H-Decompositions of r-graphs when H is an r-graph with exactly 2 edges

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Abstract

Given two r-graphs G and H, an H-*decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H. The minimum number of parts in an H-decomposition of G is denoted by $\phi_H^r(G)$. By a 2-*edge-decomposition* of an r-graph we mean an H-decomposition for any fixed r-graph H with exactly 2 edges. In the special case where the two edges of H intersect in exactly $1, 2$ or $r - 1$ vertices these 2-edge-decompositions will be called *bowtie*, *domino* and *kite* respectively. The value of the function $\phi_H^r(n)$ will be obtained for bowtie, domino and kite decompositons of r-graphs.

1 Introduction

An *hypergraph* is a (finite) set $V = V(G)$, called the *vertices* of G together with a set $E = E(G)$ of non-empty subsets (of any cardinality) of V, called the *hyperedges* or *edges*. When all the edges of an hypergraph are distinct we say that the hypergraph is *simple*. If in addition all the edges have the same cardinality $r \geqslant 2$ then G is said to be a *uniform hypergraph* or an *r-graph*. Thus, graphs are special hypergraphs. The number of vertices of an hypergraph is its *order* and is denoted by $v(G)$. The number of edges in an hypergraph is its *size* and is denoted by $e(G)$. The *complete r-graph* on n vertices is the hypergraph that consists of all r-subsets of V and it will be denoted by K_n^r. We denote by $[n]$ the set of the first n integers.

Given two r-graphs G and H, with $r \geqslant 2$, an H-*decomposition* of G is a partition of the edge set of G such that each part is either a single edge or forms a graph isomorphic to H. The minimum number of parts in an H-decomposition of G is denoted by $\phi_H^r(G)$.

For $r \geqslant 2$, we are interested in the value of the function

$$\phi_H^r(n) = \max\{\phi_H^r(G) \mid v(G) = n\},$$

which is the smallest number such that any r-graph G of order n admits an H-decomposition with at most $\phi_H^r(n)$ parts.

In the special case of $r = 2$, i.e., when we have graphs, the asymptotic value of the function $\phi_H^2(n)$, simply denoted by $\phi_H(n)$, was obtained by Pikhurko and Sousa [3] for all graphs H. However, the value of the function $\phi_H^r(n)$ is still open for $r \geqslant 3$. We will present some results about the value of the function $\phi_H^r(n)$ for some special cases of H.

By a 2-*edge-decomposition* of an r-graph we mean an H-decomposition for any fixed r-graph H with exactly 2 edges. In the special case where the two edges of H intersect in exactly $1, 2$ or $r - 1$ vertices these 2-edge-decompositions will be called *bowtie*, *domino* and *kite* respectively. In this paper the 2-edge-decomposition problem is solved asymptotically and we will also obtain the exact value $\phi_H^r(n)$ for some special cases of 2-edge-decompositions.

2   2-Edge-Decompositions of r-Graphs

We start this section with the following simple result.

Lemma 2.1. *For any non-empty r-graph H with m edges and any integer n, we have*

$$\phi_H^r(n) \leqslant \frac{1}{m}\binom{n}{r} + \frac{m-1}{m}\operatorname{ex}(n, r, H), \tag{2.1}$$

where $\operatorname{ex}(n, r, H)$ denotes the maximum number of edges that an r-graph with n vertices can have without containing a copy of H.

Proof. To prove (2.1) remove greedily one by one the edge-sets of H-subgraphs of a given r-graph G and then remove the remaining edges. The bound (2.1) follows as at most $\operatorname{ex}(n, r, H)$ parts are single edges. $\square$

Let G be an r-graph with vertex set $V(G)$ and edge set $E(G)$. We say that G is t-colorable if there is a map

$$c : V(G) \to \{1, \ldots, t\}$$

such that no edge $X \in E(G)$ in monochromatic.

Let H be a fixed r-colorable r-graph with m edges. Erdős [2] proved that

$$\operatorname{ex}(n, r, H) = o(n^r).$$

We have,

$$\phi_H^r(n) = \left(\frac{1}{m} + o(1)\right)\binom{n}{r}. \tag{2.2}$$

The upper bound in (2.2) follows directly from (2.1) and the result of Erdős [2] and the lower bound follows from $\phi_H^r(n) \geqslant \phi_H^r(K_n^r) \geqslant \frac{1}{m}\binom{n}{r}$.

Observe that the 2-edge-decomposition problem is asymptotically solved in (2.2) for r-colorable r-graphs.

In this section we will find the exact value $\phi_H^r(n)$ for some special cases of 2-edge-decompositions. Let H be a fixed r-graph with exactly 2 edges. We will start by proving that $\phi_H^r(n) \leqslant \phi_H^r(K_n^r)$. Thus, it suffices to study 2-edge-decompositions of complete r-graphs. Then, we will also prove that the trivial lower bound

$$\phi_H^r(K_n^r) \geqslant \left\lceil \frac{1}{2}\binom{n}{r} \right\rceil$$

is in fact optimal for bowtie, domino and kite decompositions of r-graphs, for some values of r.

Definition 2.2. *Let H be a fixed r-graph with 2 edges and $\mathcal{F}$ an H-decomposition of an r-graph G. We say that $\mathcal{F}$ is* optimal *if it contains exactly $\phi_H^r(G)$ elements.*

Theorem 2.3. *Let H be a fixed r-graph with 2 edges and G an r-graph with n vertices. Then,*

$$\phi_H^r(G) \leqslant \phi_H^r(K_n^r).$$

Proof. Let $\mathcal{F}$ be an optimal H-decomposition of K_n^r. We will extract from $\mathcal{F}$ an H-decomposition of G, $\mathcal{C}$, having at most $|\mathcal{F}|$ elements.

Let F be an element of $\mathcal{F}$. If F has two edges and both are edges of G then we add F to $\mathcal{C}$. If exactly on edge of F is an edge of G then add that single edge to $\mathcal{C}$. If the edges of F are not edges of G we discard F. After every element of $\mathcal{F}$ has been considered, the set $\mathcal{C}$ is clearly an H-decomposition of G and we have

$$\phi_H^r(G) \leqslant |\mathcal{C}| \leqslant |\mathcal{F}| = \phi_H^r(K_n^r),$$

as required. $\square$

Theorem 2.3 implies that, when H has exactly 2 edges, it suffices to study H-decompositions of complete r-graphs . In the sequel, and for the sake of simplicity, the edges of an r-graph will be written as $x_1 \cdots x_r$ instead of $\{x_1, \ldots, x_r\}$.

2.1 Bowtie Decomposition of the Complete r-Graph

Theorem 2.4. *Let $r = 3$ and $n \geqslant 5$ or $4 \leqslant r \leqslant \frac{1}{6}\sqrt{n}$. Then any optimal bowtie decomposition of the complete r-graph of order n has $\left\lceil \frac{1}{2}\binom{n}{r} \right\rceil$ elements.*

Before proving the theorem we will need to introduce the tools.

Theorem 2.5 (Dirac 1952, [1])**.** *Every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ has a hamiltonian cycle.*

Theorem 2.6 (Tutte 1947, [1])**.** *A graph G has a 1-factor (perfect matching) if and only if $odd(G - S) \leqslant |S|$ for all $S \subseteq V(G)$, where $odd(G - S)$ denotes the number of odd components of the graph $G - S$.*

Theorem 2.7 (Ray-Chaudhuri,Wilson 1975, [4])**.** *Let L be a set consisting of s non-negative integers and let G be an r-graph with n vertices. If $|A \cap B| \in L$ for any distinct edges $A, B \in E(G)$, then $e(G) \leqslant \binom{n}{s}$.*

Proof of Theorem 2.4. The lower bound follows immediately, thus it suffices to find a bowtie decomposition of K_n^r with $\left\lceil \frac{1}{2}\binom{n}{r} \right\rceil$ elements. Let $[n]$ be the vertex set of K_n^r. Let G_r be the graph with $V(G_r) = E(K_n^r)$ and $E(G_r) = \{uv \mid u, v \in E(K_n^r) \text{ and } |u \cap v| = 1\}$. Then, the theorem follows if we prove that G_r has a Hamiltonian cycle or a matching that saturates at least $v(G_r) - 1$ vertices.

Note that G_r is a connected $r\binom{n-r}{r-1}$-regular graph. For $r = 3$ and $n = 5, 6$ we can see by inspection that G_3 contains a perfect matching. If $r = 3$ and $n \in \{7, \dots, 11\}$ then Theorem 2.5 holds so G_3 has a Hamiltonian cycle. Suppose that $r = 3$ and $n \geqslant 12$ or $4 \leqslant r \leqslant \frac{1}{6}\sqrt{n}$. In this case we will see that Tutte's Theorem holds. If necessary remove one vertex from $V(G_r)$ so that we have $v(G_r)$ even.

Definition 2.8. *Let G be an r-graph. The* independence number *of G, denoted by $\alpha(G)$, is the maximum number of pairwise non adjacent vertices of G.*

Claim 1. $\alpha(G_r) \leqslant \binom{n}{r-1}$.

Proof. Let $\mathcal{A} \subseteq V(G_r)$ be an independent set. Then, $|A \cap B| \in \{0, 2, \dots, r-1\}$ for all distinct $A, B \in \mathcal{A}$. Considering the r-graph $([n], \mathcal{A})$, we have $|\mathcal{A}| \leqslant \binom{n}{r-1}$ by Theorem 2.7. $\qquad \square$

Claim 1 implies that for every $S \subseteq V(G_r)$ we have $odd(G_r - S) \leqslant \binom{n}{r-1}$. The following claim completes the proof of the theorem.

Claim 2. $|S| \geqslant odd(G_r - S)$, *for all* $S \subseteq V(G_r)$.

Proof. Let $S \subseteq V(G_r)$. If $S = \emptyset$ then there is nothing to prove since G_r is connected. Assume that $S \neq \emptyset$. Let $F_1, \dots, F_t$ be the components of $G_r - S$ and $v_1, \dots, v_t$ be fixed elements of $F_1, \dots, F_t$ respectively.

Assume first that $v_1, \dots, v_t$ are pairwise disjoint. Thus, $odd(G_r - S) \leqslant t \leqslant n/r$. Let $v_1 = x_1 \cdots x_r$ and $v_2 = y_1 \cdots y_r$. Then, for all $w_1, \dots, w_{r-2} \in [n] - \{v_1 \cup v_2\}$ and for all $i, j \in \{1, \dots, r\}$ the vertices $x_i y_j w_1 \cdots w_{r-2}$ are adjacent to both v_1 and v_2 and thus must be in S. Therefore $|S| \geqslant r^2\binom{n-2r}{r-2} - 1$ (recall that we might have removed one vertex from $V(G_r)$). Easy calculations show that $r^2\binom{n-2r}{r-2} - 1 \geqslant n/r \geqslant odd(G_r - S)$ as required.

Now suppose that there are i, j with $1 \leqslant i < j \leqslant t$ such that $|v_i \cap v_j| = k$ for some $k \in \{2, \dots, r-1\}$. Without loss of generality let $v_i = a_1 \cdots a_k x_{k+1} \cdots x_r$ and $v_j = a_1 \cdots a_k y_{k+1} \cdots y_r$. Then, for all $w_1 \dots, w_{r-1} \in [n] - \{v_i \cup v_j\}$, the vertices

$$a_m w_1 \cdots w_{r-1} \text{ and } x_p y_q w_1 \cdots w_{r-2},$$

where $1 \leqslant m \leqslant k$ and $p, q \in \{k+1, \ldots, r\}$, are adjacent to both v_i and v_j and thus must be in S. Therefore, $|S| \geqslant k\binom{n-2r+k}{r-1} + (r-k)^2\binom{n-2r+k}{r-2} - 1$.

It remains to prove that

$$k\binom{n-2r+k}{r-1} + (r-k)^2\binom{n-2r+k}{r-2} - 1 \geqslant \binom{n}{r-1}. \tag{2.3}$$

Let $r = 3$ and $n \geqslant 12$. We have $k = 2$ and (2.3) holds for $n \geqslant 13$. Let us consider the case $n = 12$. Then, there is $m \in [t] - \{i, j\}$ such that $v_m - \{v_i \cup v_j\} \neq \emptyset$. Otherwise, for all $m \in [t] - \{i, j\}$ we have $v_m \subseteq v_i \cup v_j = \{a_1, a_2, x_3, y_3\}$, but then we have only two choices for distinct v_m's which implies $t \leqslant 4$. Let $z \in v_m - \{v_i \cup v_j\}$. Then, for all $w \in [n] - \{v_i \cup v_j \cup v_m\}$ the edges $a_1 z w$ are adjacent to both v_i and v_m and hence are vertices of S. Note that these vertices have not been considered before. Since $|[n] - \{v_i \cup v_j \cup v_m\}| \geqslant n - 7$ and $n = 12$ we have $|S| \geqslant (n-4)^2 + (n-7) - 1 \geqslant \binom{n}{2} \geqslant t$.

Now assume that $4 \leqslant r \leqslant \frac{1}{6}\sqrt{n}$, that is, $n \geqslant 576$. We have,

$$k\binom{n-2r+k}{r-1} + (r-k)^2\binom{n-2r+k}{r-2} \geqslant 2\binom{n-2r+k}{r-1} \geqslant 2\binom{n-2r}{r-1}.$$

Therefore, to prove (2.3) it is enough to show that

$$2\binom{n-2r}{r-1} \geqslant \binom{n}{r-1}. \tag{2.4}$$

Easy calculations show that to prove (2.4) it is enough to show that

$$\frac{2(n-r+1)\cdots(n-3r+2)}{n\cdots(n-2r+1)} \geqslant 1. \tag{2.5}$$

Observe that the following inequalities hold.

$$\frac{2(n-r+1)\cdots(n-3r+2)}{n\cdots(n-2r+1)} \geqslant 2\left(\frac{n-3r+2}{n}\right)^{2r}$$

$$\geqslant 2\left(\frac{n-3r}{n}\right)^{2r}$$

$$\geqslant 2\left(1 - \frac{1}{2\sqrt{n}}\right)^{2r}$$

$$\geqslant 2\left(1 - \frac{1}{2\sqrt{n}}\right)^{\frac{1}{3}\sqrt{n}}, \quad \text{since } r \leqslant \frac{1}{6}\sqrt{n}.$$

Since $n \geqslant 576$ we have that

$$2\left(1 - \frac{1}{2\sqrt{n}}\right)^{\frac{1}{3}\sqrt{n}} \geqslant 1.$$

Therefore, inequality (2.5) holds, thus (2.3) is proved. $\qquad\square$

2.2 Domino Decomposition of the Complete 4-graph

Recall that a domino decomposition of an r-graph is a decomposition of its edge set into a fixed r-graph with two edges that intersect in exactly two vertices and single edges.

Theorem 2.9. *Any optimal domino decomposition of the complete 4-graph with $n \geqslant 6$ vertices has $\left\lceil \frac{1}{2}\binom{n}{4} \right\rceil$ elements.*

Proof. Clearly any optimal domino decomposition of K_n^4 has at least that many elements. We will now prove by induction that this lower bound is optimal. Let $n = 6$. Then

$$\{\{1234, 1256\}, \{1235, 2456\}, \{1245, 2346\}, \{1246, 3456\},$$
$$\{1345, 2356\}, \{1346, 2345\}, \{1236, 1456\}, \{1356\}\}$$

is a domino decomposition of K_6^4. Let $n \geqslant 7$ and assume that the result holds for $n-1$. Let $\mathcal{F}$ be an optimal domino decomposition of K_{n-1}^4. The result will follow if we show how to extend $\mathcal{F}$ into a domino decomposition of K_n^4 with the prescribed number of elements. Observe that $\mathcal{E} := E(K_n^4) - E(K_{n-1}^4) = \{e \cup \{n\} \mid e \in E(K_{n-1}^3)\}$ and a domino decomposition of $\mathcal{E}$ is the same as a bowtie decomposition of K_{n-1}^3. Let $\mathcal{B}$ be an optimal bowtie decomposition of K_{n-1}^3 and $\mathcal{F}'$ its extension to a domino decomposition of $\mathcal{E}$. If K_{n-1}^4 has an even number of edges then $\mathcal{F} \cup \mathcal{F}'$ is clearly a domino decomposition of K_n^4 with the required number of elements. Now suppose that K_{n-1}^4 has an odd number of edges. In this case $\mathcal{F}$ has an element that is a single edge, say $xyzw$. If K_{n-1}^3 has an even number of edges then we are done as before. So suppose that K_{n-1}^3 has an odd number of edges. Then $\mathcal{B}$ has an element that is a single edge. From the proof of Theorem 2.4 it follows that we can always choose $\mathcal{B}$ such that xya, for some $a \in [n-1] - \{x, y, z, w\}$ is the single edge in $\mathcal{B}$. Then, $\mathcal{F} \cup \mathcal{F}' \cup \{xyzw, xyan\}$ is a domino decomposition of K_n^4 with $\left\lceil \frac{1}{2}\binom{n}{4} \right\rceil$ elements. $\qquad \square$

2.3 Kite Decompositions of the complete 3-Graph and 4-Graph

Recall that a kite decomposition of an r-graph is a decomposition of its edge set into a fixed r-graph with two edges that intersect in exactly $r-1$ vertices and single edges.

Theorem 2.10. *Any optimal kite decomposition of the complete 3-graph with $n \geqslant 4$ vertices has $\left\lceil \frac{1}{2}\binom{n}{3} \right\rceil$ elements.*

Proof. The lower bound follows immediately, thus it suffices to find a kite decomposition of K_n^3 with $\left\lceil \frac{1}{2}\binom{n}{3} \right\rceil$ elements. This is done by induction on n. If $n = 4$ then $\{\{123, 234\}, \{124, 134\}\}$ is a kite decomposition of K_4^3. Let $n \geqslant 5$ and assume that the result holds for $n-1$. Let $\mathcal{F}$ be an optimal kite decomposition of K_{n-1}^3. The result follows by induction if we show how to extend $\mathcal{F}$ into a kite decomposition of K_n^3 with

the required number of elements. We first write $E(K_n^3) - E(K_{n-1}^3)$ as follows:

$$
\begin{array}{cccccc}
12n & 13n & 14n & \cdots & & 1(n-1)n \\
 & 23n & 24n & \cdots & & 2(n-1)n \\
 & & 34n & \cdots & & 3(n-1)n \\
 & & & \cdots & & \\
 & & & (n-3)(n-2)n & (n-3)(n-1)n & \\
 & & & & (n-2)(n-1)n &
\end{array}
$$

To proceed we have to consider two different cases.

Case I: Assume that K_{n-1}^3 has an even number of edges. Then all the elements of $\mathcal{F}$ have two edges, thus it suffices to pair the edges above. This can be done easily by pairing the edges in the same row, starting from left to right, so if a row has an odd number of elements then its element in the last column will be left unpaired. We end up this procedure by pairing the elements left in the last column, leaving exactly one single edge if and only if K_n^3 has an odd number of edges.

Case II: Assume that K_{n-1}^3 has an odd number of edges. Then $\mathcal{F}$ has an element that is a single edge, say xyz with $x < y < z$. We first pair this edge with the edge xyn and then proceed as in Case I. □

Theorem 2.11. *Any optimal kite decomposition of the complete 4-graph with $n \geqslant 5$ vertices has $\left\lceil \frac{1}{2}\binom{n}{4} \right\rceil$ elements.*

Proof. The lower bound follows immediately, thus it suffices to find a kite decomposition of K_n^4 with $\left\lceil \frac{1}{2}\binom{n}{4} \right\rceil$ elements. This is done by induction on n. If $n = 5$ then $\{\{1234, 1245\}, \{1235, 1345\}, \{2345\}\}$ is a kite decomposition of K_5^4. Let $n \geqslant 6$ and assume that the result holds for $n-1$. Let $\mathcal{F}$ be an optimal kite decomposition of K_{n-1}^4. The result follows by induction if we show how to extend $\mathcal{F}$ into a a kite decomposition of K_n^4. We first write $E(K_n^4) - E(K_{n-1}^4)$ in $n-3$ groups as follows:

$$
\begin{array}{lcccccc}
M_1: & 123n & 124n & 125n & \cdots & & 12(n-1)n \\
 & & 134n & 135n & \cdots & & 13(n-1)n \\
 & & & & \cdots & & \\
 & & & 1(n-3)(n-2)n & 1(n-3)(n-1)n & \\
 & & & & 1(n-2)(n-1)n & \\
 & & & & & \\
M_2: & & 234n & 235n & \cdots & & 23(n-1)n \\
 & & & 245n & \cdots & & 24(n-1)n \\
 & & & & \cdots & & \\
 & & & 2(n-3)(n-2)n & 2(n-3)(n-1)n & \\
 & & & & 2(n-2)(n-1)n & \\
 & & & & & \ddots & \\
M_{n-3}: & & & & & (n-3)(n-2)(n-1)n &
\end{array}
$$

To proceed we have to consider two different cases.

Case I: K_{n-1}^4 has an even number of edges. Then all the elements of $\mathcal{F}$ have two edges, thus it suffices to pair the edges above. This can be done using the following procedure.

> For $1 \leqslant i \leqslant n-3$, we consider the group M_i and pair the edges in the same row, starting from left to right, so if a row has an odd number of elements then its element in the last column will be left unpaired. We then pair the elements left in the last column starting from top to bottom.

After all groups have been considered all the edges left are of the form $x(n-2)(n-1)n$, with $x \in [n]$. We finish this procedure by pairing these edges, leaving exactly one single edge if and only if K_n^4 has an odd number of edges.

Case II: Assume that K_{n-1}^4 has an odd number of edges. Then $\mathcal{F}$ has an element that is a single edge, say $xyzw$ with $x < y < z < w$. We first pair this edge with the edge $xyzn$ and then proceed as in Case I. $\qquad\square$

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