \nmid c$.

We define a 2-coloring $\chi$ as $\chi(x) = \left(\left\lfloor \frac{u}{k} \right\rfloor\right)_2$, where $x = (u,v,w)_p$.

Let $x_1, x_2, x_3, x_4$ be a monochromatic sequence that satisfies the recurrence $ax_i + bx_{i+1} + cx_{i+2} = 0$. Suppose that $x_i = (u_i, v_i, w_i)_p$, for some $u_i, v_i, w_i \in \mathbb{Z}$, $1 \leqslant i \leqslant 4$. Then

$$Ap^{u_1+k}(pv_1 + w_1) + Bp^{u_2+k}(pv_2 + w_2) = -cp^{u_3}(pv_3 + w_3) \quad (1)$$
$$Ap^{u_2+k}(pv_2 + w_2) + Bp^{u_3+k}(pv_3 + w_3) = -cp^{u_4}(pv_4 + w_4) \quad (2)$$

If $u_1 < u_2$ then $p^{u_1+k}(Apv_1 + Aw_1 + Bp^{u_2-u_1}(pv_2 + w_2)) = -cp^{u_3}(pv_3 + w_3)$ and, since $w_1 \neq 0$ and $w_3 \neq 0$, it follows that $u_1 + k = u_3$. Thus

$$\left\lfloor \frac{u_3}{k} \right\rfloor = 1 + \left\lfloor \frac{u_1}{k} \right\rfloor.$$

This contradicts our assumption that $\chi(x_1) = \chi(x_3)$. Similarly we conclude that $u_2 < u_1$ is not possible. Hence, we must have $u_1 = u_2 = u_3$. Then, since $k \geqslant 1$, $p^{u_3+1}$ divides the left-hand side of (1) but not the right-hand side, a contradiction.

The proof in the case when $p^k$ divides $a$ and $c$ is similar to the proof above. □

An immediate consequence of Theorem 10 is the following claim:

**Corollary 11.** *If $gcd(a,b,c) = 1$ and if there is a prime $p$ that divides exactly two elements of the set $\{a,b,c\}$ to the same power then $k_0(4;a,b,c) = 1$.*

## 3 The cases $S(a, -p^k q, c)$ and $S(a, b, -p^k q)$

In this section we consider the case when only one of the coefficients is divisible by a prime $p$.

**Theorem 12.** *Let $p$ be a prime and let $a, c$, and $q$ be arbitrary integers not divisible by $p$. Let $C \equiv_p -c/a$ with $C \not\equiv_p 1$. Then, for any $k \geqslant 1$,*

*(i) If $p$ is odd and $o_p(C)$ is even then $S(a, -p^k q, c) \leqslant 3$.*

*(ii) If $p$ is odd and $o_p(C)$ is odd then $S(a, -p^k q, c) \leqslant 5$.*

*(iii) If $p = 2$, $k \geqslant 2$ and $a \equiv_4 c$ then $S(a, -2^k q, c) \leqslant 3$.*

*Proof.* We start with the definition of a 2-coloring of $\mathbb{Z}_p^*$ that we will use to prove claims (i) and (ii).

For $l \in \mathbb{Z}$ such that $p \nmid l$ let $H$ be the cyclic subgroup generated by $l$ and let $\{a_1, a_2, \ldots, a_t\}$ be a complete set of representatives in $\mathbb{Z}_p^*/H$. Recall that $d = o_p(l)$ denotes the order of $l$ in the multiplicative group $\mathbb{Z}_p^*$.

A 2-coloring $\psi_{(p,l)} : \mathbb{Z}_p^* \to \{0, 1\}$ is defined as $\psi_{(p,l)}(x) = (i)_2$ if $x = a_j l^i$ for some $1 \leqslant i \leqslant d-1$ and $1 \leqslant j \leqslant t$.

Thus

$$\psi_{(p,l)}(x) = \psi_{(p,l)}(lx) \Leftrightarrow (d)_2 = 1 \text{ and } x = a_j l^{d-1} \text{ for some } j. \tag{3}$$

We define a coloring $\chi : \mathbb{N} \to \{0, 1\}$ by $\chi(x) = \psi_{(p,C)}(w)$, where $x = (u, v, w)_p$.

Proof of claim (i): Assume that a $\chi$-monochromatic sequence $x_1, x_2, x_3, x_4$ satisfies the recurrence $ax_i - p^k q x_{i+1} + c x_{i+2} = 0$. For $1 \leqslant i \leqslant 4$, let $u_i, v_i$ and $w_i$ be such that $x_i = (u_i, v_i, w_i)_p$. Then $\chi(x_i) = \psi_{(p,C)}(w_i)$, i.e., the set $\{w_1, w_2, w_3, w_4\}$ is $\psi_{(p,C)}$-monochromatic and

$$\begin{aligned} ap^{u_1}(pv_1 + w_1) + cp^{u_3}(pv_3 + w_3) &= p^{u_2+k} q (pv_2 + w_2) &(4)\\ ap^{u_2}(pv_2 + w_2) + cp^{u_4}(pv_4 + w_4) &= p^{u_3+k} q (pv_3 + w_3). &(5)\end{aligned}$$

If $u_1 < u_3$ then $u_1 = u_2 + k$, by (4), which together with (5) implies $u_2 = u_4$ and hence

$$p^{u_2}\left(p(av_2 + cv_4) + aw_2 + cw_4\right) = p^{u_3+k}(pv_3 + w_3).$$

Since $u_2 < u_3 + k$, this is possible only if $w_2 \equiv_p C w_4$. But since $o_p(C)$ is even and $\psi_{(p,C)}(w_2) = \psi_{(p,C)}(w_4)$, this contradicts (3).

Similarly $u_3 < u_1$ is not possible.

Assume $u_1 = u_3$. Since $\psi_{(p,C)}(w_1) = \psi_{(p,C)}(w_3)$, by (3) $aw_1 + cw_3 \not\equiv_p 0$. By (4), $u_1 = u_3 = u_2 + k$, which implies that $u_2 < u_3 + k$ and thus contradicts (5).

Hence, in the case of $p$ odd and $d$ even we have that $S_2(a, -p^k l, c) \leqslant 3$.

Proof of claim (ii): Assume that a $\chi$-monochromatic sequence $x_1, x_2, x_3, x_4, x_5, x_6$, $x_i = (u_i, v_i, w_i)_p$, satisfies the recurrence $ax_i - p^k q x_{i+1} + c x_{i+2} = 0$. Then $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ is a monochromatic set under $\psi_{(p,C)}$ and in addition to (4) and (5) we have

$$\begin{aligned} ap^{u_3}(pv_3 + w_3) + cp^{u_5}(pv_5 + w_5) &= p^{u_4+k} q (pv_4 + w_4) &(6)\\ ap^{u_4}(pv_4 + w_4) + cp^{u_6}(pv_6 + w_6) &= p^{u_5+k} q (pv_5 + w_5). &(7)\end{aligned}$$

If $u_1 < u_3$ then $u_1 = u_2 + k$, $u_2 = u_4$ and $w_2 \equiv_p Cw_4$. Since $u_2 + k = u_1 < u_3$, it follows that $u_4 + k < u_3$ and, from (6), we get $u_5 < u_3$. Similarly, by using the equations (6) and (7), we get $w_4 \equiv_p Cw_6$. Therefore $w_2 \equiv_p Cw_4 \equiv_p C^2 w_6$ and $\psi_{(p,C)}(w_2) = \psi_{(p,C)}(w_4) = \psi_{(p,C)}(w_6)$, which contradicts (3).

Cases $u_3 < u_1$ and $u_1 = u_3$, with $w_1 \not\equiv_p Cw_3$, are handled in the same way.

If $u_1 = u_3$, with $w_1 \equiv_p Cw_3$, then $u_1 = u_3 < u_2 + k$. Therefore, from (5), we get $u_4 + k > u_3$. But this implies $u_3 = u_5$ and $w_3 \equiv_p Cw_5$. Hence $w_1 \equiv_p Cw_3 \equiv_p C^2 w_5$ and $\psi_{(p,C)}(w_1) = \psi_{(p,C)}(w_3) = \psi_{(p,C)}(w_5)$, contradicting (3). This completes the proof of (ii).

Proof of claim (iii): Define $\chi : \mathbb{N} \to \{0, 1\}$ as $\chi(x) = (v)_2$, where $x = (u, v, 1)_2$. Assume that a monochromatic sequence $x_1, x_2, x_3, x_4$ satisfies the recurrence $ax_i - 2^k qx_{i+1} + cx_{i+2} = 0$. Let $x_i = (u_i, v_i, 1)$ for some $u_i, v_i \geqslant 0$, $1 \leqslant i \leqslant 4$. It follows, since $\chi(x_1) = \chi(x_2) = \chi(x_3) = \chi(x_4)$, that $v_1, v_2, v_3$ and $v_4$ are all of the same parity and

$$\begin{aligned} 2^{u_1} a(2v_1 + 1) + 2^{u_3} c(2v_3 + 1) &= 2^{u_2+k} q(2v_2 + 1) \quad (8)\\ 2^{u_2} a(2v_2 + 1) + 2^{u_4} c(2v_4 + 1) &= 2^{u_3+k} q(2v_3 + 1). \quad (9) \end{aligned}$$

If $u_1 < u_3$ then, from (8), $u_1 = u_2 + k$ and, from (9), $u_2 = u_4$. Hence,

$$2^{u_2} \left(2(av_2 + bv_4) + a + c\right) = 2^{u_3+k} q(2v_3 + 1).$$

Since $a$ and $c$ are both odd and since $a \equiv_4 c$ we conclude that $a + c \equiv_4 2$. Hence,

$$2^{u_2+1} \left(av_2 + bv_4 + \frac{a+c}{2}\right) = 2^{u_3+k} q(2v_3 + 1).$$

Since $av_2 + bv_4$ is even and $\frac{a+c}{2}$ is odd it follows that $u_3 + k = u_2 + 1 < u_3 - k + 1$. This is not possible since $k \geqslant 2$.

Similarly, if $u_3 < u_1$ then we obtain that $u_3 = u_2 + k$, $u_2 = u_4$, and $u_2 + 1 = u_3 + k$, which is again not possible.

So, assume $u_1 = u_3$. Then

$$2^{u_3+1} \left(av_1 + bv_3 + \frac{a+c}{2}\right) = 2^{u_2+k} q(2v_2 + 1)$$

which implies that $u_3 + k > u_2 + 1$. But from (9), we have $u_2 + 1 \geqslant u_3 + k$, a contradiction.

Hence, in the case of $k \geqslant 2$ and $a \equiv_4 c$, $S(a, -2^k l, c) \leqslant 3$. □

Next we consider the recurrence $ax_i + bx_{i+1} = p^k qx_{i+2}$, where $p$ is a prime number, $a, b, q$ are integers not divisible by $p$, and $k$ is a positive integer. For $m \geqslant 3$ and a sequence $x_1, x_2, \ldots, x_m$, $x_i = (u_i, v_i, w_i)_p$, that satisfies this recurrence we have that, for all $i \in [1, m-2]$,

$$ap^{u_i}(pv_i + w_i) + bp^{u_{i+1}}(pv_{i+1} + w_{i+1}) = p^{u_{i+2}+k} q(pv_{i+2} + w_{i+2}).$$

This implies that if $u_1 < u_2$ then

$$\begin{array}{l} u_1 = u_3 + k \text{ and } aw_1 \equiv_p qw_3 \\ u_i = u_{i+1} + k \text{ and } bw_i \equiv_p qw_{i+1} \quad \text{for all } i \geqslant 3, \end{array} \quad (10)$$

if $u_2 < u_1$ then

$$u_i = u_{i+1} + k \text{ and } bw_i \equiv_p qw_{i+1} \quad \text{for all } i \geqslant 2, \tag{11}$$

and if $u_1 = u_2$ and $aw_1 + bw_2 \not\equiv_p 0$ then

$$\begin{array}{ll} u_1 = u_2 = u_3 + k & \\ u_i = u_{i+1} + k & \text{for all } i \geqslant 2 \\ bw_i \equiv_p qw_{i+1} & \text{for all } i \geqslant 3. \end{array} \tag{12}$$

These facts will be used in the proof of the following theorem.

**Theorem 13.** *Let $k$ be a positive integer and let $p$ be an odd prime. Let $a, b, q \in \mathbb{Z}$ be such that $a \equiv_p 1$ and that $b$ and $q$ are not divisible by $p$. For $B \equiv_p -b$, $L \equiv_p q/b$, $s = o_p(B)$, $d = o_p(L)$, and $t = gcd(s, d)$ we have that if $s$ is even then*

$$S(a, b, -p^k q) \leqslant \begin{cases} 3 & \text{if } s/t \text{ is even} \\ 3 & \text{if } s/t \text{ and } d/t \text{ are both odd} \\ 4 & \text{if } s/t \text{ is odd and } d/t \text{ is even} \end{cases}$$

*Proof.* Let $H = \{1, L, L^2, \ldots, L^{d-1}\}$ and $K = \{1, B, B^2, \ldots, B^{s-1}\}$, let $G = HK$, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a complete set of representatives of classes in $\mathbb{Z}_p^*/G$. Fix an integer $n$ such that $\gcd(n, t) = 1$ and $B^{s/t} \equiv_p L^{n(d/t)}$ and note that if $B^i \equiv_p L^j$, for some $i, j \in \mathbb{Z}$, then $(s/t)|\, i$ and $(d/t)|\, j$.

**Case 1:** Assume that $s/t$ is even.

We 2-color the group $G$ by $f(B^i L^j) = (i)_2$ for $i, j \in \mathbb{Z}$. Now, if $B^{i_1} L^{j_1} \equiv_p B^{i_2} L^{j_2}$ for some $i_1, i_2, j_1, j_2$, then $B^{i_1 - i_2} = L^{j_2 - j_1}$ and $(s/t)|\,(i_1 - i_2)$. Since $s/t$ is even, this implies $i_1 \equiv_2 i_2$. Therefore, $f(B^{i_1} L^{j_1}) = f(B^{i_2} L^{j_2})$ and $f$ is well-defined. Now, we extend this coloring to a 2-coloring of $\mathbb{Z}_p^*$ by $F(x) = f\left(x\alpha_j^{-1}\right)$ if $x \in G\alpha_j$. Note that, for any $x \in \mathbb{Z}_p^*$,

$$F(Bx) \neq F(x) \tag{13}$$
$$F(Lx) = F(x) \tag{14}$$

We define $\chi : \mathbb{N} \to \{0, 1\}$ by $\chi(x) = \left(\left\lfloor \frac{u}{k} \right\rfloor + F(w)\right)_2$, where $x = (u, v, w)_p$.

Suppose that a $\chi$-monochromatic sequence $x_1, x_2, x_3, x_4$ satisfies the recurrence $ax_i + bx_{i+1} = p^k qx_{i+2}$. As before, $x_i = (u_i, v_i, w_i)_p$.

If $u_1 \neq u_2$ then, from (10) and (11), $u_3 = u_4 + k$ and $w_3 \equiv_p Lw_4$. Hence,

$$\left\lfloor \frac{u_3}{k} \right\rfloor + F(w_3) = 1 + \left\lfloor \frac{u_4}{k} \right\rfloor + F(Lw_4) = 1 + \chi(x_4),$$

which is not possible because $\chi(x_3) = \chi(x_4)$.

Assume that $u_1 = u_2$. Then $F(w_1) = F(w_2)$ and, from (13), it follows $w_1 \not\equiv_p Bw_2$, i.e., $aw_1 + bw_2 \not\equiv_p 0$. Hence, from (12), $u_3 = u_4 + k$ and $w_3 \equiv_p Lw_4$, which is not possible.

Therefore, in the case of $s/t$ even, $S(a, b, -p^k q) \leqslant 3$.

**Case 2:** Assume that $s/t$ and $d/t$ are both odd. Since $s$ is even, $t$, and hence $d$, must also be even.

We define a 2-coloring on the group $G$ by $f\left(B^i L^j\right) = (i+j)_2$ for $i, j \in \mathbb{Z}$. If $B^{i_1} L^{j_1} \equiv_p B^{i_2} L^{j_2}$ for some $i_1, i_2, j_1, j_2$, then $B^{i_1-i_2} \equiv_p L^{j_2-j_1}$. Hence, $(s/t) | (i_1-i_2)$ and $(d/t) | (j_2-j_1)$ and, since $s/t$ and $d/t$ are both odd, we conclude that $i_1 - i_2 \equiv_2 j_2 - j_1$. Therefore $f$ is well-defined.

We extend $f$ to a 2-coloring $F$ of $\mathbb{Z}_p^*$ in the same way as in Case 1. Now, for any $x \in \mathbb{Z}_p^*$,

$$F(Bx) \neq F(x) \tag{15}$$
$$F(Lx) \neq F(x) \tag{16}$$

This time we define $\chi : \mathbb{N} \to \{0, 1\}$ by $\chi(x) = F(w)$, where $x = (u, v, w)_p$.

Suppose that $x_1, x_2, x_3, x_4$, with $x_i = (u_i, v_i, w_i)_p$, is a $\chi$-monochromatic sequence that satisfies $ax_i + bx_{i+1} = p^k q x_{i+2}$.

If $u_1 \neq u_2$ then, from (10) and (11), $w_3 \equiv_p Lw_4$. This implies $\chi(x_3) = F(w_3) = F(Lw_4) \neq \chi(x_4)$. If $u_1 = u_2$ then $F(w_1) = F(w_2)$ and, from (15), we obtain $w_1 \not\equiv_p Bw_2$.

**Case 3:** Assume that $s/t$ is odd and $d/t$ is even.

We color the group $G$ by $f\left(B^i L^j\right) = \left(i + \left\lfloor \frac{j}{d/t} \right\rfloor\right)_2$, for $i, j \in \mathbb{Z}$. Now, if $B^{i_1-i_2} \equiv_p L^{j_2-j_1}$, for some $i_1, i_2, j_1, j_2$, then $i_1 - i_2 = (s/t)m_1$ and $j_1 - j_2 = (d/t)m_2$, for some $m_1, m_2 \in \mathbb{Z}$. It follows that

$$L^{j_2-j_1} \equiv_p B^{m_1(s/t)} \equiv_p L^{m_1 n(d/t)}$$

and $m_1 n(d/t) + j_1 - j_2 = (d/t)(m_1 n + m_2)$ is a multiple of $d$. Hence, $m_1 n + m_2$ is divisible by $t$ and $m_1 n + m_2 \equiv_2 0$, since $t$ is even. Also, because $\gcd(n, t) = 1$, $n$ must be odd. Next we observe that

$$m_1 n + m_2 = \frac{i_1 - i_2}{s/t} n + \frac{j_1 - j_2}{d/t} = \frac{i_1 - i_2}{s/t} n + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor$$

and

$$\frac{i_1 - i_2}{s/t} n + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor \equiv_2 (i_1 - i_2) + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor,$$

since $s/t$ and $n$ are both odd.

Hence,

$$(i_1 - i_2) + \left\lfloor \frac{j_1}{d/t} \right\rfloor - \left\lfloor \frac{j_2}{d/t} \right\rfloor \equiv_2 0$$

which implies

$$i_1 + \left\lfloor \frac{j_1}{d/t} \right\rfloor \equiv_2 i_2 + \left\lfloor \frac{j_2}{d/t} \right\rfloor.$$

Therefore, $f\left(B^{i_1} L^{j_1}\right) = f\left(B^{i_2} L^{j_2}\right)$ and $f$ is well-defined. We extend this coloring to the coloring $F$ as above. For any $x \in \mathbb{Z}_p^*$, we have

$$F(Bx) \neq F(x)$$

and

$$F(Lx) \neq F(x) \Rightarrow F(L^2x) = F(Lx),$$

since $d/t > 0$ is even.

In this case we define $\chi : \mathbb{N} \to \{0,1\}$ by $\chi(x) = \left(\left\lfloor \frac{u}{k} \right\rfloor + F(w)\right)_2$, where $x = (u,v,w)_p$.

Suppose that a $\chi$-monochromatic sequence $x_1, x_2, x_3, x_4, x_5$, with $x_i = (u_i, v_i, w_i)_p$, satisfies the recurrence $ax_i + bx_{i+1} = p^k q x_{i+2}$.

If $u_1 \neq u_2$ then from (10) and (11) $u_3 = u_4 + k$, $u_4 = u_5 + k$, $w_3 \equiv_p Lw_4$ and $w_4 \equiv_p Lw_5$. Hence,

$$\chi(x_3) \equiv_2 \left\lfloor \frac{u_3}{k} \right\rfloor + F(w_3) \equiv_2 1 + \left\lfloor \frac{u_4}{k} \right\rfloor + F(Lw_4) \equiv_2 1 + \chi(x_4) + F(Lw_4) + F(w_4).$$

But since $\chi(x_3) = \chi(x_4)$, we must have $F(Lw_4) \neq F(w_4)$. In the same way, we must have $F(Lw_5) \neq F(w_5)$. This contradicts (3), since $w_4 \equiv_p Lw_5$.

Assume that $u_1 = u_2$. Then $F(w_1) = F(w_2)$ and, from (3), $aw_1 + bw_2 \not\equiv_p 0$. Hence, from (12), $u_3 = u_4 + k$, $u_4 = u_5 + k$, $w_3 \equiv_p Lw_4$ and $w_4 \equiv_p Lw_5$, which is a contradiction.

Therefore, in the case of $s/t$ odd and $d/t$ even, $S(a,b,-p^kq) \leqslant 4$. □

The upper bounds for the values of $S(a,b,c)$ on some additional classes of triples $(a,b,c)$ easily follow.

**Corollary 14.** *Let $p$ be an odd prime, let $k \geqslant 1$, and let $a,b,q \in \mathbb{Z}$ be such that none of $a,b,q$ is divisible by $p$. Let $B \equiv_p -b/a$, $L \equiv_p q/b$, $s = o_p(B)$, $d = o_p(L)$ and $t = gcd(s,d)$. Then, if $s$ is even*

$$S(a,b,-p^kq) \leqslant \begin{cases} 3 & \text{if } s/t \text{ is even} \\ 3 & \text{if } s/t \text{ and } d/t \text{ are both odd} \\ 4 & \text{if } s/t \text{ is odd and } d/t \text{ is even} \end{cases}$$

*Proof.* Let $A \in \mathbb{Z}$ be such that $aA \equiv_p 1$, and let $a' = aA$, $b' = bA$ and $l' = lA$. Then

$$S(a,b,-p^kq) = S(aA, bA, -p^kqA) = S(a', b', -p^kq'),$$

since $-b/a = -b'/a' \equiv_p -b'$ and $q/b = q'/b'$, $B \equiv_p -b'$ and $L \equiv_p q'/b'$. Therefore, the claim follows from Theorem 13. □

What happens when $p = 2$? We are able to describe the case of the recurrence $ax_i + bx_{i+1} = 2^kqx_{i+2}$ if $a$ and $b$ are odd numbers congruent modulo 4.

**Theorem 15.** *Let $a,b,q \in \mathbb{Z}$ be odd integers with $a \equiv_4 b$.*

*(i) If $k = 2$ then $S(a,b,-2^kq) \leqslant 4$.*

*(ii) If $k \geqslant 3$ then $S(a,b,-2^kq) \leqslant 3$.*

*Proof.* Let $L = q - b$.

Proof of claim (i): Assume $L \equiv_4 0$ and define a 2-coloring $\chi : \mathbb{N} \to \{0,1\}$ as $\chi(x) = \left(\left\lfloor \frac{u}{2} \right\rfloor + v\right)_2$, where $x = (u, v, 1)_2$. Suppose that a monochromatic sequence $x_1, x_2, x_3, x_4, x_5$, $x_i = (u_i, v_i, 1)_2$, satisfies the recurrence $ax_i + bx_{i+1} = 2^k q x_{i+2}$.

If $u_1 \neq u_2$ then from (10) and (11), $u_3 = u_4 + 2$ and $4a(2v_2+1) + b(2v_3+1) = q(2v_4+1)$. Hence, $2a(2v_2+1) + bv_3 = qv_4 + L/2$. Since $L \equiv_4 0$, this implies $v_3 \equiv_2 v_4$. But then,

$$\chi(x_3) \equiv_2 \left\lfloor \frac{u_3}{2} \right\rfloor + v_3 \equiv_2 1 + \left\lfloor \frac{u_4}{2} \right\rfloor + v_4 \equiv_2 1 + \chi(x_4),$$

a contradiction.

If $u_1 = u_2$, since $\chi(x_1) = \chi(x_2)$, it follows that $v_1 \equiv_2 v_2$. Hence

$$2^{u_2}\left(2(av_1 + bv_2) + a + b\right) = 2^{u_3+2} q(2v_3 + 1).$$

Since $a \equiv_4 b$ it follows that $u_2 = u_3 + 1 > u_3$. Then, from (11), we get $u_4 = u_5 + 2$ and $v_4 \equiv_2 v_5$. This fact implies $\chi(x_4) \neq \chi(x_5)$.

Now assume $L \equiv_4 2$ and define a 2-coloring $\chi$ as $\chi(x) = (v)_2$, where $x = (u, v, 1)_2$.

Suppose that $x_1, x_2, x_3, x_4, x_5$, with $x_i = (u_i, v_i, 1)_2$, is a monochromatic sequence that satisfies $ax_i + bx_{i+1} = 2^k q x_{i+2}$. Then all $v_i$'s are of the same parity.

If $u_1 \neq u_2$ then, as before, $2a(2v_2 + 1) + bv_3 = qv_4 + L/2$. Since $L \equiv_4 2$, this implies $v_3 \equiv_2 1 + v_4$, a contradiction. If $u_1 = u_2$, we get $v_4 \equiv_2 1 + v_5$, a contradiction again.

Proof of claim (ii): Assume that $k \geqslant 3$.

If $L \equiv_4 0$ we define a 2-coloring $\chi$ as $\chi(x) = \left(\left\lfloor \frac{u}{k} \right\rfloor + v\right)_2$, where $x = (u, v, 1)_2$.

Suppose that $x_1, x_2, x_3, x_4$, $x_i = (u_i, v_i, 1)_2$, is a monochromatic sequence that satisfies $ax_i + bx_{i+1} = 2^k q x_{i+2}$.

If $u_1 \neq u_2$ then $u_2 \geqslant u_3 + k$ and $u_3 = u_4 + k$. It follows that $2^{u_2 - u_3 - 1} a(2v_2 + 1) + bv_3 = qv_4 + L/2$. Since $L \equiv_4 0$ and $u_2 - u_3 \geqslant k \geqslant 3$ we conclude that $v_3 \equiv_2 v_4$. Hence

$$\chi(x_3) \equiv_2 \left\lfloor \frac{u_3}{k} \right\rfloor + v_3 \equiv_2 1 + \left\lfloor \frac{u_4}{k} \right\rfloor + v_4 \equiv_2 1 + \chi(x_4),$$

which contradicts our assumption that $x_3$ and $x_4$ are of the same color.

If $u_1 = u_2$ then $v_1 \equiv_2 v_2$ and $2^{u_2}\left(2(av_1 + bv_2) + a + b\right) = 2^{u_3+2} l(2v_3 + 1)$. Since $a + b \equiv_4 2$ it follows that $u_2 = u_3 + k - 1 > u_3 + 1$. Then, from (11), we obtain $u_3 = u_4 + 2$ and $v_3 \equiv_2 v_4$. This implies $\chi(x_4) \neq \chi(x_5)$.

If $L \equiv_4 2$ we define a 2-coloring $\chi$ of positive integers by $\chi(x) = (v)_2$, where $x = (u, v, 1)_2$.

Reasoning similar to one demonstrated above leads to the conclusion that there is no 4-term monochromatic sequence that satisfies the recurrence $ax_i + bx_{i+1} = 2^k q x_{i+2}$. $\square$

As we mentioned in the introduction, this paper was inspired by results obtained by Harborth and Maasberg in [3], [4], and [5]. The following theorem extends Harborth and Maasberg's result from [4] that $k_0(4; 1, 1, -p) = 1$ for all odd primes $p$.

**Theorem 16.** *Let $r$ and $m$ be positive odd integers and let $k$ be a non-negative integer. Then $S(1, 1, -r^m(kr + 1)) = 3$.*

*Proof.* We consider a recurrence $x_n + x_{n+1} = r^m(kr+1)x_{n+2}$ where $r$, $m$, and $k$ are as above.

A 2-coloring $\varphi$ is defined in the following way. For $q \in \{1, ..., r-1\}$ the coloring $\varphi$ colors all $m \equiv_r q$ by 0 if $q \in \{1, ..., (r-1)/2\}$ and $\varphi$ colors all $m \equiv_r q$ by 1 if $q \in \{(r-1)/2, ..., r-1\}$. If $m$ is a multiple of $r$ then $\varphi(m) \neq \varphi(m/r)$.

Suppose that there is a $\varphi$-monochromatic sequence $x_1$, $x_2$, $x_3$, $x_4$ that satisfies the recurrence and that $x_1$ is the smallest possible. Thus

$$r^m(kr+1)x_3 = x_2 + x_1 \text{ and } r^m(kr+1)x_4 = x_3 + x_2.$$

Since $x_2 + x_1 \equiv_{r^m} x_3 + x_2 \equiv_{r^m} 0$ and since $x_1$, $x_2$, and $x_3$ are of the same color we conclude that $x_1$, $x_2$, and $x_3$ are multiples of $r^m$. Let $x_1 = r^m y_1$, $x_2 = r^m y_2$, and $x_3 = r^m y_3$. Then

$$r^m(kr+1)y_3 = y_2 + y_1 \text{ and } (kr+1)x_4 = y_3 + y_2.$$

Note that $y_1$, $y_2$, and $y_3$ are of the same color that is different than the color of $x_4$ since $m$ is odd. As before, the first equality implies that $y_1$ and $y_2$ are multiples of $r$. Hence $y_3 \equiv_r x_4$. Since $\{y_3, x_4\}$ is not monochromatic this implies that both of them are multiples of $r$. Say, $x_4 = ry_4$. But then $y_1$, $y_2$, $y_3$, $y_4$ is a monochromatic sequence that satisfy the original recurrence with $y_1 < x_1$ which contradicts our assumption that $x_1$ is the smallest possible. □

Now we are in a position to describe what happens with the recurrence $x_i + x_{i+1} = cx_{i+2}$, if $c$ is a positive integer.

**Corollary 17.** *Let $c \in \mathbb{N}$. Then*

$$S(1,1,-c) = \begin{cases} 4 & \text{if } c = 1 \text{ or } c = 4, \\ \infty & \text{if } c = 2, \\ 3 & \text{if } c \equiv_8 0, \\ 3 & \text{if } c = r^m(kr+1) \text{ for some odd } r, \geqslant 0 \text{ and } k \in \mathbb{N}, \\ 3 & \text{if } c = p^k q \text{ for some odd prime } p,\ k \geqslant 0 \text{ and } q \in \mathbb{N} \\ & \text{such that } p \nmid q \text{ and } o_p(q) \not\equiv_4 0. \end{cases}$$

*In all other cases,* $3 \leqslant S(1,1,-c) \leqslant 4$.

We note that if $p$ is a prime such that $p \equiv_4 3$ and if $q \in \mathbb{N}$ is such that $p \nmid q$ then either $o_p(q)$ is odd or $o_p(q) \equiv_4 2$. Thus, for such $p$ and $q$, $S(1,1,-p^k q) = 3$, $k \in \mathbb{N}$.

# 4 More Values for $S(2; a, b, c)$: Another Technique

In this section we introduce a new technique which gives upper bounds for $S(a, b, c)$ for some of the cases not covered in Sections 2 and 3, as well as some of the cases which have been already covered. We introduce this technique through the case of the recurrence $8x_i - 6x_{i+1} + x_{i+2} = 0$.

**Theorem 18.** $S(-8,6,-1) \leqslant 5$.

*Proof.* Let $\pi$ be a permutation on $\mathbb{Z}_{11}^2$ defined by

$$\pi(a,b) = (b, 6b - 8a).$$

We consider the recurrence $-8x_i + 6x_{i+1} = x_{i+2}$ modulo 11. Excluding the trivial cycle $(0,0)$, we represent the cycles of this permutation by the following Table 1.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 6 | 6 | 10 | 1 | 3 | 10 | 3 | 4 |
| 0 | 2 | 1 | 1 | 9 | 2 | 6 | 9 | 6 | 8 |
| 0 | 3 | 7 | 7 | 8 | 3 | 9 | 8 | 9 | 1 |
| 0 | 4 | 2 | 2 | 7 | 4 | 1 | 7 | 1 | 5 |
| 0 | 5 | 8 | 8 | 6 | 5 | 4 | 6 | 4 | 9 |
| 0 | 6 | 3 | 3 | 5 | 6 | 7 | 5 | 7 | 2 |
| 0 | 7 | 9 | 9 | 4 | 7 | 10 | 4 | 10 | 6 |
| 0 | 8 | 4 | 4 | 3 | 8 | 2 | 3 | 2 | 10 |
| 0 | 9 | 10 | 10 | 2 | 9 | 5 | 2 | 5 | 3 |
| 0 | 10 | 5 | 5 | 1 | 10 | 8 | 1 | 8 | 7 |
| 1 | 2 | 4 | 8 | 5 | 10 | 9 | 7 | 3 | 6 |
| 1 | 4 | 5 | 9 | 3 | | | | | |
| 2 | 8 | 10 | 7 | 6 | | | | | |

Table 1: The cycles of the permutation $\pi$ of $\mathbb{Z}_{11}^2$.

(In the first row, 0 1 6 6 . . . means $(0,1) \xrightarrow{\pi} (1,6) \xrightarrow{\pi} (6,6) \xrightarrow{\pi} \ldots$.)

Let $f$ be a 2-coloring of $\mathbb{Z}_{11}$ such that

$$f(m) = \begin{cases} 0 & \text{if } m \in \{1,2,3,5,7\} \\ 1 & \text{if } m \in \{4,6,8,9,10\} \end{cases}$$

and assume that 0 is colored by both colors.

We observe that no 6 consecutive elements in any of the cycles have the same color, but that there is a cycle with five consecutive elements colored by the same color; 6 9 6 8 0 or 7 5 7 2 0, for example. Also, we note that a single 0 is among any five consecutive elements of the same color, in any of the cycles.

Let $\chi : \mathbb{N} \to \{0,1\}$ be such that $\chi(x) = f(w)$ if $x = (u,v,w)_{11}$ for some $u, v \geqslant 0$ and $1 \leqslant w \leqslant 10$. It is not difficult to see that, under this coloring there is no monochromatic 6-term sequence $x_1, x_2, x_3, x_4, x_5, x_6$ satisfing the recurrence $x_{n+2} = 6x_{n+1} - 8x_n$. ☐

The above proof also implies that if $a, b, c \in \mathbb{Z}$ are such that $a \equiv_{11} 8$, $b \equiv_{11} -6$ and $c \equiv_{11} -1$ then $S(a,b,c) \leqslant 5$.

The method of the above theorem can be summarized as follows.

Given the recurrence relation $ax_i + bx_{i+1} = cx_{i+2}$, we choose a prime number $p$ such that $p \nmid c$ and consider the recurrence as a permutation on $\mathbb{Z}_p^2$ defined by $\pi(x,y) = (y, \alpha a x + \alpha b y)$ where $\alpha \in \mathbb{Z}$ is such that $\alpha c \equiv_p 1$. Then we find a 2-coloring of $\mathbb{Z}_p$ in a way that we minimize the length of the longest monochromatic interval in any cycle of the permutation $\pi$, assuming that 0 is colored by both colors.

We repeat this process for several primes and choose the best one among them.

Some computer generated results of this method are summarized in Tables 2 and 3. We observe that some of the bounds in those two tables are tighter than the bound given by Theorem 7.

| $p$ | $a\ (mod\ p)$ | $b\ (mod\ p)$ | $S(a,b,-1)$ |
|---|---|---|---|
| 3 | 1 | 1, 2 | $\leqslant 4$ |
| | 2 | 0 | $\leqslant 3$ |
| | 2 | 1, 2 | $\leqslant 4$ |
| 5 | 1 | 1, 4 | $\leqslant 5$ |
| | 1 | 2, 3 | $\leqslant 3$ |
| | 2, 4 | 0 | $\leqslant 3$ |
| | 2 | 2, 3 | $\leqslant 4$ |
| | 3 | 0 | $\leqslant 3$ |
| | 3 | 1, 4 | $\leqslant 4$ |
| | 4 | 1 | $\leqslant 4$ |
| | 4 | 3 | $\leqslant 6$ |
| 7 | 1 | 1, 2, 5, 6 | $\leqslant 5$ |
| | 1 | 4 | $\leqslant 4$ |
| | 2 | 0, 1, 3, 4 | $\leqslant 5$ |
| | 2 | 2 | $\leqslant 4$ |
| | 2 | 5 | $\leqslant 4$ |
| | 3, 5, 6 | 0 | $\leqslant 3$ |
| | 3 | 1 | $\leqslant 5$ |
| | 3 | 4 | $\leqslant 6$ |
| | 3 | 6 | $\leqslant 4$ |
| | 4 | 0, 2, 3, 5 | $\leqslant 5$ |
| | 4 | 1, 6 | $\leqslant 4$ |
| | 5 | 1 | $\leqslant 6$ |
| | 5 | 2 | $\leqslant 5$ |
| | 5 | 4, 5, 6 | $\leqslant 4$ |
| | 6 | 1, 3, 5 | $\leqslant 4$ |
| | 6 | 4 | $\leqslant 5$ |

Table 2: Some more bounds for $S(a,b,-1)$

# 5 Concluding Remarks

It is a very interesting fact that $S(a,b,c) \leqslant 6$ in all cases that we have considered, except when $a+b+c=0$, in which case $S(a,b,c)=\infty$. We wonder if $a+b+c \neq 0$ implies

| $p$ | $a$ (*mod* $p$) | $b$ (*mod* $p$) | $S(a,b,-1)$ |
|---|---|---|---|
| | 1 | 1, 4, 7, 10 | $\leqslant 6$ |
| | 1 | 2, 5, 5, 6, 9 | $\leqslant 5$ |
| | 1 | 3, 8 | $\leqslant 4$ |
| | 2, 6, 7 | 0 | $\leqslant 3$ |
| | 2 | 1, 2, 6, 8 | $\leqslant 5$ |
| | 2 | 3, 4, 5, 7 | $\leqslant 4$ |
| | 3 | 0, 1, 2, 3, 5, 6, 10 | $\leqslant 5$ |
| | 3 | 4, 7 | $\leqslant 4$ |
| 11 | 3 | 8 | $\leqslant 6$ |
| | 4 | 0, 2, 3, 4, 7, 9, 10 | $\leqslant 5$ |
| | 4 | 1 | $\leqslant 6$ |
| | 4 | 5, 6 | $\leqslant 4$ |
| | 5 | 0, 2, 3, 4, 5, 8, 9 | $\leqslant 5$ |
| | 5 | 1, 10 | $\leqslant 4$ |
| | 5 | 6 | $\leqslant 6$ |
| | 6 | 2, 3, 4, 9 | $\leqslant 4$ |
| | 6 | 5, 7, 8, 10 | $\leqslant 5$ |

Table 3: Some more bounds for $S(a,b,-1)$

$S(a,b,c) \leqslant 6$.

From Corollary 17 we see that the only two values of $c$ for which the value of $S(1,1,-c)$ equals 4 are $c = 1$ and $c = 4$. The case $c = 1$ is discussed in Corollary 8 and the case $c = 4$ was done in [5] with the help of computer by showing that any 2-coloring of the interval $[1,71]$ contains a monochromatic 4-term sequence satisfying the recurrence $x_i + x_{i+1} = 4x_{i+2}$. We ask if there are other values of $c$ for which $S(1,1,-c) = 4$. For example, is it true that $S(1,1,-10) = 4$? (The case $c = 10$ is the smallest value of $c$ for which the exact value of $S(1,1,-c)$ is unknown. The next case is $c = 26$.)

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