Congruence classes of orientable 2-cell embeddings of bouquets of circles and dipoles*

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Abstract

Two 2-cell embeddings $\imath : X \to S$ and $\jmath : X \to S$ of a connected graph X into a closed orientable surface S are *congruent* if there are an orientation-preserving surface homeomorphism $h : S \to S$ and a graph automorphism γ of X such that $\imath h = \gamma \jmath$. Mull et al. [Proc. Amer. Math. Soc. 103(1988) 321–330] developed an approach for enumerating the congruence classes of 2-cell embeddings of a simple graph (without loops and multiple edges) into closed orientable surfaces and as an application, two formulae of such enumeration were given for complete graphs and wheel graphs. The approach was further developed by Mull [J. Graph Theory 30(1999) 77–90] to obtain a formula for enumerating the congruence classes of 2-cell embeddings of complete bipartite graphs into closed orientable surfaces. By considering automorphisms of a graph as permutations on its dart set, in this paper Mull et al.'s approach is generalized to any graph with loops or multiple edges, and by using this method we enumerate the congruence classes of 2-cell embeddings of a bouquet of circles and a dipole into closed orientable surfaces.

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1 Introduction

Let X be a finite connected graph allowing loops and multiple edges with vertex set $V(X)$ and edge set $E(X)$. An edge in $E(X)$ connecting vertices u and v (if the edge is a loop then $u = v$) gives rise to a pair of opposite *darts*, initiated at u and v respectively, and two darts are said to be *adjacent* if they are initiated at the same vertex. Denote by $D(X)$ the dart set of X. An automorphism of X is a permutation on $D(X)$ that preserves the adjacency of darts and maps any pair of opposite darts to a pair of opposite darts. All automorphisms of X form a permutation group on $D(X)$ which is called the *automorphism group* of X and denoted by $\mathrm{Aut}(X)$. Clearly, if the graph X is simple, that is if X has no loops or multiple edges, then $\mathrm{Aut}(X)$ acts faithfully on the vertex set $V(X)$ and hence can be considered as a permutation group on $V(X)$.

An *embedding* of X into a closed surface S is a homeomorphism $\imath : X \to S$ of X (as a one-dimensional simplicial complex in the 3-space $\mathbb{R}^3$) into S. If every component of $S - \imath(X)$ is a 2-cell, then $\imath$ is said to be a *2-cell embedding*. Basic terminologies for graph embeddings are referred to White [12], Gross and Tucker [5] or Biggs and White [2]. In this paper we are concerned with 2-cell embeddings of connected graphs into closed orientable surfaces and for convenience of statement, an embedding of a graph always means a 2-cell embedding of the connected graph into a closed orientable surface unless otherwise stated.

Two 2-cell embeddings $\imath : X \to S$ and $\jmath : X \to S$ of a graph X into a closed orientable surface S are *congruent* if there are an orientation-preserving surface homeomorphism $h : S \to S$ and a graph automorphism γ of X such that $\imath h = \gamma \jmath$. When we restrict γ as the identity in this definition, the two embeddings $\imath$ and $\jmath$ are called *equivalent*. In other words, the equivalence (congruence resp.) classes of embeddings of a graph X is the isomorphism classes of embeddings of a labeled (an unlabeled resp.) graph X. Enumerating unlabeled objects is technically more difficult than enumerating labeled ones. Likewise, enumerating the congruence classes of embeddings of a graph is more difficult than enumerating the equivalence classes of them.

Each equivalence class of embeddings of X into an orientable surface corresponds uniquely to a *combinatorial map* $\mathcal{M} = (X; \rho)$ (see Biggs and White [2, Chapter 5]), where ρ is a permutation on the dart set $D(X)$ such that each cycle of ρ gives the ordered list of darts encountered in an oriented trip on the surface around a vertex of X. The permutation ρ is called the *rotation* of the map $\mathcal{M}$. Conversely, a permutation ρ' on the dart set $D(X)$ whose orbits coincide with the sets of darts initiated at the same vertex, called a *rotation* of the graph X, gives rise to a map $\mathcal{M}' = (X; \rho')$ which corresponds to an equivalence class of embeddings of X into a closed orientable surface. Let ρ be a rotation of X. In the cycle decomposition of ρ, the cycle permuting the darts initiated at a vertex v is said to be the *local rotation* ρ_v at v. Clearly, ρ and ρ_v are permutations in $S_{D(X)}$, the symmetric group on $D(X)$, and $\rho = \prod_{v \in V(X)} \rho_v$. Denote by $R(X)$ the set of all rotations of X. Then for any $\rho \in R(X)$ and $h \in \mathrm{Aut}(X)$, ρh is the composition of permutations ρ and h on $D(X)$ in $S_{D(X)}$ (for convenience, all permutations and functions are composed from left to right).

By contrast, it is known [2] that two embeddings of X into an orientable surface are congruent if and only if their corresponding pairs $\mathcal{M}_1 = (X;\rho_1)$ and $\mathcal{M}_2 = (X;\rho_2)$ are isomorphic, that is, there is a graph automorphism $\phi \in \mathrm{Aut}(X)$ such that $\rho_1\phi = \phi\rho_2$. If $\rho_1 = \rho_2 = \rho$ then ϕ is called an *automorphism* of the map $\mathcal{M} = (X;\rho)$ and all automorphisms of the map $\mathcal{M} = (X;\rho)$ form the *automorphism group* of the map $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$. It is well-known that $\mathrm{Aut}(\mathcal{M})$ is semiregular on $D(X)$ (for example see [2, Chapter 5]), that is, the stabilizer of any arc of $D(X)$ in $\mathrm{Aut}(\mathcal{M})$ is the identity group. In particular, the map $\mathcal{M}$ is *regular* if $\mathrm{Aut}(\mathcal{M})$ is transitive on the dart set $D(X)$.

Mull et al. [11] enumerated the congruence classes of embeddings of the complete graphs and the wheel graphs into orientable surfaces, and Mull [10] did the same work for the complete bipartite graphs. Kwak and Lee [8] gave a similar but extended method for enumerating the congruence classes of embeddings of graphs with a given group of automorphisms into orientable and also into nonorientable surfaces.

As a distribution problem of the equivalence (or congruence) classes of embeddings of a graph into each surface, the genus distributions for the bouquet B_n and the dipole D_n into orientable surfaces were done in [4] and [7], respectively, and a similar work into nonorientable surfaces was done by Kwak and Shim [9].

For more results related to embeddings of connected graphs, see [2, 3, 5]. The enumerating approach in [11] was developed for simple graphs. In this paper it is generalized to any graph with loops or multiple edges. With this generalization, we give formulae for the numbers of congruence classes of embeddings of the bouquet B_n, the graph with one vertex and n loops, and the dipole D_n, the graph with two vertices and n multiple edges.

2 Enumerating formula

In this section, we generalize Mull et al.'s method for enumerating the congruence classes of embeddings of simple graphs to any graph with loops or multiple edges. This generalization can be easily proved by a similar method given in [11], and we omit the detailed proof. For a graph X, since the automorphism group $\mathrm{Aut}(X)$ is defined as a permutation group on the dart set of X, $\mathrm{Aut}(X)$ acts on its rotation set $R(X)$ by conjugacy action, that is, $\rho^\alpha = \alpha^{-1}\rho\alpha$ for all $\alpha \in \mathrm{Aut}(X)$ and $\rho \in R(X)$. Corresponding to Theorem 5.2.4(ii) of [2], we have the following proposition which is just Burnside's Lemma for the present context.

Proposition 2.1 *The number of congruence classes of embeddings of a connected graph X is*

$$|C(X)| = \frac{1}{|\mathrm{Aut}(X)|} \sum_{\alpha \in \mathrm{Aut}(X)} |\mathrm{Fix}(\alpha)|, \tag{1}$$

where $\mathrm{Fix}(\alpha) = \{\rho \in R(X) \mid \alpha^{-1}\rho\alpha = \rho\}$ *is the fixed set of* α.

Let $C\ell(\alpha_i)$, $1 \leqslant i \leqslant m$, denote the conjugacy classes of $\mathrm{Aut}(X)$ with α_i $(1 \leqslant i \leqslant m)$ as representatives. It is easy to see that $|\mathrm{Fix}(\alpha)| = |\mathrm{Fix}(\alpha_i)|$ for every $\alpha \in C\ell(\alpha_i)$. Thus, Eq. (1) can be further written as the following form.

$$|C(X)| = \frac{1}{|\mathrm{Aut}(X)|}\sum_{i=1}^{m}|\mathrm{Fix}(\alpha_i)||C\ell(\alpha_i)|. \qquad (2)$$

For $\beta \in \mathrm{Aut}(X)$ which fixes $v \in V(X)$, we define the *fixed set* $\mathrm{Fix}_v(\beta)$ *at* v of β to be the set of local rotations at v fixed by β under conjugacy action, that is,

$$\mathrm{Fix}_v(\beta) \;=\; \{\rho_v \mid \rho_v^\beta = \rho_v,\ \rho_v \text{ is a local rotation at } v\}.$$

Let $\alpha \in \mathrm{Aut}(X)$. Consider the natural action of α on the vertex set $V(X)$. Let $\ell(v)$ denote the length of the orbit of $\langle\alpha\rangle$ containing v acting on $V(X)$. Then $\mathrm{Fix}_v(\alpha^{\ell(v)})$ is well defined because $\alpha^{\ell(v)}$ fixes v. Denote by $N(v)$ the set of darts initiated at v, and by $\alpha^{\ell(v)}|_{N(v)}$ the restriction of $\alpha^{\ell(v)}$ on $N(v)$, respectively. Let $|N(v)| = n$ and ϕ the Euler function. A permutation α on a set is said to be *semiregular* if the cyclic group $\langle\alpha\rangle$ acts semiregularly on the set, that is, $\langle\alpha\rangle$ has the trivial stabilizer at each vertex. The following proposition corresponds to Theorems 4 and 5 of [11].

Proposition 2.2 *Let $\alpha \in \mathrm{Aut}(X)$ and let S be the set of representatives of the orbits of $\langle\alpha\rangle$ acting on $V(X)$. Then,*

(1) $|\mathrm{Fix}(\alpha)| = \prod_{v\in S}|\mathrm{Fix}_v(\alpha^{\ell(v)})|$,

(2) $|\mathrm{Fix}_v(\alpha^{\ell(v)})| = \begin{cases} \phi(d)(\frac{n}{d}-1)!d^{\frac{n}{d}-1} & \text{if } \alpha^{\ell(v)}|_{N(v)} \text{ is semiregular and has order } d,\\ 0 & \text{otherwise.}\end{cases}$

3 Embeddings of a bouquet of circles

In this section we enumerate the congruence classes of embeddings of B_n, the bouquet with n loops. For a real number x, denote by $\lfloor x\rfloor$ the largest integer that is not greater than x. For an edge e of B_n, let e^+ and e^- be the two opposite darts corresponding to e. Denote by

$$E(B_n) = \{e_1, e_2, \ldots, e_n\},$$
$$D(B_n) = \{e_1^+, e_1^-, \ldots, e_n^+, e_n^-\},$$

the edge set and the dart set of B_n, respectively. Let $1 \leqslant \ell \leqslant n$. To construct automorphisms of B_n, we divide the edge set $\{e_1, e_2, \ldots, e_r\}$ with $r = \ell\lfloor\frac{n}{\ell}\rfloor$ into $\lfloor\frac{n}{\ell}\rfloor$ blocks of size ℓ as follows:

$$\{e_1, e_2, \ldots, e_\ell\}, \{e_{\ell+1}, e_{\ell+2}, \ldots, e_{2\ell}\}, \ldots, \{e_{(\lfloor\frac{n}{\ell}\rfloor-1)\ell+1}, e_{(\lfloor\frac{n}{\ell}\rfloor-1)\ell+2}, \ldots, e_r\}$$

and we define

$$g_i^\ell = (e^+_{(i-1)\ell+1}\ e^+_{(i-1)\ell+2}\ \cdots\ e^+_{i\ell})(e^-_{(i-1)\ell+1}\ e^-_{(i-1)\ell+2}\ \cdots\ e^-_{i\ell}), \quad 1 \leqslant i \leqslant \lfloor\tfrac{n}{\ell}\rfloor,$$
$$h_i^\ell = (e^+_{(i-1)\ell+1}\ e^+_{(i-1)\ell+2}\ \cdots\ e^+_{i\ell}\ e^-_{(i-1)\ell+1}\ e^-_{(i-1)\ell+2}\ \cdots\ e^-_{i\ell}), \quad 1 \leqslant i \leqslant \lfloor\tfrac{n}{\ell}\rfloor$$

as permutations of the arcs whose underlying edges are in the i-th block, respectively. Then for each $1 \leqslant \ell \leqslant n$ and $1 \leqslant i \leqslant \lfloor \frac{n}{\ell} \rfloor$, g_i^ℓ and h_i^ℓ are automorphisms of B_n with orders ℓ and 2ℓ, respectively. Set

$$
\begin{array}{ll}
a_s = \prod_{i=1}^{\frac{n}{s}} g_i^s & \text{when } s \text{ is an odd divisor of } n, \\
b_{t,j} = \prod_{i=1}^{j} g_i^{2t} \cdot \prod_{i=2j+1}^{\frac{n}{t}} h_i^t, \quad 0 \leqslant j \leqslant \lfloor n/2t \rfloor & \text{when } t \text{ is a divisor of } n.
\end{array}
$$

In particular,

$$
\begin{aligned}
b_{t,0} &= (e_1^+ \ \cdots \ e_t^+ \ e_1^- \ \cdots \ e_t^-) \cdots (e_{n-t+1}^+ \ \cdots \ e_n^+ \ e_{n-t+1}^- \ \cdots \ e_n^-); \\
b_{t,\lfloor \frac{n}{2t} \rfloor} &= \begin{cases} (e_1^+ \ \cdots \ e_{2t}^+)(e_1^- \ \cdots \ e_{2t}^-) \cdots (e_{n-2t+1}^+ \ \cdots \ e_n^+)(e_{n-2t+1}^- \ \cdots \ e_n^-) & \text{if } \ 2t | n; \\ (e_1^+ \ \cdots \ e_{2t}^+)(e_1^- \ \cdots \ e_{2t}^-) \cdots (e_{n-3t+1}^+ \ \cdots \ e_{n-t}^+) & \\ \qquad \times (e_{n-3t+1}^- \ \cdots \ e_{n-t}^-)(e_{n-t+1}^+ \ \cdots \ e_n^+ \ e_{n-t+1}^- \ \cdots \ e_n^-) & \text{if } \ 2t \nmid n. \end{cases}
\end{aligned}
$$

Clearly, for an odd divisor s and any divisor t of n, a_s and $b_{t,j}$ $(0 \leqslant j \leqslant \lfloor \frac{n}{2t} \rfloor)$ are semiregular automorphisms of B_n of orders s and $2t$, respectively.

For example, if $n = 5$, then s and t are 1 or 5. In this case, all possible permutations $g_i^s, a_s, b_{t,j}$ on the set $D(B_5)$ are as follows.

$$
\begin{aligned}
& g_i^1 = 1 \ (1 \leqslant i \leqslant 5), \quad g_1^5 = (e_1^+ \ e_2^+ \ \cdots \ e_5^+)(e_1^- \ e_2^- \ \cdots \ e_5^-), \\
& a_1 = \textstyle\prod_{i=1}^5 g_i^1 = 1, \quad a_5 = (e_1^+ \ e_2^+ \ \cdots \ e_5^+)(e_1^- \ e_2^- \ \cdots \ e_5^-), \\
& b_{1,0} = \textstyle\prod_{i=1}^5 h_i^1 = (e_1^+ \ e_1^-)(e_2^+ \ e_2^-)(e_3^+ \ e_3^-)(e_4^+ \ e_4^-)(e_5^+ \ e_5^-), \\
& b_{1,1} = \textstyle\prod_{i=1}^1 g_i^2 \cdot \prod_{i=3}^5 h_i^1 = (e_1^+ \ e_2^+)(e_1^- \ e_2^-)(e_3^+ \ e_3^-)(e_4^+ \ e_4^-)(e_5^+ \ e_5^-), \\
& b_{1,2} = \textstyle\prod_{i=1}^2 g_i^2 \cdot \prod_{i=5}^5 h_i^1 = (e_1^+ \ e_2^+)(e_1^- \ e_2^-)(e_3^+ \ e_4^+)(e_3^- \ e_4^-)(e_5^+ \ e_5^-), \\
& b_{5,0} = (e_1^+ \ e_2^+ \ e_3^+ \ e_4^+ \ e_5^+ \ e_1^- \ e_2^- \ e_3^- \ e_4^- \ e_5^-).
\end{aligned}
$$

Note that these are all semiregular automorphisms of B_5.

Let $k_i = (e_i^+ \ e_i^-)$ $(1 \leqslant i \leqslant n)$ and $K = \langle k_1 \rangle \times \cdots \times \langle k_n \rangle$. Then $K \cong \mathbb{Z}_2^n$. Set $A = \mathrm{Aut}(B_n)$. Clearly, A induces an action on the edge set E. The kernel of this action is K and $A/K \cong S_n$. In fact, the automorphism group $\mathrm{Aut}(B_n)$ is the wreath product $\mathbb{Z}_2 \wr S_n$ and $|\mathrm{Aut}(B_n)| = 2^n n!$. For an element g of a group A, denote by $o(g)$ the order of g in A, by $C_A(g)$ the centralizer of g in A and by $C\ell(g)$ the conjugacy class of A containing g.

Let $n > 2$ and $\Omega = \{1, 2, \ldots, n\}$. Let S_n be the symmetric group on Ω. For a $g \in S_n$, the *cycle type* of g is the n-tuple whose k-th entry is the number of k-cycles presented in the disjoint cycle decomposition of g. By elementary group theory, two permutations in

S_n are conjugate if and only if they have the same cycle type. Furthermore, if $g \in S_n$ has cycle type $(t_1, t_2, \ldots, t_n)$ then the conjugacy class $C\ell(g)$ of S_n containing g has cardinality

$$|C\ell(g)| = \frac{n!}{\prod_{i=1}^{n} i^{t_i}(t_i)!}. \tag{3}$$

and the size of the centralizer of g in S_n is $n!/|C\ell(g)|$.

The following lemma describes the conjugacy class structure of semiregular elements of $\mathrm{Aut}(B_n)$, which is essential to enumerate the congruence classes of embeddings of a bouquet B_n of n circles.

Lemma 3.1 *Let $A = \mathrm{Aut}(B_n)$ and let g be a semiregular element in A. Then $o(g) \mid 2n$. If $o(g) = s$ is odd, then $g \in C\ell(a_s)$, and if $o(g) = 2t$ is even, then $g \in C\ell(b_{t,j})$ for some $0 \leqslant j \leqslant \lfloor \frac{n}{2t} \rfloor$. Furthermore,*

(1) *for any two odd divisors s_1, s_2 of n, $C\ell(a_{s_1}) = C\ell(a_{s_2})$ if and only if $s_1 = s_2$;*

(2) *for any two divisors t_1, t_2 of n, $C\ell(b_{t_1,j_1}) = C\ell(b_{t_2,j_2})$ if and only if $t_1 = t_2$ and $j_1 = j_2$ where $0 \leqslant j_1 \leqslant \lfloor \frac{n}{2t_1} \rfloor$ and $0 \leqslant j_2 \leqslant \lfloor \frac{n}{2t_2} \rfloor$;*

(3) $|C\ell(a_s)| = \dfrac{2^n n!}{(2s)^{\frac{n}{s}}(\frac{n}{s})!}$ *and* $|C\ell(b_{t,j})| = \dfrac{2^n n!}{2^j \cdot (2t)^{\frac{n-jt}{t}} \cdot j!(\frac{n-2jt}{t})!}$.

Proof. Let g have order p. Since g is semiregular on $D(B_n)$, one has $p \mid 2n$. First assume that each cycle in the disjoint cycle decomposition of g contains no opposite darts of an edge. Then gK is conjugate in A/K to $\prod_{i=1}^{\frac{n}{p}}(e_{(i-1)p+1} \ \cdots \ e_{ip})$ because $A/K \cong S_n$. Thus, g is conjugate in A to

$$\prod_{i=1}^{\frac{n}{p}}(e^+_{(i-1)p+1} \ e^+_{(i-1)p+2} \ \cdots \ e^+_{ip})(e^-_{(i-1)p+1} \ e^-_{(i-1)p+2} \ \cdots \ e^-_{ip}),$$

which is a_p when p is odd and $b_{\frac{p}{2},\frac{n}{p}}$ when p is even. Now assume that a cycle in the disjoint cycle decomposition of g contains the two opposite darts of an edge, say e^+ and e^-. Then there is an integer t such that $0 < t < o(g)$ and $(e^+)^{g^t} = e^-$. Thus, g^t fixes the edge e, forcing $(e^-)^{g^t} = e^+$. This means that g^{2t} fixes the dart e^+ and by the semiregularity of g, $g^{2t} = 1$, implying $o(g) \mid 2t$. Since $0 < t < o(g)$, one has $o(g) = 2t$. Note that $(e^+)^g$ and $(e^-)^g$ are opposite darts and $((e^+)^g)^{g^t} = (e^-)^g$. Then the cycle of g containing e^+ and e^- has the form $(e_{i_1}^{\delta_1} \ e_{i_2}^{\delta_2} \ \cdots e_{i_t}^{\delta_t} \ e_{i_1}^{\delta_1'} \ e_{i_2}^{\delta_2'} \ \cdots e_{i_t}^{\delta_t'})$, where $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n$, $\delta_j = \pm 1$ and $\delta_j \delta_j' = -1$ for each $1 \leqslant j \leqslant t$. The semiregularity of g implies that each cycle in the disjoint cycle decomposition of g has length $2t$. Let j be the number of cycles in the disjoint cycle decomposition of g which contains no opposite darts of an edge. Since $A/K \cong S_n$, gK is conjugate in A/K to

$$\prod_{i=1}^{j}(e_{2(i-1)t+1} \ \cdots \ e_{2it}) \cdot \prod_{i=2j+1}^{\frac{n}{t}} (e_{(i-1)t+1} \ \cdots \ e_{it})$$

and hence g is conjugate in A to

$$\prod_{i=1}^{j}(e^+_{2(i-1)t+1}\ \cdots\ e^+_{2it})(e^-_{2(i-1)t+1}\ \cdots\ e^-_{2it})\prod_{i=2j+1}^{\frac{n}{t}}(e^+_{(i-1)t+1}\ \cdots\ e^+_{it}\ e^-_{(i-1)t+1}\ \cdots\ e^-_{it}),$$

this is, x is conjugate in A to $b_{t,j}$.

For (1), let s_1 and s_2 be two odd divisors of n. Clearly, if $s_1 = s_2$, then $C\ell(a_{s_1}) = C\ell(a_{s_2})$. If $C\ell(a_{s_1}) = C\ell(a_{s_2})$, then a_{s_1} and a_{s_2} have the same order, implying $s_1 = s_2$.

For (2), let t_1, t_2 be two divisors of n. Similar argument as (1) gives that if $t_1 \neq t_2$ then $C\ell(b_{t,j_1}) \neq C\ell(b_{t,j_2})$. Let $t_1 = t_2 = t$ and $0 \leqslant j_1, j_2 \leqslant \lfloor\frac{n}{2t}\rfloor$. Clearly, if $j_1 = j_2$ then $C\ell(b_{t,j_1}) = C\ell(b_{t,j_2})$. If $C\ell(b_{t,j_1}) = C\ell(b_{t,j_2})$ then b_{t,j_1} and b_{t,j_2} are conjugate in A and hence the induced actions of b_{t,j_1} and b_{t,j_2} on E are conjugate in $A/K \cong S_n$. It follows that $j_1 = j_2$ because the induced action of b_{t,j_i} $(i = 1, 2)$ on E is a product of j_i disjoint $2t$-cycles and $\frac{n-2tj_i}{t}$ t-cycles.

To prove (3), we first prove the following fact.

Fact: Let t and s be divisors of n with s odd. Set $x = a_s$ or $b_{t,j}$, where $0 \leqslant j \leqslant \lfloor\frac{n}{2t}\rfloor$. If there exists a $k \in K$ such that $o(x) = o(xk)$ and xk is semiregular on $D(B_n)$, then xk is conjugate to x in K.

Assume that $o(x) = o(xk)$ and xk is semiregular on $D(B_n)$. Then xk and x have the same number of cycles in their disjoint cycle decompositions, which implies that k is a product of even k_i's in $K = \langle k_1\rangle \times \cdots \times \langle k_n\rangle$ because k_j is a 2-cycle for each $1 \leqslant j \leqslant n$. The lemma is clearly true for $k = 1$. Let $k = k_{i_1}k_{i_2}\cdots k_{i_{2r}}$ with $1 \leqslant i_1 < i_2 < \cdots < i_{2r} \leqslant n$.

Set $c_0 = (e^+_1\ e^+_2\ \cdots\ e^+_n)(e^-_1\ e^-_2\ \cdots\ e^-_n)$ and $c_1 = (e^+_1\ e^+_2\ \cdots\ e^+_n\ e^-_1\ e^-_2\ \cdots\ e^-_n)$. Assume that $x = c_0$ or c_1. For each $1 \leqslant j \leqslant r$, let $h_j = \prod_{m=i_{2j-1}}^{i_{2j}-1} k_m$. Then

$$x^{-1}h_jx = k_{i_{2j-1}}k_{i_{2j}}\cdot\prod_{m=i_{2j-1}}^{i_{2j}-1}k_m = k_{i_{2j-1}}k_{i_{2j}}h_j,$$

that is, $xk_{i_{2j-1}}k_{i_{2j}} = h_jxh_j^{-1} = h_j^{-1}xh_j$. Since $k = k_{i_1}k_{i_2}\cdots k_{i_{2r}}$, one has $xk = h^{-1}xh$, where $h = \prod_{j=1}^{r} h_j \in K$. Thus, xk and x are conjugate in K.

Now assume that $x \neq c_0,\ c_1$. For $1 \leqslant \ell \leqslant n$, let B_ℓ and $B_{n-\ell}$ be the bouquets with $V(B_\ell) = V(B_{n-\ell}) = V(B_n)$, $E(B_\ell) = \{e_1, \ldots, e_\ell\}$ and $E(B_{n-\ell}) = \{e_{\ell+1}, \ldots, e_n\}$. If $x = a_s = \prod_{i=1}^{\frac{n}{s}} g_i^s$ then $\frac{n}{s} > 1$ because $x \neq c_0$. Let $x_1 = g_1^s$ and $x_2 = \prod_{i=2}^{\frac{n}{s}} g_i^s$. Then, x_1 and x_2 are semiregular automorphisms of B_s and B_{n-s} respectively with $o(x_1) = o(x_2) = o(x) = s$. Let $x = b_{t,j} = \prod_{i=1}^{j} g_i^{2t}\cdot\prod_{i=2j+1}^{\frac{n}{t}} h_i^t$ for some $0 \leqslant j \leqslant \lfloor\frac{n}{2t}\rfloor$. If $j \geqslant 1$ let $x_1 = g_1^{2t}$ and $x_2 = \prod_{i=2}^{j} g_i^{2t}\cdot\prod_{i=2j+1}^{\frac{n}{t}} h_i^t$. Since $x \neq c_0$, one has $x_2 \neq 1$. Then $o(x_1) = o(x_2) = o(x) = 2t$, and x_1 and x_2 are semiregular automorphisms of the bouquets B_{2t} and B_{n-2t}, respectively. If $j = 0$ then $\frac{n}{t} > 1$ because $x \neq c_1$. Let $x_1 = h_1^t$ and $x_2 = \prod_{i=2}^{\frac{n}{t}} h_i^t$. Then, $o(x_1) = o(x_2) = o(x) = 2t$, and x_1 and x_2 are semiregular automorphisms of the bouquets B_t and B_{n-t}, respectively. Thus, for $x = a_s$ or $b_{t,j}$ $(0 \leqslant j \leqslant \lfloor\frac{n}{2t}\rfloor)$ there always exist some

$1 < m < n$ and semiregular automorphisms x_1 and x_2 of the bouquets B_m and B_{n-m} respectively such that $x = x_1x_2$ and $o(x_1) = o(x_2) = o(x)$.

Let $k = h_1h_2$ be such that $h_1 \in \langle k_1\rangle \times \cdots \times \langle k_m\rangle$ and $h_2 \in \langle k_{m+1}\rangle \times \cdots \times \langle k_n\rangle$. Since xk is a semiregular automorphism of B_n, x_1h_1 and x_2h_2 must be semiregular automorphisms of the bouquets B_m and B_{n-m} with the same order as x because $xk = (x_1h_1)(x_2h_2)$. By induction on n, there exist $h_1' \in \langle k_1\rangle \times \cdots \times \langle k_m\rangle$ and $h_2' \in \langle k_{m+1}\rangle \times \cdots \times \langle k_n\rangle$ such that $x_1h_1 = (h_1')^{-1}x_1h_1'$ and $x_2h_2 = (h_2')^{-1}x_2h_2'$. Let $k' = h_1'h_2'$. Then,

$$xk = (x_1h_1)(x_2h_2) = [(h_1')^{-1}x_1h_1'][(h_2')^{-1}x_2h_2'] = (h_1'h_2')^{-1}x_1x_2(h_1'h_2') = (k')^{-1}xk'.$$

This completes the proof of the Fact.

Now assume $g \in C_K(a_s) = C_A(a_s) \cap K$. Recall that $a_s = \prod_{i=1}^{\frac{n}{s}} g_i^s$ and $K = \langle k_1\rangle \times \cdots \times \langle k_n\rangle$ where $k_i = (e_i^+ \ e_i^-)$ for $1 \leqslant i \leqslant n$. Since $g \in K$, g commutes with g_i^s for each $1 \leqslant i \leqslant \frac{n}{s}$. It follows that

$$C_K(a_s) = \langle x_1\rangle \times \cdots \times \langle x_{\frac{n}{s}}\rangle,$$

where $x_i = \prod_{m=(i-1)s+1}^{is} k_m$ for each $1 \leqslant i \leqslant \frac{n}{s}$. Similarly, for each $0 \leqslant j \leqslant \lfloor\frac{n}{2t}\rfloor$ one has

$$C_K(b_{t,j}) = \langle y_1\rangle \times \cdots \times \langle y_j\rangle \times \langle z_{2j+1}\rangle \times \cdots \times \langle z_{\frac{n}{t}}\rangle,$$

where $y_i = \prod_{m=2(i-1)t+1}^{2it} k_m$ for each $1 \leqslant i \leqslant j$ and $z_i = \prod_{m=(i-1)t+1}^{it} k_m$ for each $2tj+1 \leqslant i \leqslant \frac{n}{t}$. Thus, $C_K(a_s) \cong \mathbb{Z}_2^{\frac{n}{s}}$ and $C_K(b_{t,j}) \cong \mathbb{Z}_2^j \times \mathbb{Z}_2^{\frac{n-2tj}{t}}$.

Set $x = a_s$ or $b_{t,j}$ $(0 \leqslant j \leqslant \lfloor\frac{n}{2t}\rfloor)$. It is straightforward to check $C_A(x)K/K \leqslant C_{A/K}(xK)$. Conversely, take $yK \in C_{A/K}(xK)$. Then $yxK = xyK$, that is, $x^{-1}b^{-1}xb = k'$ for some $k' \in K$, implying that $xk' = y^{-1}xy$ is semiregular and has the same order as x. By the above Fact, there exists a $k \in K$ such that $xk' = k^{-1}xk$ and hence $(yk)^{-1}x(yk) = x$ $(k = k^{-1})$, implying $yK = ykK \in C_A(x)K/K$. It follows that

$$C_A(x)K/K = C_{A/K}(xK).$$

Note that K is the kernel of the induced action of A on the edge set $E = \{e_1, e_2, \ldots, e_n\}$. One may view A/K as a permutation group on E. Denote by xK the induced permutation of x on E. If $x = a_s$ then xK is a semiregular permutation of order s on E. Since $A/K \cong S_n$, by Eq (3) one has

$$|C_{A/K}(a_sK)| = s^{\frac{n}{s}}(\frac{n}{s})!.$$

If $x = b_{t,j}$ then $b_{t,j}K$ is a product of j disjoint $2t$-cycles and $\frac{n-2jt}{t}$ disjoint t-cycles. Thus,

$$|C_{A/K}(b_{t,j}K)| = (2t)^j j! \cdot t^{\frac{n-2tj}{t}}(\frac{n-2tj}{t})!.$$

On the other hand,

$$C_A(x)K/K \cong C_A(x)/(C_A(x) \cap K) = C_A(x)/C_K(x).$$

Since $|C_K(a_s)| = 2^{\frac{n}{s}}$, we have

$$\begin{aligned}|C_A(a_s)| &= |C_K(a_s)| \cdot |C_A(a_s)/C_K(a_s)| \\ &= |C_K(a_s)| \cdot |C_A(a_s)K/K| \\ &= 2^{\frac{n}{s}}|C_{A/K}(a_sK)| \\ &= (2s)^{\frac{n}{s}}(\frac{n}{s})!.\end{aligned}$$

Similarly, one has

$$|C_A(b_{t,j})| = 2^j 2^{\frac{n-2tj}{t}} \cdot (2t)^j j! \cdot t^{\frac{n-2tj}{t}}(\frac{n-2tj}{t})! = 2^j (2t)^{\frac{n-jt}{t}} j!(\frac{n-2tj}{t})!.$$

As a result, one has

$$\begin{aligned}|C\ell(a_s)| &= \frac{|A|}{|C_A(a_s)|} &= \frac{2^n n!}{(2s)^{\frac{n}{s}}(\frac{n}{s})!}, \\ |C\ell(b_{t,j})| &= \frac{|A|}{|C_A(b_{t,j})|} &= \frac{2^n n!}{2^j \cdot (2t)^{\frac{n-jt}{t}} \cdot j!(\frac{n-2jt}{t})!}.\end{aligned}$$

□

Theorem 3.2 *Let $C(B_n)$ be the set of congruence classes of embeddings of a bouquet B_n of n circles. Then*

$$|C(B_n)| = \sum_{\substack{s \mid n \\ s \text{ odd}}} \frac{\phi(s)(\frac{2n}{s}-1)! s^{\frac{n}{s}-1}}{2^{\frac{n}{s}}(\frac{n}{s})!} + \sum_{t \mid n} \sum_{j=0}^{\lfloor \frac{n}{2t} \rfloor} \frac{\phi(2t) t^{j-1}(\frac{n}{t}-1)!}{2j!(\frac{n-2tj}{t})!}.$$

Proof. By Proposition 2.1 and Eq. (2),

$$|C(B_n)| = \frac{1}{|\mathrm{Aut}(B_n)|}\sum_{i=1}^{m} |C\ell(g_i)||\mathrm{Fix}(g_i)|.$$

Note that $|\mathrm{Fix}(g_i)| \neq 0$ only for semiregular automorphisms g_i because each rotation in $R(B_n)$ is a $2n$-cycle. By Lemma 3.1 (1) and (2), we have

$$|C(B_n)| = \frac{1}{2^n n!}\left(\sum_{\substack{s \mid n \\ s \text{ odd}}} |C\ell(a_s)||\mathrm{Fix}(a_s)| + \sum_{t \mid n}\sum_{j=0}^{\lfloor \frac{n}{2t} \rfloor} |C\ell(b_{2t,j})||\mathrm{Fix}(b_{2t,j})|\right).$$

By Lemma 3.1 (3),

$$\begin{aligned}|C\ell(a_s)| &= \frac{|A|}{|C_A(a_s)|} &= \frac{2^n n!}{(2s)^{\frac{n}{s}}(\frac{n}{s})!}, \\ |C\ell(b_{t,j})| &= \frac{|A|}{|C_A(b_{t,j})|} &= \frac{2^n n!}{2^j \cdot (2t)^{\frac{n-jt}{t}} \cdot j!(\frac{n-2jt}{t})!}.\end{aligned}$$

By Proposition 2.2,

$$\begin{aligned}|\mathrm{Fix}(a_s)| &= \phi(s)(\tfrac{2n}{s}-1)!s^{\frac{2n}{s}-1},\\ |\mathrm{Fix}(b_{t,j})| &= \phi(2t)(\tfrac{n}{t}-1)!(2t)^{\frac{n}{t}-1}.\end{aligned}$$

Thus,

$$\begin{aligned}|C(B_n)| &= \tfrac{1}{2^n n!}\left(\sum_{\substack{s \mid n\\ s\ odd}} \frac{2^n n!\cdot\phi(s)(\frac{2n}{s}-1)!s^{\frac{2n}{s}-1}}{(2s)^{\frac{n}{s}}(\frac{n}{s})!} + \sum_{t \mid n}\sum_{j=0}^{\lfloor\frac{n}{2t}\rfloor} \frac{2^n n!\cdot\phi(2t)(\frac{n}{t}-1)!(2t)^{\frac{n}{t}-1}}{2^j\cdot(2t)^{\frac{n-jt}{t}}\cdot j!(\frac{n-2jt}{t})!}\right)\\ &= \sum_{\substack{s \mid n\\ s\ odd}} \frac{\phi(s)(\frac{2n}{s}-1)!s^{\frac{n}{s}-1}}{2^{\frac{n}{s}}(\frac{n}{s})!} + \sum_{t \mid n}\sum_{j=0}^{\lfloor\frac{n}{2t}\rfloor} \frac{\phi(2t)t^{j-1}(\frac{n}{t}-1)!}{2j!(\frac{n-2tj}{t})!}.\end{aligned}$$

□

Let $n=p$ be an odd prime. Then in Theorem 3.2, s and t should be 1 or p. Furthermore, the formula in Theorem 3.2 can be simplified as follows.

Corollary 3.3 *Let p be a prime and let $C(B_p)$ be the set of congruence classes of embeddings of a bouquet B_p of p circles. Then*

$$|C(B_p)| = \begin{cases} 2 & p=2\\ \frac{p^2-1}{2p}+\frac{1}{2^p}\prod_{i=1}^{p-1}(2p-i)+\frac{(p-1)!}{2}\sum_{j=0}^{\frac{p-1}{2}}\frac{1}{j!(p-2j)!} & p\geqslant 3\end{cases}$$

When $n=1,2,3,4,5,6,7$ or 8, the number $|C(B_n)|$ is 1, 2, 5, 18, 105, 902, 9749 or 127072, which grows rapidly. The following theorem estimates how the number $|C(B_n)|$ varies rapidly.

Theorem 3.4

$$\lim_{n\to\infty}\frac{|C(B_n)|}{(2n-1)!/2^n n!}=1.$$

Proof. We first give two facts without proof, of which the second one is well known.

Fact 1: The function $f(x)=x^{\frac{n}{x}-1}$ defined on $(e,+\infty)$ is strictly monotone decreasing.

Fact 2: For a positive integer n, $\sum_{d \mid n}\phi(d)=n$.

Set

$$a_n=\sum_{\substack{s \mid n\\ s>1\ odd}}\frac{\phi(s)(\frac{2n}{s}-1)!s^{\frac{n}{s}-1}}{2^{\frac{n}{s}}(\frac{n}{s})!} \quad\text{and}\quad b_n=\sum_{t \mid n}\sum_{j=0}^{\lfloor\frac{n}{2t}\rfloor}\frac{\phi(2t)t^{j-1}(\frac{n}{t}-1)!}{2j!(\frac{n-2tj}{t})!}.$$

Then, $|C(B_n)| = (2n-1)!/2^n n! + a_n + b_n$.

By Fact 1, $s^{\frac{n}{s}-1} \leqslant 3^{\frac{n}{3}-1}$ for all $s \geqslant 3$. It follows

$$a_n \leqslant (n-1)! 3^{\frac{n}{3}-1} \sum_{\substack{s \mid n \\ s>1 \ odd}} \phi(s),$$

and by Fact 2, $a_n \leqslant n! 3^{\frac{n}{3}-1}$. Now

$$\lim_{n\to\infty} \frac{a_n}{(2n-1)!/2^n n!} \leqslant \lim_{n\to\infty} \frac{n! 3^{\frac{n}{3}-1}}{(2n-1)!/2^n n!} = \lim_{n\to\infty} 2n^2 3^{\frac{n}{3}-1} \cdot \frac{(n-1)!}{(2n-1)!!}.$$

Since $\frac{(n-1)!}{(2n-1)!!} = \frac{1}{1} \cdot \frac{1}{3} \cdot \frac{2}{5} \cdots \frac{n-1}{2n-1} \leqslant \frac{1}{2^{n-1}}$, one has

$$\lim_{n\to\infty} 2n^2 3^{\frac{n}{3}-1} \cdot \frac{(n-1)!}{(2n-1)!!} \leqslant \lim_{n\to\infty} \frac{2n^2 3^{\frac{n}{3}-1}}{2^{n-1}} \leqslant \lim_{n\to\infty} \frac{n^2 2^{\frac{2n}{3}-1}}{2^{n-1}} = \lim_{n\to\infty} \frac{n^2}{2^{\frac{n}{3}}} = 0.$$

Thus, $\lim\limits_{n\to\infty} \dfrac{a_n}{(2n-1)!/2^n n!} = 0$.

It is easy to see that

$$\begin{aligned} b_n &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\phi(2)(n-1)!}{2j!(n-2j)!} + \sum_{\substack{t \mid n \\ t>1}} \sum_{j=0}^{\lfloor \frac{n}{2t} \rfloor} \frac{\phi(2t)(2t)^{j-1}(\frac{n}{t}-1)!}{2^j j!(\frac{n-2tj}{t})!} \\ &\leqslant \tfrac{n!}{2} + \sum_{\substack{t \mid n \\ t>1}} \sum_{j=0}^{\lfloor \frac{n}{2t} \rfloor} \phi(2t)(2t)^{j-1}(\frac{n}{t}-1)! \\ &\leqslant \tfrac{n!}{2} + \sum_{\substack{t \mid n \\ t>1}} n\phi(2t)(2t)^{\frac{n}{2t}-1}(\frac{n}{t}-1)!. \end{aligned}$$

Again by Facts 1 and 2, one has $b_n \leqslant n!/2 + n^2 4^{\frac{n}{4}-1}(n-1)!$. Then,

$$\lim_{n\to\infty} \frac{n!/2 + n^2 4^{\frac{n}{4}-1}(n-1)!}{(2n-1)!/2^n n!} = \lim_{n\to\infty} (n^2 + n^3 2^{\frac{n}{2}-1}) \frac{(n-1)!}{(2n-1)!!}.$$

Noting that $\frac{(n-1)!}{(2n-1)!!} \leqslant \frac{1}{2^{n-1}}$, one has

$$\lim_{n\to\infty} (n^2 + n^3 2^{\frac{n}{2}-1}) \frac{(n-1)!}{(2n-1)!!} \leqslant \lim_{n\to\infty} \frac{n^2}{2^{n-1}} + \lim_{n\to\infty} \frac{n^3}{2^{\frac{n}{2}-1}} = 0.$$

As a result, $\lim\limits_{n\to\infty} \dfrac{b_n}{(2n-1)!/2^n n!} = 0$ and hence

$$\lim_{n\to\infty} \frac{|C(B_n)|}{(2n-1)!/2^n n!} = 1 + \lim_{n\to\infty} \frac{a_n + b_n}{(2n-1)!/2^n n!} = 1.$$

□

Note that $|R(B_n)|/|\mathrm{Aut}(B_n)| = \frac{(2n-1)!}{2^n n!}$. We have the following corollary.

Corollary 3.5 *Asymptotically,* $|C(B_n)| = |R(B_n)|/|\mathrm{Aut}(B_n)|$.

4 Embeddings of a dipole

In this section we enumerate the congruence classes of embeddings of D_n, the dipole with two vertices and n multiple edges. For an edge e of D_n, let e^+ and e^- be the two opposite darts corresponding to e, respectively. The following theorem is the main result of this section.

Theorem 4.1 *The number $|C(D_n)|$ of congruence classes of embeddings of the dipole D_n is*

$$\begin{cases} \frac{1}{2n}\sum_{t\mid n}\phi(t)^2(\frac{n}{t}-1)!t^{\frac{n}{t}-1}+\sum_{s\mid n}\sum_{j=0}^{\lfloor\frac{n}{2s}\rfloor}\frac{\phi(s)(\frac{n}{s}-1)!s^{j-1}}{2^{j+1}j!(\frac{n}{s}-2j)!} & \text{if } n \text{ is odd;}\\ \frac{1}{2n}\sum_{t\mid n}\phi(t)^2(\frac{n}{t}-1)!t^{\frac{n}{t}-1}+\sum_{\substack{s\mid n\\ s\ odd}}\sum_{j=0}^{\frac{n}{2s}-1}\frac{\phi(s)(\frac{n}{s}-1)!s^{j-1}}{2^{j+1}j!(\frac{n}{s}-2j)!} & \\ \qquad +\sum_{\substack{r\mid n\\ r\ even}}\frac{\phi(\frac{r}{2})(\frac{2n}{r}-1)!r^{\frac{n}{r}-1}}{2^{\frac{2n}{r}}(\frac{n}{r})!} & \text{if } n \text{ is even.}\end{cases}$$

Proof. Let

$$\begin{aligned} V(D_n) &= \{u,v\},\\ E(D_n) &= \{e_1,e_2,\ldots,e_n\},\\ D(D_n) &= \{e_1^+,e_1^-,e_2^+,e_2^-,\ldots,e_n^+,e_n^-\}.\end{aligned}$$

Furthermore, assume that e_1^+, e_2^+, …, e_n^+ initiate at a given vertex of D_n, say u. For each $1\leqslant \ell\leqslant n$, define

$$c_i^\ell=(e^+_{(i-1)\ell+1}\ e^+_{(i-1)\ell+2}\ \cdots\ e^+_{i\ell})(e^-_{(i-1)\ell+1}\ e^-_{(i-1)\ell+2}\ \cdots\ e^-_{i\ell}),\quad 1\leqslant i\leqslant \lfloor\tfrac{n}{\ell}\rfloor.$$

Clearly, c_i^ℓ is an automorphism of D_n of order ℓ. Set

$$\begin{aligned} &g_{s,j}=\textstyle\prod_{i=1}^{j}c_i^{2s}\cdot\prod_{i=2j+1}^{\frac{n}{s}}c_i^s, 0\leqslant j\leqslant\lfloor\frac{n}{2s}\rfloor && \text{when } s \text{ is an odd divisor of } n,\\ &h_t=\textstyle\prod_{i=1}^{\frac{n}{t}}c_i^t && \text{when } t \text{ is a divisor of } n.\end{aligned}$$

Let $A=\mathrm{Aut}(D_n)$. Let H and K be the kernels of A acting on the vertex set $V(D_n)$ and edge set $E(D_n)$, respectively. Then, $A/H\cong\mathbb{Z}_2$ and $A/K\cong S_n$, the symmetric group of degree n. It follows that $A=H\times K\cong S_n\times\mathbb{Z}_2$, where $K=\langle k\rangle$ with $k=(e_1^+\ e_1^-)(e_2^+\ e_2^-)\cdots(e_n^+\ e_n^-)$. Clearly, H can be viewed as a symmetric group on the dart set $D^+(D_n)=\{e_1^+,e_2^+,\ldots,e_n^+\}$. For $g\in\mathrm{Aut}(D_n)$, denote by $C\ell(g)$ the conjugacy class of A containing g.

Let $g\in A$ and $\rho\in R(D_n)$ be such that $g^{-1}\rho g=\rho$. As $A=H\cup kH$, one has $g\in H$ or kH. First assume $g\in H$. Then, g fixes the vertices u and v, and $g^{-1}\rho g=\rho$

implies that $g^{-1}\rho_u g = \rho_u$. Since ρ_u is an n-cycle on the set $D^+(D_n)$ and since H can be viewed as a symmetric group on $D^+(D_n)$, $g^{-1}\rho_u g = \rho_u$ implies that g, as a permutation on $D^+(D_n)$, is semiregular. Then, g, as a permutation on $D^+(D_n)$, is conjugate to $\prod_{i=1}^{\frac{n}{t}}(e^+_{(i-1)t+1}\ e^+_{(i-1)t+2}\ \cdots\ e^+_{it})$ for some divisor t of n because $A/K \cong S_n$. And as a permutation on $D(D_n)$, g is conjugate to

$$h_t = \prod_{i=1}^{\frac{n}{t}}(e^+_{(i-1)t+1}\ e^+_{(i-1)t+2}\ \cdots\ e^+_{it})(e^-_{(i-1)t+1}\ e^-_{(i-1)t+2}\ \cdots\ e^-_{it}).$$

Now assume $g = kh \in kH$. Then $g^{-1}\rho g = \rho$ implies $g^{-2}\rho g^2 = \rho$, that is, $h^{-2}\rho h^2 = \rho$. Since $h \in H$ fixes each vertex of D_n, $h^{-2}\rho_u h^2 = \rho_u$. Noting that ρ_u is an n-cycle on $D^+(D_n)$, h^2 must be semiregular, implying that h is either semiregular or has two kinds of cycles in the disjoint cycle decomposition of h which have length an odd integer s or length $2s$. If h is semiregular of order t then $h \in C\ell(h_t)$ and $g \in C\ell(kh_t)$. If h has two kinds of cycles of length s and $2s$, let h be of j disjoint $2s$-cycles in the disjoint cycle decomposition of h. Then $h \in C\ell(g_{s,j})$ and $g \in C\ell(kg_{s,j})$ for $1 \leqslant j \leqslant \lfloor\frac{n}{2s}\rfloor$. Note that $g_{s,0} = h_s$ and if n is even then $g_{s,\frac{n}{2s}} = h_{2s}$ is semiregular. By Proposition 2.1,

$$|C(D_n)| = \frac{1}{|\mathrm{Aut}(D_n)|}\sum_{i=1}^{m}|C\ell(g_i)||\mathrm{Fix}(g_i)|,$$

where $C\ell(g_i)(1 \leqslant i \leqslant m)$ are the conjugacy classes of $\mathrm{Aut}(D_n)$ with representatives $g_i(1 \leqslant i \leqslant m)$ and $\mathrm{Fix}(g_i) = \{\rho \in R(D_n) \mid g_i^{-1}\rho g_i = \rho\}$. Thus,

$$|C(D_n)| = \begin{cases} \frac{1}{2n!}\Big(\sum_{t \mid n}|C\ell(h_t)||\mathrm{Fix}(h_t)| + \sum_{s \mid n}\sum_{j=0}^{\lfloor\frac{n}{2s}\rfloor}|C\ell(kg_{s,j})||\mathrm{Fix}(kg_{s,j})|\Big) & \text{if } n \text{ is odd,} \\ \frac{1}{2n!}\Big(\sum_{t \mid n}|C\ell(h_t)||\mathrm{Fix}(h_t)| + \sum_{\substack{s \mid n \\ s \text{ odd}}}\sum_{j=0}^{\frac{n}{2s}-1}|C\ell(kg_{s,j})||\mathrm{Fix}(kg_{s,j})| & \\ \qquad + \sum_{\substack{r \mid n \\ r \text{ even}}}|C\ell(kh_r)||\mathrm{Fix}(kh_r)|\Big) & \text{if } n \text{ is even.} \end{cases}$$

Note that $A = H \times K \cong S_n \times \mathbb{Z}_2$. Let s and t be divisors of n with s odd. Then, $|C\ell(kg_{s,j})| = |C\ell(g_{s,j})| = \frac{n!}{(2s)^j j! \cdot s^{\frac{n}{s}-2j}(\frac{n}{s}-2j)!}$ for each $0 \leqslant j \leqslant \lfloor\frac{n}{2s}\rfloor$ and $|C\ell(kh_t)| = |C\ell(h_t)| = \frac{n!}{t^{\frac{n}{t}}(\frac{n}{t})!}$. By Proposition 2.2, one has

$$\begin{aligned} |\mathrm{Fix}(h_t)| &= |\mathrm{Fix}_u(h_t)||\mathrm{Fix}_v(h_t)| = (\phi(t)(\tfrac{n}{t}-1)!t^{\frac{n}{t}-1})^2, \\ |\mathrm{Fix}(kg_{s,j})| &= |\mathrm{Fix}_u(g_{s,j}^2)| = \phi(s)(\tfrac{n}{s}-1)!s^{\frac{n}{s}-1}, \\ |\mathrm{Fix}(kh_r)| &= |\mathrm{Fix}_u(h_r^2)| = \phi(\tfrac{r}{2})(\tfrac{2n}{r}-1)!(\tfrac{r}{2})^{\frac{2n}{r}-1}. \end{aligned}$$

As a result, we have $|C(D_n)| =$

$$\begin{cases} \frac{1}{2n}\sum_{t \mid n}\phi(t)^2(\frac{n}{t}-1)!t^{\frac{n}{t}-1}+\sum_{s \mid n}\sum_{j=0}^{\lfloor\frac{n}{2s}\rfloor}\frac{\phi(s)(\frac{n}{s}-1)!s^{j-1}}{2^{j+1}j!(\frac{n}{s}-2j)!} & \text{if } n \text{ is odd;} \\ \frac{1}{2n}\sum_{t \mid n}\phi(t)^2(\frac{n}{t}-1)!t^{\frac{n}{t}-1}+\sum_{\substack{s \mid n \\ s \text{ odd}}}\sum_{j=0}^{\frac{n}{2s}-1}\frac{\phi(s)(\frac{n}{s}-1)!s^{j-1}}{2^{j+1}j!(\frac{n}{s}-2j)!} & \\ \qquad +\sum_{\substack{r \mid n \\ r \text{ even}}}\frac{\phi(\frac{r}{2})(\frac{2n}{r}-1)!r^{\frac{n}{r}-1}}{2^{\frac{2n}{r}}(\frac{n}{r})!} & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. □

For some small n, the numbers $|C(D_n)|$ are $|C(D_3)| = 2$, $|C(D_4)| = 3$, $|C(D_5)| = 7$, $|C(D_6)| = 19$, $|C(D_7)| = 71$, $|C(D_8)| = 369$, $|C(D_9)| = 2393$ and $|C(D_{10})| = 18644$. Furthermore, a similar analysis to Theorem 3.4 and Corollary 3.5 gives rise to

$$\lim_{n\to\infty}\frac{|C(D_n)|}{[(n-1)!]^2/2n}=\lim_{n\to\infty}\frac{|C(D_n)|}{|R(D_n)|/|\mathrm{Aut}(D_n)|}=1.$$

Remark: Genus distribution of the equivalence classes of 2-cell embeddings of some graphs such as bouquets of circles and dipoles are known. However, the genus distribution of congruence classes of 2-cell embeddings of graphs is unknown, except a stemmed bouquet, a bouquet with an attaching edge. In fact, Gross, Robbins and Tucker [4] gave a recurrence formula for the genus distribution of congruence classes of a stemmed bouquet.

References

[1] N.L. Biggs, Automorphisms of imbedded graphs, J. Combin. Theory Ser. B 11 (1971) 132–138.

[2] N.L. Biggs, A.T. White, Permutation groups and combinatorial structures, Cambridge University Press, Cambridge-New York, 1979.

[3] E. Flapan, N. Weaver, Intrinsic chirality of complete graphs, Proc. Amer. Math. Soc. 115 (1992) 233–236.

[4] J.L. Gross, D.P. Robbins, T.W. Tucker, Genus distributions for bouquets of circles, J. Combin. Theory Ser. B 47 (1989) 292–306.

[5] J.L. Gross, T.W. Tucker, Topological Graph Theory, John Wiley and Sons, New York, 1987.

[6] L.D. James, G.A. Jones, Regular orientable imbeddings of complete graphs, J. Combin. Theory Ser. B 39 (1985) 353–367.

[7] J.H. Kwak, J. Lee, Genus polynomials of dipoles, Kyungpook Math. J. 33 (1993) 115–125.

[8] J.H. Kwak, J. Lee, Enumeration of graph embeddings, Discrete Math. 135 (1994) 129–151.

[9] J.H. Kwak, S.H. Shim, Total embedding distributions for bouquets of circles, Discrete Math. 248 (2002) 93–108.

[10] B.P. Mull, Enumerating the orientable 2-cell imbeddings of complete bipartite graph, J. Graph Theory 30 (1999) 77–90.

[11] B.P. Mull, R.G. Rieper, A.T. White, Enumerating 2-cell imbeddings of connected graphs, Proc. Amer. Math. Soc. 103 (1988) 321–330.

[12] A.T. White, Graphs, groups and surfaces, second edition, North-Holland, Amsterdam, 1984.