Satisfying states of triangulations of a convex n-gon

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Abstract

In this work we count the number of satisfying states of triangulations of a convex n-gon using the transfer matrix method. We show an exponential (in n) lower bound. We also give the exact formula for the number of satisfying states of a strip of triangles.

1 Introduction

A classic theorem of Petersen claims that every cubic (each degree 3) graph with no cutedge has a perfect matching. A well-known conjecture of Lovasz and Plummer from the

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mid-1970's, still open, asserts that for every cubic graph G with no cutedge, the number of perfect matchings of G is exponential in $|V(G)|$. The assertion of the conjecture was proved for the $k-$regular bipartite graphs by Schrijver [Sch98] and for the planar graphs by Chudnovsky and Seymour [CS08]. Both of these results are difficult. In general, the conjecture is widely open; see [KSS08] for a linear lower bound obtained so far.

We suggest to study the conjecture of Lovasz and Plummer in the dual setting. This relates the conjecture to a phenomenon well-known in statistical physics, namely to the degeneracy of the Ising model on totally frustrated triangulations of $2-$dimensional surfaces.

In order to explain this we need to start with another well-known conjecture, namely the directed cycle double cover conjecture of Jaeger (see [Jae00]): *Every cubic graph with no cutedge can be embedded in an orientable surface so that each face is homeomorphic to an open disc (i.e., the embedding defines a map) and the geometric dual has no loop.*

By a slight abuse of notation we say that a map in a $2-$dimensional surface is a *triangulation* if each face is bounded by a cycle of length 3 (in particular there is no loop); hence we allow multiple edges. We say that a set S of edges of a triangulation T is *intersecting* if S contains exactly one edge of each face of T.

Assuming the directed cycle double cover conjecture, we can reformulate the conjecture of Lovasz and Plummer as follows: *Each triangulation has an exponential number of intersecting sets of edges*.

We next consider the Ising model. Given a triangulation $T = (V, E)$, we associate the *coupling constant* $c(e) = -1$ with each edge $e \in E$. A *spin-assignment* of $U \subseteq V$ is a function $\sigma : U \to \{+, -\}$ where + denotes 1 and - denotes -1. Each spin-assignment of U is naturally identified with an element from $\{+, -\}^{|U|}$. A *state* of the Ising model is any spin-assignment of V. The energy of a state s is defined as $-\sum_{\{u,v\}\in E} c(uv) \cdot \sigma(u) \cdot \sigma(v)$. The states of minimum energy are called *groundstates*. The number of groundstates is usually called the *degeneracy* of T, denoted g(T), and it is an extensively studied quantity (for regular lattices T) in statistical physics (see for example [LV03]). Moreover, a basic tool in the degeneracy study is the transfer matrix method.

We further say that a state σ *frustrates* edge $\{u, v\}$ if $\sigma(u) = \sigma(v)$. Clearly, each state frustrates at least one edge of each face of T, and a state is a groundstate if it frustrates the smallest possible number of edges. We say that a state σ is *satisfying* if σ frustrates exactly one edge of each face of T. Hence, the set of the frustrated edges of any satisfying state is an intersecting set defined above, and we observe: The number of the satisfying states is at most twice the number of the intersecting sets of edges. Moreover, the converse also holds for planar triangulations: if we delete an intersecting set of edges from a planar triangulation, we get a bipartite graph and its bipartition defines a pair of satisfying states.

We finally note that a satisfying state does not need to exist, but if it exists, then the set of the satisfying states is the same as the set of the groundstates.

Summarizing, half the number of satisfying states is a lower bound to the number of intersecting sets. We can also formulate the result of Chudnovsky and Seymour by: *Each planar triangulation has an exponential degeneracy.* This motivates the problem we study as well as the (transfer matrix) method we use.

Given C_n a convex n-gon, a *triangulation of* C_n is a plane graph obtained from C_n by adding $n-3$ new edges so that C_n is its boundary (boundary of its outer face). We denote by $\Delta(\mathrm{C}_n)$ the set of all triangulations of C_n. An *almost-triangulation* is a plane graph so that all its inner faces are triangles. Note that if $n \geqslant 3$, then $\Delta(\mathrm{C}_n)$ is a subset of the set of almost-triangulations with $n-2$ inner faces. For T an almost-triangulation, we say that a state σ is *satisfying* if σ frustrates exactly one edge of each triangular face of T. We denote by s(T) the number of satisfying states of an almost-triangulation T. The main goal of this work is to show that the number of satisfying states of any triangulation of a convex n-gon is exponential in n.

Organization: We first recall, in Section 2, a known and simple bijection between triangulations of a convex n-gon and plane ternary trees with $n-2$ internal vertices. We then formally state the main results of this work. In Section 3 we give a constructive step by step procedure that given a plane ternary tree Γ with $n-2$ internal vertices, sequentially builds a triangulation T of a convex n-gon by repeatedly applying one of three different elementary operations. Finally, in Section 4 we interpret each elementary operation in terms of operations on matrices. Then, we apply the transfer matrix method to obtain, for each triangulation of a convex n-gon T, an expression for a matrix whose coordinates add up to the number of satisfying states of T. We then derive a closed formula for the number of satisfying states of a natural subclass of $\Delta(\mathrm{C}_n)$; the class of "triangle strips". Finally, we establish an exponential lower bound for the number of satisfying states of triangulations of a convex n-gon. Future research directions are discussed in Section 5.

2 Structure of the class of triangulations of a convex n-gon

Let T be a triangulation of a convex n-gon. Denote by F(T) the set of inner faces of T and let $\{\mathrm{I(T)}, \mathrm{O(T)}\}$ be the partition of F(T) such that $\Delta \in \mathrm{I(T)}$ if and only if no edge of Δ belongs to the boundary of T (i.e. to C_n). We henceforth refer to the elements of I(T) by *interior triangles of* T. Consider now the bijection Γ between $\Delta(\mathrm{C}_n)$ and the set of all plane ternary trees with $n-2$ internal vertices and n leaves that maps T to Γ_{T} so that:

(i) $\{\gamma_\Delta, \gamma_{\Delta'}\}$ is an edge of Γ_{T} if and only if Δ and Δ' are inner faces of T that share an edge, and

(ii) e is a leaf of Γ_{T} adjacent to γ_Δ if and only if e is an edge of C_n that belongs to Δ.

(See Figure 1 for an illustration of how Γ acts on an element of $\Delta(\mathrm{C}_n)$.) The bijection Γ induces another bijection, say γ, from the inner faces of T (i.e. F(T)), to the internal vertices of Γ_{T}. In particular, inner faces Δ and Δ' of T share an edge if and only if $\{\gamma_\Delta, \gamma_{\Delta'}\}$ is an edge of Γ_{T} which is not incident to a leaf. Hence, γ identifies interior triangles of T with internal vertices of Γ_{T} that are not adjacent to leaves.

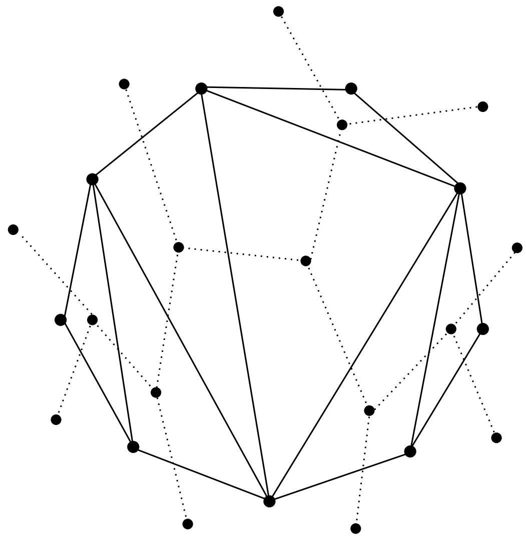

Figure 1: A triangulation of a convex 9-gon T and the associated tree Γ_{T}.

2.1 Main results

Say a triangulation of a convex n-gon T is a *strip of triangles* provided $|\mathrm{I}(\mathrm{T})| = 0$. Our first result is an exact formula for the number of satisfying states of any strip of triangles. Our second main contribution gives an exponential lower bound for the number of satisfying states of any triangulation of a convex n-gon. Specifically, denoting by F_k the k-th Fibonacci number and $\varphi = (1+\sqrt{5})/2 \approx 1.61803$ the golden ratio, we establish the following results:

Theorem 1 *If* T *is a triangulation of a convex* n*-gon with* $|\mathrm{I}(\mathrm{T})| = 0$*, then* $\mathrm{s}(\mathrm{T}) = 2F_{n+1}$.

Theorem 2 *If* T *is a triangulation of a convex* n*-gon, then* $\mathrm{s}(\mathrm{T}) \geqslant \varphi^2(\sqrt{\varphi})^n$*. Moreover,* $\sqrt{\varphi} \approx 1.27202$*.*

3 Construction of triangulations of a convex n-gon

In this section we discuss how to iteratively construct any triangulation of a convex n-gon. First, we introduce two basic operations whose repeated application allows one to build strips of triangles. Then, we describe a third operation which is crucial for recursively building triangulations with a non-empty set of interior triangles from triangulations with fewer interior triangles.

3.1 Basic operations

Let $\mathrm{T} = (\mathrm{V}, \mathrm{E})$ be a triangulation of a convex n-gon. We will often distinguish a boundary edge of T to which we shall refer as *bottom edge of* T and denote by $\lfloor \mathrm{T} \rfloor$.

We now define two elementary operations (see Figure 2 for an illustration):

Operation W
Input: $(\mathrm{T}, \lfloor \mathrm{T} \rfloor)$ where $\mathrm{T} \in \Delta(\mathrm{C}_n)$ and $\lfloor \mathrm{T} \rfloor = (\beta_1, \beta_2)$.
Output: $(\widehat{\mathrm{T}}, \lfloor \widehat{\mathrm{T}} \rfloor)$, where $\widehat{\mathrm{T}} \in \Delta(\mathrm{C}_{n+1})$ is a triangulation obtained from T by adding a new vertex $\widehat{\beta}_1$ to T and two new edges $\{\widehat{\beta}_1, \beta_1\}$ and $\{\widehat{\beta}_1, \beta_2\}$. Moreover, $\lfloor \widehat{\mathrm{T}} \rfloor = (\widehat{\beta}_1, \beta_2)$.

Operation Z
Input: $(\mathrm{T}, \lfloor \mathrm{T} \rfloor)$ where $\mathrm{T} \in \Delta(\mathrm{C}_n)$ and $\lfloor \mathrm{T} \rfloor = (\beta_1, \beta_2)$.
Output: $(\widehat{\mathrm{T}}, \lfloor \widehat{\mathrm{T}} \rfloor)$, where $\widehat{\mathrm{T}} \in \Delta(\mathrm{C}_{n+1})$ is a triangulation obtained from T by adding a new vertex $\widehat{\beta}_2$ to T and two new edges $\{\beta_1, \widehat{\beta}_2\}$ and $\{\widehat{\beta}_2, \beta_2\}$. Moreover, $\lfloor \widehat{\mathrm{T}} \rfloor = (\beta_1, \widehat{\beta}_2)$.

Henceforth, we also view operations W and Z as maps from inputs to outputs. Abusing terminology, we consider two nodes joined by an edge to be a *degenerate triangulation* whose bottom edge is its unique edge. Let T_0 be a degenerate triangulation. Say that $\lfloor \mathrm{T}_0 \rfloor$ is the *top edge of* T, denoted $\lceil \mathrm{T} \rceil$ (see Figure 2), if there is a sequence $\mathrm{R}_1, \ldots, \mathrm{R}_l \in \{\mathrm{W}, \mathrm{Z}\}$ such that $(\mathrm{T}, \lfloor \mathrm{T} \rfloor)$ is obtained by evaluating $\mathrm{R}_l \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1$ at $(\mathrm{T}_0, \lfloor \mathrm{T}_0 \rfloor)$. When bottom edges are clear from context, we shall simply write

$$\mathrm{T} = \mathrm{R}_l \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1(\mathrm{T}_0)\,.$$

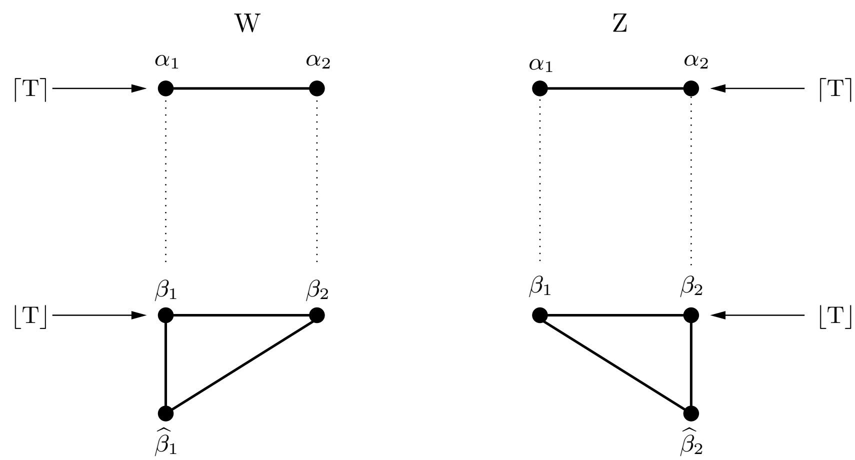

Figure 2: An arbitrary strip of triangles T with $\lceil \mathrm{T} \rceil = (\alpha_1, \alpha_2)$ and $\lfloor \mathrm{T} \rfloor = (\beta_1, \beta_2)$. Operations W and Z evaluated at $(\mathrm{T}, \lfloor \mathrm{T} \rfloor)$.

3.2 The $|\mathrm{I}(\mathrm{T})| = 0$ case

Our goal in this section is to show that any triangulation of a convex n-gon with no interior triangles can be obtained by sequentially applying basic operations of type W and Z starting from a degenerate triangulation.

Let T be a triangulation such that $|\mathrm{I}(\mathrm{T})| = 0$. Note that each internal vertex of Γ_{T} is adjacent to at least one leaf. Hence, Γ_{T} has two internal vertices each one adjacent to exactly two leaves, and $n-4$ internal vertices adjacent to exactly one leaf. This implies that Γ_{T} is made up of a path $P = \gamma_{\Delta^1} \dots \gamma_{\Delta^{n-2}}$ with two leaves connected to each γ_{Δ^1} and $\gamma_{\Delta^{n-2}}$, and one leaf connected to each internal vertex of the path P (see Figure 3). To obtain T from Γ_{T} we choose one of the two endnodes of the path (say γ_{Δ^1}) and sequentially add the triangles $\Delta^1, \dots, \Delta^{n-2}$ one by one, according to the bijection γ, starting from γ_{Δ^1} and following the trajectory of the path P. Consequently, we can construct T from a pair of vertices (α_1, α_2) of Δ^1 by applying a sequence of $n-2$ operations $\mathrm{R}_1, \mathrm{R}_2, \dots, \mathrm{R}_{n-2} \in \{\mathrm{W}, \mathrm{Z}\}$, where the choice of each operation depends on the structure of Γ_{T}. For example, for the triangulation in Figure 3, provided $\lceil \mathrm{T} \rceil = (\alpha_1, \alpha_2)$ and $\lfloor \mathrm{T} \rfloor = (\beta_1, \beta_2)$, we have that $\mathrm{R}_1 = \mathrm{W}$, $\mathrm{R}_2 = \mathrm{Z}$, $\mathrm{R}_3 = \mathrm{Z}$, and so on and so forth.

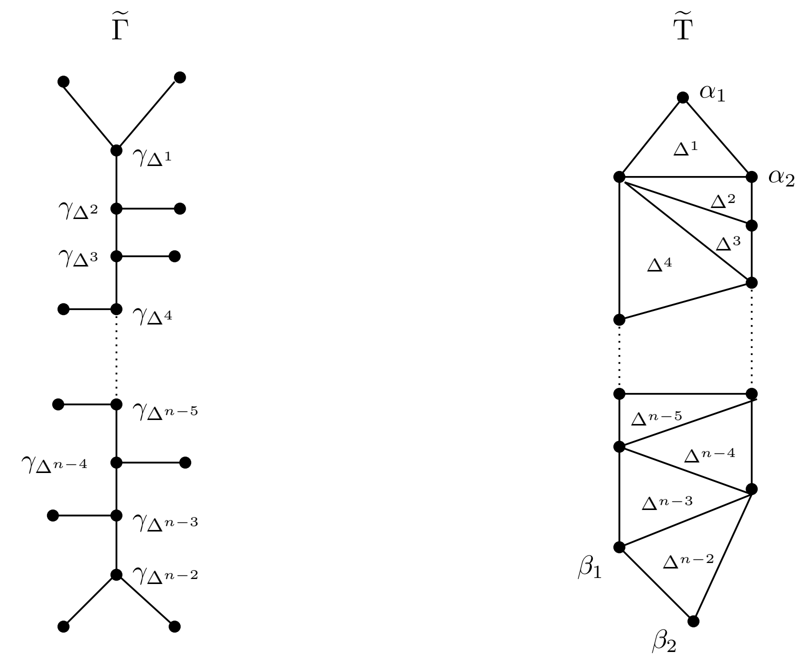

Figure 3: A tree $\widetilde{\Gamma}$ in the range of bijection Γ and construction of triangulation $\widetilde{\mathrm{T}}$ such that $\Gamma_{\widetilde{\mathrm{T}}} = \widetilde{\Gamma}$.

The next result summarizes the conclusion of the previous discussion.

Lemma 3 *For any* $\mathrm{T} \in \Delta(\mathrm{C}_n)$ *it holds that* $|\mathrm{I}(\mathrm{T})| = 0$ *if and only if there is a degenerate triangulation* T_0 *and basic operations* $\mathrm{R}_1, \mathrm{R}_2, \dots, \mathrm{R}_{n-2} \in \{\mathrm{W}, \mathrm{Z}\}$ *such that*

$$\mathrm{T} = \mathrm{R}_{n-2} \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1(\mathrm{T}_0)\,.$$

In fact, there are non–negative integers $w_1, \dots, w_m, z_1, \dots, z_m$ *adding up to* $n-2$ *such that* $w_j \geqslant 1$ *for* $j \neq 1$*,* $z_j \geqslant 1$ *for* $j \neq m$*, and*

$$\mathrm{T} = \mathrm{Z}^{z_m} \circ \mathrm{W}^{w_m} \circ \cdots \circ \mathrm{Z}^{z_2} \circ \mathrm{W}^{w_2} \circ \mathrm{Z}^{z_1} \circ \mathrm{W}^{w_1}(\mathrm{T}_0)\,.$$

3.3 The $|\mathrm{I}(\mathrm{T})| \geqslant 1$ case

We now consider the following additional basic operation (see Figure 4 for an illustration):

Operation $\bullet$
Input: $(\mathrm{T}_i, \lfloor \mathrm{T}_i \rfloor)$ where $\mathrm{T}_i \in \Delta(\mathrm{C}_{n_i})$, $i \in \{1,2\}$ and $\lfloor \mathrm{T}_i \rfloor = (\beta_1^i, \beta_2^i)$.
Output: $(\mathrm{T}, \lfloor \mathrm{T} \rfloor)$, where $\mathrm{T} \in \Delta(\mathrm{C}_{n_1+n_2-1})$ is a triangulation obtained from T_1 and T_2 by identifying β_1^2 with β_2^1 and adding the edge $\{\beta_1^1, \beta_2^2\}$. Moreover, $\lfloor \mathrm{T} \rfloor = (\beta_1^1, \beta_2^2)$.

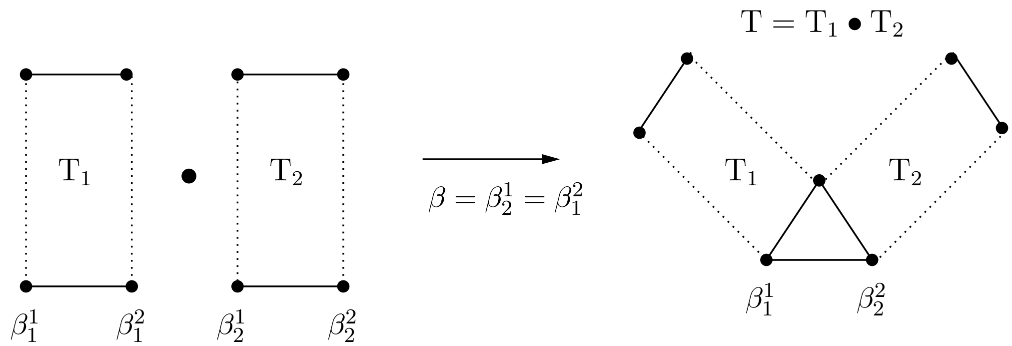

Figure 4: Building an interior triangle by means of operation $\bullet$.

Assume T is such that $|\mathrm{I}(\widetilde{\mathrm{T}})| = 1$. In particular, let $\mathrm{I}(\mathrm{T}) = \{\Delta\}$. Clearly, the tree Γ_T contains exactly one internal vertex that is not adjacent to a leaf. Hence, in Γ_T there must be three internal vertices each of them adjacent to two leaves, and $n-6$ internal vertices adjacent to exactly one leaf. Thus, we can identify in Γ_T three paths $P_1 = \gamma_{\Delta_1^1} \ldots \gamma_{\Delta_{n_1}^1}$, $P_2 = \gamma_{\Delta_1^2} \ldots \gamma_{\Delta_{n_2}^2}$, and $P_3 = \gamma_{\Delta_{n_3}^3} \ldots \gamma_{\Delta_1^3}$ with end-vertices $\gamma_{\Delta_{n_1}^1} = \gamma_{\Delta_{n_2}^2} = \gamma_{\Delta_{n_3}^3} = \gamma_\Delta$, and such that: (1) $n_1 + n_2 + n_3 = n$ and $n_1, n_2, n_3 \geqslant 2$, (2) each $\gamma_{\Delta_1^j}$ with $j \in \{1,2,3\}$ is adjacent to two leaves of Γ_T, and (3) each $\gamma_{\Delta_{i_j}^j}$ with $j \in \{1,2,3\}$ and $i_j \in \{2, \ldots, n_j - 1\}$ is adjacent to a single leaf of Γ_T.

Given Γ_T, we can construct T by means of the following iterative step by step procedure:

1. For $i \in \{1,2\}$, add triangles $\Delta_1^i, \ldots, \Delta_{n_i-1}^i$ according to the bijection following the trajectory from $\gamma_{\Delta_1^i}$ to $\gamma_{\Delta_{n_i-1}^i}$ given by P_i, thus obtaining a triangulation T_i such that Γ_{T_i} is the minimal subtree of Γ_T containing $P_i \setminus \gamma_\Delta$. Moreover, note that $\mathrm{T}_i \in \Delta(\mathrm{C}_{n_i+1})$ is such that $|\mathrm{I}(\mathrm{T}_i)| = 0$, and that there is a degenerate triangulation $\mathrm{T}_{i,0}$ which is an edge of triangle Δ_1^i, and basic operations $\mathrm{R}_1^i, \ldots, \mathrm{R}_{n_i-1}^i \in \{\mathrm{W}, \mathrm{Z}\}$ such that
$$\mathrm{T}_i = \mathrm{R}_{n_i-1}^i \circ \ldots \circ \mathrm{R}_2^i \circ \mathrm{R}_1^i(\mathrm{T}_{i,0})\,.$$
Also, note that $\lfloor \mathrm{T}_i \rfloor$ is an edge of $\Delta_{n_i-1}^i$.

2. Apply operation $\bullet$ in order to construct $\widehat{\mathrm{T}} = \mathrm{T}_1 \bullet \mathrm{T}_2 \in \Delta(\mathrm{C}_{n_1+n_2+1})$. Note that $\Delta \in \mathrm{F}(\widehat{\mathrm{T}})$ and $\lfloor \widehat{\mathrm{T}} \rfloor$ is the unique edge of Δ which is in the boundary of $\widehat{\mathrm{T}}$.

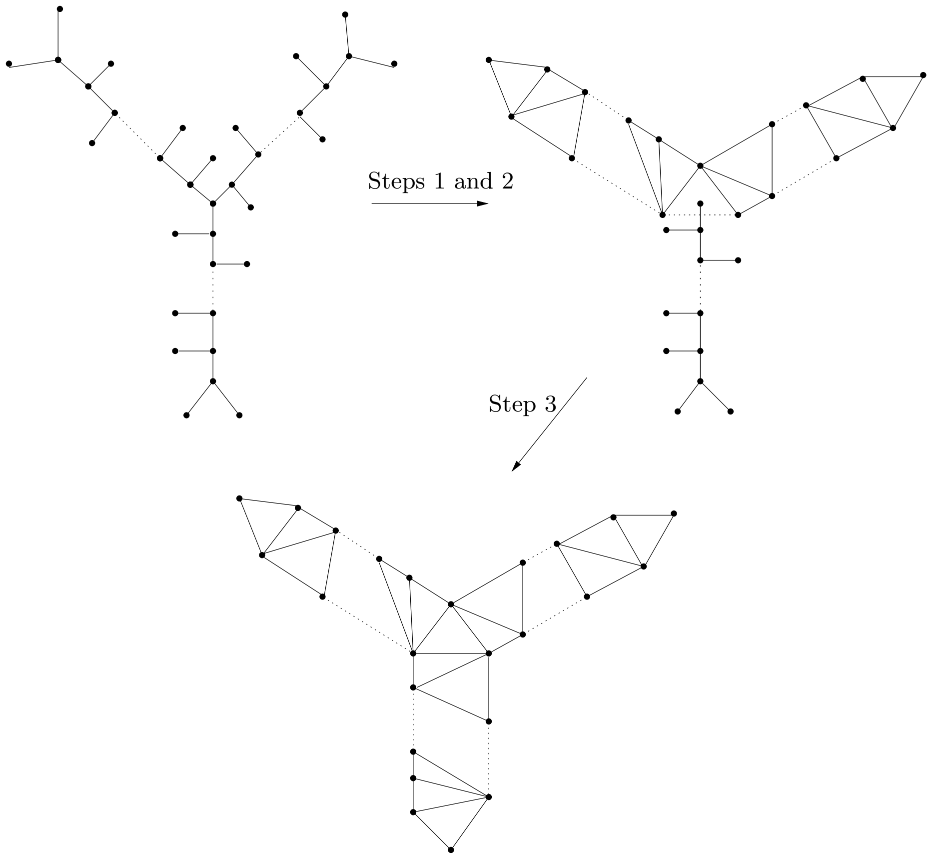

Figure 5: Sketch of construction of an arbitrary T with $|\mathrm{I}(\mathrm{T})| = 1$.

3. Finally, starting from $\widehat{\mathrm{T}}$ add triangles associated to vertices of the path P_3. This is done by performing a sequence of $n_3 - 1$ operations W and Z along $P_3 \setminus \gamma_\Delta$ starting from $(\widehat{\mathrm{T}}, \lfloor\widehat{\mathrm{T}}\rfloor)$. Given that $\widehat{\mathrm{T}} \in \Delta(\mathrm{C}_{n_1+n_2+1})$, we obtain $\mathrm{T} \in \Delta(\mathrm{C}_{n_1+n_2+n_3})$ (recall that $n_1 + n_2 + n_3 = n$).

We summarize the previous discussion as follows:

Lemma 4 *Let* T *be a triangulation of a convex n-gon such that* $|\mathrm{I}(\mathrm{T})| = 1$. *For some* $n_1, n_2, n_3 \geqslant 2$ *such that* $n_1+n_2+n_3 = n$, *there are triangulations* T_1 *and* T_2 *of convex* (n_1+1) *and* (n_2+1)*-gons such that* $|\mathrm{I}(\mathrm{T}_1)| = |\mathrm{I}(\mathrm{T}_2)| = 0$, *and basic operations* $\mathrm{R}_1, \ldots, \mathrm{R}_{n_3-1} \in \{\mathrm{W}, \mathrm{Z}\}$ *such that*

$$\mathrm{T} = \mathrm{R}_{n_3-1} \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1(\mathrm{T}_1 \bullet \mathrm{T}_2)\,.$$

Now, we state the main result concerning the recursive construction of an arbitrary triangulation of a convex n-gon that we will need.

Lemma 5 *Let* T *be a triangulation of a convex n-gon such that* $|\mathrm{I}(\mathrm{T})| = m \geqslant 2$. *Then, there are* $\widehat{n} \geqslant 5$, $\widetilde{n} \geqslant 3$ *and* $l \geqslant 1$ *such that* $\widetilde{n}+\widehat{n}+l-1 = n$, *and triangulations* $\widetilde{\mathrm{T}} \in \Delta(\mathrm{C}_{\widetilde{n}})$ *and* $\widehat{\mathrm{T}} \in \Delta(\mathrm{C}_{\widehat{n}})$ *satisfying:*

1. $|\mathrm{I}(\widetilde{\mathrm{T}})| = 0$*,*

2. $(\widehat{\mathrm{T}}, \lfloor\widehat{\mathrm{T}}\rfloor)$ *is either:*

   (a) The output of operation W *or* Z *and* $|\mathrm{I}(\widehat{\mathrm{T}})| = m-1$*, or*

   (b) The output of operation $\bullet$ *and* $|\mathrm{I}(\widehat{\mathrm{T}})| = m-2$*.*

3. There are basic operations $\mathrm{R}_1, \ldots, \mathrm{R}_l \in \{\mathrm{W}, \mathrm{Z}\}$ *for which* $\mathrm{T} = \mathrm{R}_l \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1(\widetilde{\mathrm{T}} \bullet \widehat{\mathrm{T}})$*.*

Proof: Observe that there must be an internal vertex of Γ_{T}, say γ_Δ, such that if $\Gamma_{\widehat{\mathrm{T}}}$, $\Gamma_{\widetilde{\mathrm{T}}}$ and $\Gamma_{\mathrm{T}_{l+2}}$ are the three sub-trees of Γ_{T} rooted in γ_Δ, then all internal vertices of $\Gamma_{\widetilde{\mathrm{T}}} \setminus \gamma_\Delta$ and $\Gamma_{\mathrm{T}_{l+2}} \setminus \gamma_\Delta$ are adjacent to at least one leaf. In particular, $|\mathrm{I}(\widetilde{\mathrm{T}})| = |\mathrm{I}(\mathrm{T}_{l+2})| = 0$, and condition 1 of the statement of the lemma is satisfied.

Let $\gamma_{\widehat{\Delta}}$ be the neighbor of γ_Δ in $\Gamma_{\widehat{\mathrm{T}}}$. Note that one of the following two situations must occur:

> **Case 1:** In $\Gamma_{\widehat{\mathrm{T}}} \setminus \gamma_\Delta$, the vertex $\gamma_{\widehat{\Delta}}$ is adjacent to a leaf (see Figure 6.(a)). In particular, $\Gamma_{\widehat{\mathrm{T}}}$ has exactly $m-1$ internal vertices which are not adjacent to any leaf, or
>
> **Case 2:** None of the neighbors of $\gamma_{\widehat{\Delta}}$ in $\Gamma_{\widehat{\mathrm{T}}} \setminus \gamma_\Delta$ are adjacent to leaves (see Figure 6.(b)). In particular, $\Gamma_{\widehat{\mathrm{T}}}$ has exactly $m-2$ internal vertices which are not adjacent to any leaf.

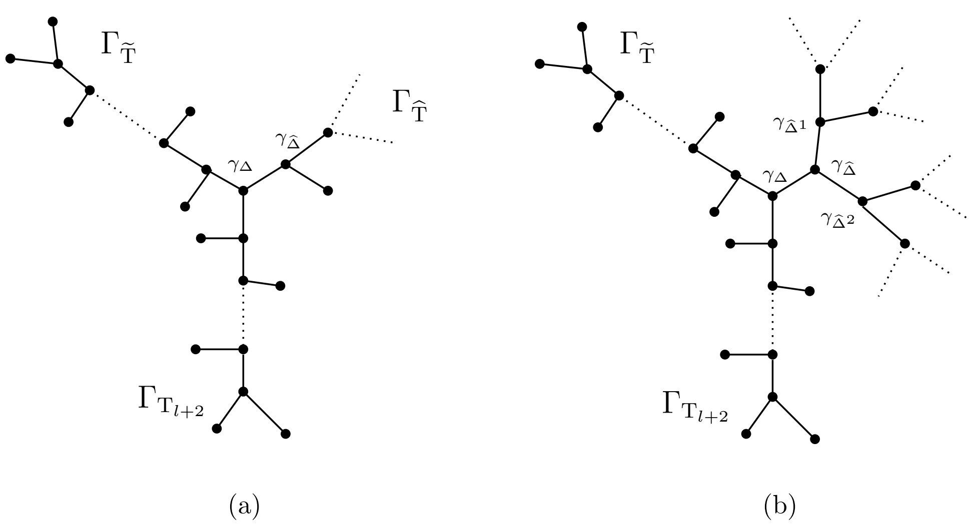

Figure 6: Structure of Γ_{T} depending on the one of subtree $\Gamma_{\widehat{\mathrm{T}}}$.

Assume that the first case holds. Recall that $|\mathrm{I}(\widehat{\mathrm{T}})| = m-1$. Let $\widehat{\mathrm{T}}_0$ be the triangulation such that $\Gamma_{\widehat{\mathrm{T}}_0}$ is the ternary tree obtained from $\Gamma_{\widehat{\mathrm{T}}} \setminus \gamma_\Delta$ by deleting the neighbor of $\gamma_{\widehat{\Delta}}$ which is a leaf. Let $\lfloor\widehat{\mathrm{T}}_0\rfloor$ be the edge of $\widehat{\mathrm{T}}_0$ corresponding to the unique edge incident to

$\gamma_{\widehat{\Delta}}$ in $\Gamma_{\widehat{\mathrm{T}}_0}$. Note that applying one basic operation of type W or Z we can obtain $(\widehat{\mathrm{T}}, \lfloor\widehat{\mathrm{T}}\rfloor)$ from $(\widehat{\mathrm{T}}_0, \lfloor\widehat{\mathrm{T}}_0\rfloor)$. Therefore, $(\widehat{\mathrm{T}}, \lfloor\widehat{\mathrm{T}}\rfloor)$ satisfies condition 2a of the statement of the lemma.

Suppose now that the second case holds. Recall that $|\mathrm{I}(\widehat{\mathrm{T}})| = m-2$. Let $\gamma_{\widehat{\Delta}^1}$ and $\gamma_{\widehat{\Delta}^2}$ be the vertices in $\Gamma_{\widehat{\mathrm{T}}} \setminus \gamma_{\Delta}$ that are neighbors of $\gamma_{\widehat{\Delta}}$. Let $\Gamma_{\widehat{\mathrm{T}},1}$ and $\Gamma_{\widehat{\mathrm{T}},2}$ be the trees obtained from $\Gamma_{\widehat{\mathrm{T}}\setminus\gamma_{\Delta}}$ by removing the trees rooted at $\gamma_{\widehat{\Delta}^2}$ and $\gamma_{\widehat{\Delta}^1}$, respectively. Consider $i \in \{1,2\}$ and note that $\Gamma_{\widehat{\mathrm{T}},i}$ is a ternary tree since by hypothesis neither $\gamma_{\widehat{\Delta}^1}$ nor $\gamma_{\widehat{\Delta}^2}$ are adjacent to leaves of $\Gamma_{\widehat{\mathrm{T}}} \setminus \gamma_{\Delta}$. Let $\widehat{\mathrm{T}}_i$ be the triangulation that is in bijective correspondence with $\Gamma_{\widehat{\mathrm{T}},i}$. Define $\lfloor\widehat{\mathrm{T}}_i\rfloor$ to be the edge of triangulation $\widehat{\mathrm{T}}_i$ which is in bijection with the edge $(\gamma_{\widehat{\Delta}}, \gamma_{\widehat{\Delta}^i})$ of $\Gamma_{\widehat{\mathrm{T}},i}$. Note that $(\widehat{\mathrm{T}}, \lfloor\widehat{\mathrm{T}}\rfloor)$ may be obtained as $\widehat{\mathrm{T}}_1 \bullet \widehat{\mathrm{T}}_2$. Therefore, $(\widehat{\mathrm{T}}, \lfloor\widehat{\mathrm{T}}\rfloor)$ satisfies condition 2b of the statement of the lemma.

To finish the construction of T it suffices to apply an appropriate sequence of l operations from the set $\{\mathrm{W}, \mathrm{Z}\}$ starting from $(\widetilde{\mathrm{T}} \bullet \widehat{\mathrm{T}}, \lfloor\widetilde{\mathrm{T}} \bullet \widehat{\mathrm{T}}\rfloor)$. The result follows. ■

4 Satisfying States

In this section we first present a technique, the so called Transfer Matrix Method. The technique is usually applied in situations where there is an underlying regular lattice, and gives formulas for its degeneracy. We adapt the technique to the context where instead of a lattice there is a triangulation of a convex n-gon T and use it to determine s(T). Then, we apply the method to derive an exact formula for the number of satisfying states of strips of triangles. Finally, we extend our arguments in order to establish an exponential lower bound for s(T) of any T triangulation of a convex n-gon.

4.1 Transfer matrices and satisfying matrix

Henceforth, the index of rows and columns of all 4×4 matrices we consider will be assumed to belong to $\{+,-\}^2$. Let T be a triangulation of a convex n-gon such that $|\mathrm{I}(\mathrm{T})| = 0$. From now on, let $\boldsymbol{1}$ denote the 4×1 vector all of whose coordinates are 1, i.e. $\boldsymbol{1} = (1,1,1,1)^t$. Our immediate goal is to obtain a matrix $\mathcal{M} = \mathcal{M}(\mathrm{T})$ of type 4×4 that satisfies the following two conditions:

> **Condition 1:** Columns and rows of $\mathcal{M}$ are indexed by spin-assignments of the top and bottom node pairs of T, respectively.
>
> **Condition 2:** For $\phi, \psi \in \{+,-\}^2$, the value $\mathcal{M}[\phi, \psi]$ is equal to the number of satisfying states of T if the spin-assignments of the top and bottom node pairs of T are ψ and ϕ, respectively.

Matrix $\mathcal{M}$ is called the *satisfying matrix* of T. It immediately follows that

$$\mathrm{s}(\mathrm{T}) = \boldsymbol{1}^t \cdot \mathcal{M} \cdot \boldsymbol{1}\,.$$

By Lemma 3, each triangulation $\mathrm{T} \in \Delta(\mathrm{C}_n)$ such that $|\mathrm{I}(\mathrm{T})| = 0$ may be constructed by applying a sequence of $n-2$ operations of type W or Z starting from T's top edge. To each operation $\mathrm{R} \in \{\mathrm{W}, \mathrm{Z}\}$ we associate a so called *transfer matrix* of type 4×4, say $\mathcal{R} \in \{\mathcal{W}, \mathcal{Z}\}$ such that:

- Columns of $\mathcal{R}$ are indexed by spin-assignments of the bottom node pair of T.
- Rows are indexed by spin-assignments of the bottom node pair of R(T).
- For $\phi, \psi \in \{+,-\}^2$, matrix $\mathcal{R}$ satisfies

$$\mathcal{R}[\phi,\psi] = \begin{cases} 1\,, & \text{if by setting the spin-assignments of the bottom node pairs of T and R(T) to } \psi \text{ and } \phi \text{ respectively, the state of the triangle created by the application of R is satisfying,} \\ 0\,, & \text{otherwise.} \end{cases}$$

Proposition 6 *Let $n \geqslant 3$ and T_0 be a degenerate triangulation. Let $\mathrm{T} \in \Delta(\mathrm{C}_n)$ be such that $\mathrm{T} = \mathrm{R}_{n-2} \circ \cdots \mathrm{R}_2 \circ \mathrm{R}_1(\mathrm{T}_0)$. If $\mathcal{R}_i$ denotes the transfer matrix associated to $\mathrm{R}_i \in \{\mathrm{W}, \mathrm{Z}\}$, then $\mathcal{M}(\mathrm{T}) = \mathcal{R}_{n-2} \cdots \mathcal{R}_2 \cdot \mathcal{R}_1$.*

Proof: We proceed by induction on n. If $n = 3$ we have that $\mathrm{T} = \mathrm{R}_1(\mathrm{T}_0)$ and the statement follows by definition of $\mathcal{M}(\mathrm{T})$ and $\mathcal{R}$. Assume $n > 3$. By inductive hypothesis the satisfying matrix of the triangulation $\widehat{\mathrm{T}} = \mathrm{R}_{n-3} \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1(\mathrm{T}_0) \in \Delta(\mathrm{C}_{n-1})$ is

$$\mathcal{M}(\widehat{\mathrm{T}}) = \mathcal{R}_{n-3} \cdot \mathcal{R}_{n-4} \cdots \mathcal{R}_2 \cdot \mathcal{R}_1\,.$$

The matrix $\mathcal{R}_{n-2} \cdot \mathcal{M}(\widehat{\mathrm{T}})$ satisfies Condition 1 since columns of the matrix $\mathcal{M}(\widehat{\mathrm{T}})$ are indexed by the spin-assignment of $\lceil \widehat{\mathrm{T}} \rceil = \lceil \mathrm{T} \rceil$ and the rows of matrix $\mathcal{R}_{n-2}$ by the spin-assignment of $\lfloor \mathrm{T} \rfloor$.

We still need to show that $\mathcal{R}_{n-2} \cdot \mathcal{M}(\widehat{\mathrm{T}})$ satisfies Condition 2. By inductive hypothesis, we have that $\mathcal{M}(\widehat{\mathrm{T}})[\chi,\psi]$ is the number of satisfying states of $\widehat{\mathrm{T}}$ if the spin-assignments for $\lfloor \widehat{\mathrm{T}} \rfloor$ and $\lceil \widehat{\mathrm{T}} \rceil$ are χ and ψ, respectively. By definition, $\mathcal{R}_{n-2}[\phi,\chi]$ may be 1 or 0 depending on whether or not the application of R_{n-2} to $(\widehat{\mathrm{T}}, \lfloor \widehat{\mathrm{T}} \rfloor)$ creates a triangle for which a satisfying state is obtained by setting the spin-assignments of $\lfloor \mathrm{T} \rfloor$ and of $\lfloor \widehat{\mathrm{T}} \rfloor$ equal to ϕ and χ, respectively. Therefore, $\mathcal{R}_{n-2}[\phi,\chi] = 1$ if and only if each satisfying state in $\widehat{\mathrm{T}}$ with spin-assignment χ and ψ for $\lfloor \widehat{\mathrm{T}} \rfloor$ and $\lceil \widehat{\mathrm{T}} \rceil$ respectively, is a satisfying state in T with spin-assignment ϕ and ψ for $\lfloor \mathrm{T} \rfloor$ and $\lceil \mathrm{T} \rceil$ respectively. By definition of $\mathcal{M}(\mathrm{T})$, it immediately follows that

$$\mathcal{M}(\mathrm{T})[\phi,\psi] = \sum_{\chi \in \{+,-\}^2} \mathcal{R}_{n-2}[\phi,\chi] \cdot \mathcal{M}(\widehat{\mathrm{T}})[\chi,\psi] = \Big(\mathcal{R}_{n-2} \cdot \mathcal{M}(\widehat{\mathrm{T}})\Big)[\phi,\psi]\,,$$

and that $\mathcal{M}(\mathrm{T}) = \mathcal{R}_{n-2} \cdot \mathcal{M}(\widehat{\mathrm{T}})$, thus concluding the inductive proof. ∎

4.2 Satisfying states of strips of triangles

We now apply the transfer matrix method to count the number of satisfying states in any triangulation of a convex n-gon T satisfying the condition $|\mathrm{I}(\mathrm{T})| = 0$. First, we observe that the matrices $\mathcal{W}$ and $\mathcal{Z}$ associated to operations W and Z, respectively, are given by:

$$\mathcal{W} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{Z} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\mathcal{W} = \Pi \cdot \mathcal{Z} \cdot \Pi$ where Π is the following permutation matrix:

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\Pi^{-1} = \Pi$, for any $k \geqslant 0$ we get that

$$\mathcal{W}^k \;=\; (\Pi \cdot \mathcal{Z} \cdot \Pi)^k \;=\; \Pi \cdot \mathcal{Z}^k \cdot \Pi\,. \tag{1}$$

Theorem 7 *Let* T_0 *be a degenerate triangulation,* $w_1, \ldots, w_m, z_1, \ldots, z_m$ *be a sequence of non–negative integers adding up to* $n-2$ *such that* $w_j \geqslant 1$ *for* $j \neq 1$ *and* $z_j \geqslant 1$ *for* $j \neq m$. *If* $\mathrm{T} = \mathrm{Z}^{z_m} \circ \mathrm{W}^{w_m} \circ \ldots \circ \mathrm{Z}^{z_1} \circ \mathrm{W}^{w_1}(\mathrm{T}_0)$ *and* $\mathcal{M} = \mathcal{M}(\mathrm{T})$*, then*

$$\mathcal{M} = \mathcal{Z}^{z_m} \cdot \Pi \cdot \mathcal{Z}^{w_m} \cdot \Pi \cdots \Pi \cdot \mathcal{Z}^{z_1} \cdot \Pi \cdot \mathcal{Z}^{w_1} \cdot \Pi\,.$$

Moreover, if F_k *denotes the* k*-th Fibonacci number, then*

$$\mathcal{M} \cdot \boldsymbol{1} = \begin{pmatrix} F_{n-1} \\ F_n \\ F_n \\ F_{n-1} \end{pmatrix}.$$

Proof: From Proposition 6 we have

$$\mathcal{M} = \mathcal{Z}^{z_m} \cdot \mathcal{W}^{w_m} \cdot \mathcal{Z}^{z_{m-1}} \cdot \mathcal{W}^{w_{m-1}} \cdots \mathcal{Z}^{z_2} \cdot \mathcal{W}^{w_2} \cdot \mathcal{Z}^{z_1} \cdot \mathcal{W}^{w_1}\,.$$

By (1), the first stated identity immediately follows.

Now, for the second part, let $k \geqslant 1$. Observe that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}.$$

It follows that,

$$\mathcal{Z}^k \cdot \boldsymbol{1} \;=\; \begin{pmatrix} F_{k-1} & F_k & 0 & 0 \\ F_k & F_{k+1} & 0 & 0 \\ 0 & 0 & F_{k+1} & F_k \\ 0 & 0 & F_k & F_{k-1} \end{pmatrix} \cdot \boldsymbol{1} \;=\; \begin{pmatrix} F_{k+1} \\ F_{k+2} \\ F_{k+2} \\ F_{k+1} \end{pmatrix}. \tag{2}$$

The first stated identity, the fact that $\Pi \cdot \mathcal{Z}^k \cdot \boldsymbol{1} = \mathcal{Z}^k \cdot \boldsymbol{1}$, and observing that $\Pi \cdot \boldsymbol{1} = \boldsymbol{1}$, we get that

$$\begin{aligned} \mathcal{M} \cdot \boldsymbol{1} &= \mathcal{Z}^{z_m} \cdot \Pi \cdot \mathcal{Z}^{w_m} \cdot \Pi \cdots \mathcal{Z}^{z_1} \cdot \Pi \cdot \mathcal{Z}^{w_1} \cdot \Pi \cdot \boldsymbol{1} \\ &= \mathcal{Z}^{z_m} \cdot \mathcal{Z}^{w_m} \cdots \mathcal{Z}^{z_1} \cdot \mathcal{Z}^{w_1} \cdot \boldsymbol{1}. \end{aligned}$$

Since $\sum_{i=1}^{m}(z_i + w_i) = n - 2$, the desired conclusion follows from (2). ■

Proof of Theorem 1: By hypothesis and Lemma 3 we have that for some degenerate triangulation T_0 there are non–negative integers $w_1, \ldots, w_m, z_1, \ldots, z_m$ adding up to $n-2$ such that $w_j \geqslant 1$ if $j \neq 1$, $z_j \geqslant 1$ if $j \neq m$, and

$$\mathrm{T} \;=\; \mathrm{Z}^{z_m} \circ \mathrm{W}^{w_m} \circ \cdots \circ \mathrm{Z}^{z_2} \circ \mathrm{W}^{w_2} \circ \mathrm{Z}^{z_1} \circ \mathrm{W}^{w_1}(\mathrm{T}_0).$$

By Theorem 7, we get that $\mathrm{s}(\mathrm{T}) = \boldsymbol{1}^t \cdot \mathcal{M}(\mathrm{T}) \cdot \boldsymbol{1} = 2(F_n + F_{n-1}) = 2F_{n+1}$. ■

We now obtain some intermediate results that we will need to prove Theorem 2: Let $\mathrm{T} \in \Delta(\mathrm{C}_n)$ and $\{\beta_1, \beta_2\}$ be an edge belonging to the boundary of T. The *satisfying vector* of T associated to node pair (β_1, β_2) denoted by $\boldsymbol{v}_{\mathrm{T}}((\beta_1, \beta_2))$ is a vector indexed by the spin-assignments $\{+,-\}^2$ of (β_1, β_2), so that $\boldsymbol{v}_{\mathrm{T}}((\beta_1, \beta_2))[\psi]$ is equal to the number of satisfying states of T if the spin-assignment of (β_1, β_2) is equal to ψ. For instance, by Theorem 7, for every triangulation T of a convex n-gon with no interior triangles,

$$\boldsymbol{v}_{\mathrm{T}}(\lfloor \mathrm{T} \rfloor) = \begin{pmatrix} F_{n-1} \\ F_n \\ F_n \\ F_{n-1} \end{pmatrix}.$$

Clearly, for every $\mathrm{T} \in \Delta(\mathrm{C}_n)$ we have that

$$\boldsymbol{v}_{\mathrm{T}}[++] = \boldsymbol{v}_{\mathrm{T}}[--], \qquad \boldsymbol{v}_{\mathrm{T}}[+-] = \boldsymbol{v}_{\mathrm{T}}[-+]. \tag{3}$$

Note that for edges $(\beta_1, \beta_2) \neq (\widehat{\beta}_1, \widehat{\beta}_2)$ belonging to the boundary of T, if

$$\boldsymbol{v}_{\mathrm{T}}((\beta_1, \beta_2)) = \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{y} \\ \mathrm{x} \end{pmatrix}, \qquad \boldsymbol{v}_{\mathrm{T}}((\widehat{\beta}_1, \widehat{\beta}_2)) = \begin{pmatrix} \widehat{\mathrm{x}} \\ \widehat{\mathrm{y}} \\ \widehat{\mathrm{y}} \\ \widehat{\mathrm{x}} \end{pmatrix},$$

then $2(\mathrm{x} + \mathrm{y}) = 2(\widehat{\mathrm{x}} + \widehat{\mathrm{y}})$, or equivalently $\mathrm{x} + \mathrm{y} = \widehat{\mathrm{x}} + \widehat{\mathrm{y}}$.

Proposition 8 *If* $\mathrm{R} \in \{\mathrm{W}, \mathrm{Z}\}$*,* $\widehat{\mathrm{T}} \in \Delta(\mathrm{C}_{\widehat{n}})$*, and* $\mathrm{T} = \mathrm{R}(\widehat{\mathrm{T}})$*, then*

$$\boldsymbol{v}_{\mathrm{T}}(\lfloor \mathrm{T} \rfloor) = \mathcal{R} \cdot \boldsymbol{v}_{\widehat{\mathrm{T}}}(\lfloor \widehat{\mathrm{T}} \rfloor) .$$

Proof: Implicit in the proof of Proposition 6. ■

We now define a useful operation on satisfying vectors. Let $\bullet$ be the binary operator over $\mathbb{N}^4$ defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_2 \, y_3 \\ x_1 y_2 + x_3 y_4 \\ x_4 y_3 + x_3 y_1 \\ x_3 y_2 \end{pmatrix} .$$

Proposition 9 *Let* $\mathrm{T}_1 \in \Delta(\mathrm{C}_{n_1})$ *and* $\mathrm{T}_2 \in \Delta(\mathrm{C}_{n_2})$ *be such that* $\lfloor \mathrm{T}_1 \rfloor = (\beta_1^1, \beta_2^1)$ *and* $\lfloor \mathrm{T}_2 \rfloor = (\beta_1^2, \beta_2^2)$*. Then,*

$$\boldsymbol{v}_{\mathrm{T}_1 \bullet \mathrm{T}_2}((\beta_1^1, \beta_2^2)) = \boldsymbol{v}_{\mathrm{T}_1}((\beta_1^1, \beta_2^1)) \bullet \boldsymbol{v}_{\mathrm{T}_2}((\beta_1^2, \beta_2^2)) .$$

Proof: To simplify the notation we henceforth denote $\boldsymbol{v}_{\mathrm{T}_1 \bullet \mathrm{T}_2}((\beta_1^1, \beta_2^2))$, $\boldsymbol{v}_{\mathrm{T}_1}((\beta_1^1, \beta_2^1))$ and $\boldsymbol{v}_{\mathrm{T}_2}((\beta_1^2, \beta_2^2))$ by $\boldsymbol{v}_{\beta_1^1 \beta_2^2}$, $\boldsymbol{v}_{\beta_1^1 \beta_2^1}$, and $\boldsymbol{v}_{\beta_1^2 \beta_2^2}$, respectively. For $i \in \{1, 2\}$, we know that $\boldsymbol{v}_{\beta_1^i \beta_2^i}[\psi]$ is equal to the number of satisfying states of T_i if $\psi \in \{+, -\}^2$ is the spin-assignment for (β_1^i, β_2^i). We consider the following cases depending on the spin-assignment of (β_1^1, β_2^2).

- Spin-assignment of (β_1^1, β_2^2) is ++: Since +++ is not a satisfying assignment for the triangle $(\beta_1^1, \beta, \beta_2^2)$ of T, if the spin-assignment of $\beta = \beta_1^2 = \beta_2^1$ is +, then the state of T is not satisfying. If the spin assignment of $(\beta_1^1, \beta, \beta_2^2)$ is +−+, each satisfying state of T_1 and T_2 (with spin-assignment for (β_1^1, β_2^1) equal to +− and spin-assignment for (β_1^2, β_2^2) equal to −+) is a satisfying state for T, and

$$\boldsymbol{v}_{\beta_1^1 \beta_2^2}[{+}{+}] = \boldsymbol{v}_{\beta_1^1 \beta_2^1}[{+}{-}] \boldsymbol{v}_{\beta_1^2 \beta_2^2}[{-}{+}] .$$

- Spin-assignment of (β_1^1, β_2^2) is +−: Note that the triangle $(\beta_1^1, \beta, \beta_2^2)$ with spin-assignment ++− fulfills the condition of satisfying state. Hence, each satisfying state of T_1 and T_2 (with spin-assignment for (β_1^1, β_2^1) equal to ++ and spin-assignment for (β_1^2, β_2^2) equal to +−) is a satisfying state for T. Analogously, if the spin-assignment of β is equal to −, each satisfying state of T_1 and T_2 (with spin-assignment for (β_1^1, β_2^1) equal to +− and spin-assignment for (β_1^2, β_2^2) equal to −−) is a satisfying state for T. It follows that

$$\boldsymbol{v}_{\beta_1^1 \beta_2^2}[{+}{-}] = \boldsymbol{v}_{\beta_1^1 \beta_2^1}[{+}{+}] \boldsymbol{v}_{\beta_1^2 \beta_2^2}[{+}{-}] + \boldsymbol{v}_{\beta_1^1 \beta_2^1}[{+}{-}] \boldsymbol{v}_{\beta_1^2 \beta_2^2}[{-}{-}] .$$

Finally, by a symmetry argument, we also have that $\boldsymbol{v}_{\beta_1^1 \beta_2^2}[{-}{-}] = \boldsymbol{v}_{\beta_1^1 \beta_2^1}[{-}{+}] \boldsymbol{v}_{\beta_1^2 \beta_2^2}[{+}{-}]$ and $\boldsymbol{v}_{\beta_1^1 \beta_2^2}[{-}{+}] = \boldsymbol{v}_{\beta_1^1 \beta_2^1}[{-}{+}] \boldsymbol{v}_{\beta_1^2 \beta_2^2}[{+}{+}] + \boldsymbol{v}_{\beta_1^1 \beta_2^1}[{-}{-}] \boldsymbol{v}_{\beta_1^2 \beta_2^2}[{-}{+}]$. ■

We now recall some basic well known facts about Fibonacci numbers. Let φ denote the golden ratio. If F_n denotes the n-th Fibonacci number, it is well known that $F_{n+1} = F_n + F_{n-2}$ for all $n \geqslant 1$, and that

$$F_n \;=\; \frac{\varphi^n - (-\frac{1}{\varphi})^n}{\sqrt{5}}.$$

It immediately follows that for all $n \geqslant 1$,

$$\varphi^{n-2} \;\leqslant\; F_n \;\leqslant \frac{1 + \left(\frac{1}{\varphi}\right)^2}{\sqrt{5}}\varphi^n \;\leqslant\; \varphi^n\,. \tag{4}$$

Lemma 10 *If* T *is a triangulation of a convex* n*-gon, then* $\varphi^{n-|\mathrm{I(T)}|} \geqslant \varphi^2(\sqrt{\varphi})^n$.

Proof: Since $|\mathrm{O(T)}| \geqslant |\mathrm{I(T)}| + 2$ and $|\mathrm{O(T)}| + |\mathrm{I(T)}| = n - 2$, we get that $|\mathrm{I(T)}| \leqslant (n/2) - 2$. The claimed result immediately follows. ■

Proof of of Theorem 2: We claim that for any triangulation of a convex n-gon T such that $|\mathrm{I(T)}| = m$ it holds that $\mathrm{s(T)} \geqslant \varphi^{n-m}$. To prove this claim we proceed by induction on m. If $m = 0$, by Theorem 1 we have that $\mathrm{s(T)} = 2F_{n+1}$. Using the lower bound in (4) we obtain $\mathrm{s(T)} \geqslant 2\varphi^{n-1} \geqslant \varphi^n$. If $m = 1$, by Lemma 4 we know that for some $n_1, n_2, n_3 \geqslant 2$ such that $n_1 + n_2 + n_3 = n$ there are triangulations $\mathrm{T}_1 \in \Delta(\mathrm{C}_{n_1+1})$ and $\mathrm{T}_2 \in \Delta(\mathrm{C}_{n_2+1})$ such that $|\mathrm{I}(\mathrm{T}_1)| = |\mathrm{I}(\mathrm{T}_2)| = 0$, and basic operations $\mathrm{R}_1, \ldots, \mathrm{R}_{n_3-1} \in \{\mathrm{W}, \mathrm{Z}\}$ such that

$$\mathrm{T} = \mathrm{R}_{n_3-1} \circ \cdots \circ \mathrm{R}_2 \circ \mathrm{R}_1(\mathrm{T}_1 \bullet \mathrm{T}_2)\,.$$

By Theorem 7, for $i \in \{1, 2\}$ we know that

$$\boldsymbol{v}_{\mathrm{T}_i}(\lfloor \mathrm{T}_i \rfloor) = \begin{pmatrix} F_{n_i} \\ F_{n_i+1} \\ F_{n_i+1} \\ F_{n_i} \end{pmatrix}.$$

Now, denote $\boldsymbol{v}_{\mathrm{T}_1 \bullet \mathrm{T}_2}(\lfloor \mathrm{T}_1 \bullet \mathrm{T}_2 \rfloor)$ by $\boldsymbol{v}$. Observe that Proposition 9 and the definition of $\bullet$ imply that

$$\boldsymbol{v} = \begin{pmatrix} F_{n_1} \\ F_{n_1+1} \\ F_{n_1+1} \\ F_{n_1} \end{pmatrix} \bullet \begin{pmatrix} F_{n_2} \\ F_{n_2+1} \\ F_{n_2+1} \\ F_{n_2} \end{pmatrix} = \begin{pmatrix} F_{n_1+1}F_{n_2+1} \\ F_{n_1}F_{n_2+1} + F_{n_1+1}F_{n_2} \\ F_{n_1}F_{n_2+1} + F_{n_1+1}F_{n_2} \\ F_{n_1+1}F_{n_2+1} \end{pmatrix}.$$

Repeated application of Proposition 8 yields that

$$\mathrm{s(T)} = \boldsymbol{1}^t \cdot \mathcal{R}_{n_3-1} \cdots \mathcal{R}_2 \cdot \mathcal{R}_1 \cdot \boldsymbol{v}\,.$$

By (3) and due to the block structure of $\mathcal{Z}$, we have that $\Pi \cdot \boldsymbol{v} = \boldsymbol{v}$ and $\Pi \cdot \mathcal{Z}^q \cdot \boldsymbol{v} = \mathcal{Z}^q \cdot \boldsymbol{v}$, for every $q \geqslant 0$. Therefore, since $\mathcal{W} = \Pi \cdot \mathcal{Z} \cdot \Pi$, the last displayed identity may be rewritten as $\mathrm{s}(\mathrm{T}) = \boldsymbol{1}^t \cdot \mathcal{Z}^{n_3-1} \cdot \boldsymbol{v}$. Hence,

$$\begin{aligned}\mathrm{s}(\mathrm{T}) &= \boldsymbol{1}^t \cdot \begin{pmatrix} F_{n_3-2} & F_{n_3-1} & 0 & 0 \\ F_{n_3-1} & F_{n_3} & 0 & 0 \\ 0 & 0 & F_{n_3} & F_{n_3-1} \\ 0 & 0 & F_{n_3-1} & F_{n_3-2} \end{pmatrix} \cdot \boldsymbol{v} \\ &= 2\left(F_{n_3}F_{n_1+1}F_{n_2+1} + F_{n_3+1}(F_{n_1}F_{n_2+1} + F_{n_1+1}F_{n_2})\right) .\end{aligned}$$

Since Fibonacci numbers satisfy the identity $F_{p+q} = F_pF_{q-1} + F_{p+1}F_q$, we get that

$$\begin{aligned}\mathrm{s}(\mathrm{T}) &= 2\left(F_{n_3}(F_{n_1+1}F_{n_2+1} + F_{n_1}F_{n_2+1} + F_{n_1+1}F_{n_2}) + F_{n_3-1}(F_{n_1}F_{n_2+1} + F_{n_1+1}F_{n_2})\right) \\ &= 2\left(F_{n_3}(F_{n_1+2}F_{n_2+1} + F_{n_1+1}F_{n_2}) + F_{n_3-1}(F_{n_1+2}F_{n_2} + F_{n_1+1}F_{n_2-1} - F_{n_1-1}F_{n_2-1})\right) \\ &= 2(F_{n_3}F_{n_1+n_2+2} + F_{n_3-1}(F_{n_1+n_2+1} - F_{n_1-1}F_{n_2-1})) \\ &= 2(F_{n_1+n_2+n_3+1} - F_{n_3-1}F_{n_1-1}F_{n_2-1}) .\end{aligned}$$

Since $n = n_1 + n_2 + n_3$, $2 > \varphi$ and $\varphi^2 - 1 = \varphi$, by (4) it follows that

$$\mathrm{s}(\mathrm{T}) \geqslant 2\varphi^{n_1+n_2+n_3-1}\left(1 - \varphi^{-2}\right) \geqslant \varphi^{n-1} .$$

Now, suppose the claim holds for every triangulation $\mathrm{T} \in \Delta(\mathrm{C}_n)$ such that $|\mathrm{I}(\mathrm{T})| < m$. Let $\mathrm{T} \in \Delta(\mathrm{C}_n)$ be such that $|\mathrm{I}(\mathrm{T})| = m$.

We know from Lemma 5 that there is a $\widetilde{\mathrm{T}} \in \Delta(\mathrm{C}_{\widetilde{n}})$ such that $|\mathrm{I}(\widetilde{\mathrm{T}})| = 0$, a $\widehat{\mathrm{T}} \in \Delta(\mathrm{C}_{\widehat{n}})$ satisfying condition 2 of Lemma 5, basic operations $\mathrm{R}_1, \ldots, \mathrm{R}_l \in \{\mathrm{W}, \mathrm{Z}\}$ where $l \geqslant 1$, and $n = \widehat{n} + \widetilde{n} + l - 1$ such that

$$\mathrm{T} = \mathrm{R}_l \circ \cdots \circ \mathrm{R}_1(\widetilde{\mathrm{T}} \bullet \widehat{\mathrm{T}}) .$$

By an argument similar to the one used to handle the $m = 1$ case, we have that

$$\mathrm{s}(\mathrm{T}) = \boldsymbol{1}^t \cdot \mathcal{R}_l \cdots \mathcal{R}_2 \cdot \mathcal{R}_1 \cdot \boldsymbol{v}_{\widetilde{\mathrm{T}}\bullet\widehat{\mathrm{T}}}(\lfloor \widetilde{\mathrm{T}} \bullet \widehat{\mathrm{T}} \rfloor) .$$

Since $\widetilde{\mathrm{T}} \in \Delta(\mathrm{C}_{\widetilde{n}})$ is such that $|\mathrm{I}(\widetilde{\mathrm{T}})| = 0$, by Theorem 7 we have that

$$\boldsymbol{v}_{\widetilde{\mathrm{T}}}(\lfloor \widetilde{\mathrm{T}} \rfloor) = \begin{pmatrix} F_{\widetilde{n}-1} \\ F_{\widetilde{n}} \\ F_{\widetilde{n}} \\ F_{\widetilde{n}-1} \end{pmatrix} .$$

Let $\widehat{x}$ and $\widehat{y}$ denote $\boldsymbol{v}_{\widehat{\mathrm{T}}}(\lfloor \widehat{\mathrm{T}} \rfloor)[{+}{+}]$ and $\boldsymbol{v}_{\widehat{\mathrm{T}}}(\lfloor \widehat{\mathrm{T}} \rfloor)[{+}{-}]$ respectively. Observe that (3) implies that $\boldsymbol{v}_{\widehat{\mathrm{T}}}(\lfloor \widehat{\mathrm{T}} \rfloor)[{-}{+}] = \widehat{y}$ and $\boldsymbol{v}_{\widehat{\mathrm{T}}}(\lfloor \widehat{\mathrm{T}} \rfloor)[{-}{-}] = \widehat{x}$. Hence, by Proposition 9,

$$\boldsymbol{v}_{\widetilde{\mathrm{T}}\bullet\widehat{\mathrm{T}}}(\lfloor \widetilde{\mathrm{T}} \bullet \widehat{\mathrm{T}} \rfloor) = \begin{pmatrix} F_{\widetilde{n}-1} \\ F_{\widetilde{n}} \\ F_{\widetilde{n}} \\ F_{\widetilde{n}-1} \end{pmatrix} \bullet \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{y} \\ \widehat{x} \end{pmatrix} = \begin{pmatrix} \widehat{y}F_{\widetilde{n}} \\ \widehat{x}F_{\widetilde{n}} + \widehat{y}F_{\widetilde{n}-1} \\ \widehat{x}F_{\widetilde{n}} + \widehat{y}F_{\widetilde{n}-1} \\ \widehat{y}F_{\widetilde{n}} \end{pmatrix} .$$

Denoting $\boldsymbol{v} = \boldsymbol{v}_{\widetilde{\mathrm{T}}\bullet\widehat{\mathrm{T}}}(\lfloor\widetilde{\mathrm{T}} \bullet \widehat{T}\rfloor)$ we again observe that (3) implies that $\Pi \cdot \boldsymbol{v} = \boldsymbol{v}$ and $\Pi \cdot \mathcal{Z}^q \cdot \boldsymbol{v} = \mathcal{Z}^q \cdot \boldsymbol{v}$ for all $q \geqslant 0$. Putting everything together we conclude that

$$\begin{aligned} \mathrm{s}(\mathrm{T}) &= \boldsymbol{1}^t \cdot \mathcal{Z}^l \cdot \begin{pmatrix} \widehat{y}F_{\widetilde{n}} \\ \widehat{x}F_{\widetilde{n}} + \widehat{y}F_{\widetilde{n}-1} \\ \widehat{x}F_{\widetilde{n}} + \widehat{y}F_{\widetilde{n}-1} \\ \widehat{y}F_{\widetilde{n}} \end{pmatrix} \\ &= 2(\widehat{x}F_{l+2}F_{\widetilde{n}} + \widehat{y}(F_{l+1}F_{\widetilde{n}} + F_{l+2}F_{\widetilde{n}-1}))\,. \end{aligned}$$

The lower bound for Fibonacci numbers given in (4) and the fact that $2 > \varphi$ imply that

$$\begin{aligned} \mathrm{s}(\mathrm{T}) &\geqslant 2\left(\widehat{x}\varphi^{l+\widetilde{n}-2} + 2\widehat{y}\varphi^{l+\widetilde{n}-3}\right) \\ &\geqslant 2(\widehat{x} + \widehat{y})\varphi^{l+\widetilde{n}-2}\,. \end{aligned}$$

Recalling that $\mathrm{s}(\widehat{\mathrm{T}}) = 2(\widehat{x}+\widehat{y})$ and observing that conditions 1 and 2 of Lemma 5 guarantee that $|\mathrm{I}(\widehat{\mathrm{T}})|$ is equal to $m-1$ or $m-2$, from the inductive hypothesis we obtain that $\mathrm{s}(\widehat{\mathrm{T}}) \geqslant \varphi^{\widehat{n}-(m-1)}$. It follows that $\mathrm{s}(\mathrm{T}) \geqslant \varphi^{\widehat{n}+\widetilde{n}+l-2-(m-1)} = \varphi^{n-m}$. This concludes the inductive prove of the claim. Lemma 10 immediately implies the desired result. ■

5 Conclusion

We have established that the number of satisfying states of any triangulation of a convex n-gon is exponential in n. It would be of interest to generalize this result to more general triangulations. Two natural cases to address next are triangulations that are embedable over low genus surfaces and k-trees.

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