Hamiltonian paths in the complete graph with edge-lengths 1, 2, 3

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Submitted: May 29, 2009; Accepted: Mar 10, 2010; Published: Mar 15, 2010
Mathematics Subject Classification: 05C38

Abstract

Marco Buratti has conjectured that, given an odd prime p and a multiset L containing $p-1$ integers taken from $\{1, \dots, \frac{p-1}{2}\}$, there exists a Hamiltonian path in the complete graph with p vertices whose multiset of edge-lengths is equal to L modulo p. We give a positive answer to this conjecture in the case of multisets of the type $\{1^a, 2^b, 3^c\}$ by completely classifying such multisets that are linearly or cyclically realizable.

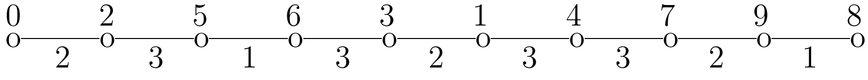
1 Introduction

Given a permutation $\sigma = (\sigma(0), \dots, \sigma(n-1))$ of the set of integers $\{0, \dots, n-1\}$, we define $d_i = \sigma(i) - \sigma(i-1)$, $i = 1, \dots, n-1$.

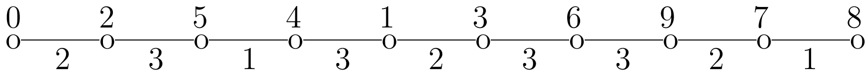
We may construct the associated multiset of differences

$$L = \{|d_1|, \dots, |d_{n-1}|\}$$

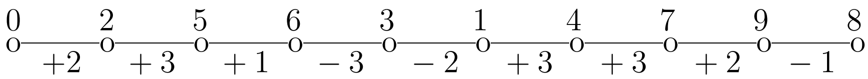
In this situation, following [1], we say that σ is a *linear realization* of the multiset L. For example, $\sigma = (0, 2, 5, 6, 3, 1, 4, 7, 9, 8)$ is a linear realization of $L = \{1^2, 2^3, 3^4\}$, where each exponent denotes the multiplicity of the base element in the multiset L. The following diagram allows us to describe both the multiset of differences and the permutation

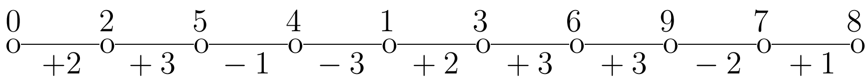


We notice however that the sequence of differences does not uniquely determine the permutation. For example,



is a different realization of the same multiset with the same sequence of differences and the same initial vertex. Let us denote with $\langle d_1, \ldots, d_{n-1}\rangle$ the sequence of signed differences. Once the first vertex is fixed, this sequence uniquely determines the permutation σ and we found it a useful device in our computation. In this paper, we shall choose $\sigma(0) = 0$ and sometimes we shall identify the permutation σ with the related sequence of signed differences by writing $(0, \sigma(1), \ldots, \sigma(n-1)) = \langle d_1, \ldots, d_{n-1}\rangle$. From now on we shall use the signed differences also in the diagrams. The previous examples become, respectively,





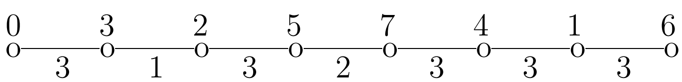
For every pair of elements i, j taken from $\{0, \ldots, n-1\}$, we define, following [1],

$$d(i, j) = \min\{|i-j|, n-|i-j|\}$$

Given a permutation of the elements of $\{0, \ldots, n-1\}$, $\sigma = (\sigma(0), \ldots, \sigma(n-1))$, we consider the list

$$L = \{d(\sigma(i-1), \sigma(i)) : i = 1, \ldots n-1\}$$

and we call σ a *cyclic realization* of L *modulo* n. For example, the linear realizations above are also cyclic realizations modulo 10 of the multiset $L = \{1^2, 2^3, 3^4\}$. However, there are cyclic realizations of a multiset L which are not linear realization of L, for example,



Notice that a cyclic realization modulo n is best viewed as a realization of a multiset of elements in $\mathbf{Z}_n$, while for a linear realization the elements are taken in $\mathbf{Z}$.

Remark 1.1 Every linear realization of a list $L = \{{d_1}^{a_1}, \ldots, {d_k}^{a_k}\}$ can be viewed as a cyclic realization modulo $k+1$, although not necessarily of the same list. For example, the sequence $(0, 4, 2, 3, 1)$ is a linear realization of $L = \{1^1, 2^2, 3^0, 4^1\}$ while it is a cyclic realization of $L' = \{1^2, 2^2, 3^0, 4^0\}$ modulo 5.

If all the elements in the list are less than or equal to $\frac{|L|}{2}$, then every linear realization of L is also a cyclic realization of the same list L. (Section 3 of [1]).

Marco Buratti conjectured that if $p = 2m+1$ is a prime number, and L is any list of $2m$ elements chosen from the set $\{1, 2, \ldots, m\}$ then there exists a cyclic realization of L. The proof of such a conjecture would be extremely useful to solve cyclic graph decomposition problems.

Consider the complete graph K_p on p vertices labelled with the elements of Z_p, the group of residue classes modulo p. Following [1], we call $d(i, j)$, $i, j \in Z_p$, the *length* of the edge ij. Buratti's conjecture can also be reformulated by saying that, for any list L of $2m$ elements taken from $\{1,\ldots,m\}$, there is a Hamiltonian path H in K_p whose multiset of edge-lengths is equal to L.

It is easy to see that if p is not prime, then one can find a multiset L which has no cyclic realization. For example, if $p = 2m$ then the multiset $L = \{2^{2m-1}\}$ is not cyclically realizable.

In recent papers, [1], [2], it was shown that the conjecture is true for lists with at most two distinct values. Moreover in [1] the conjecture was shown to be true for lists in which one of the elements occurs "sufficiently many times". The lists with only two distinct values that can be realized were also characterized in [1], when p is not necessarily prime.

It seems natural to attack the problem in the case of three lengths which, however, appears to be quite difficult. The present paper is a first step in this direction. In fact, we focused our attention to the case of lists containing only elements from the set $\{1, 2, 3\}$ and any multiplicity. We concentrate at first on linear realizations, which in most cases can be also interpreted as cyclic realizations of the same multiset (see Remark 1.1), because they could be used in an inductive argument.

In particular, to this aim, we find it useful to introduce the notion of a *perfect* linear realization: we shall say that a linear realization of the multiset L is *perfect*, and we denote it by RL, if the terminal vertex of the diagram is labelled by the largest element, otherwise we shall call it an *imperfect* realization. We denote by rL a linear realization of L which may or may not be perfect.

Given a perfect realization $RL_1 = (0, i_1, \ldots, i_{s-1}, s) = \langle d_1, \ldots, d_s\rangle$ and a second realization $rL_2 = (0, j_1, \ldots, j_t) = \langle d'_1, \ldots, d'_t\rangle$, not necessarily perfect, we may form a new realization $R(L_1 \cup L_2)$ ($L_1 \cup L_2$ union of multisets), which we denote by $RL_1 + rL_2$, by taking

$$RL_1 + rL_2 = (0, i_1, \ldots, i_{s-1}, s, j_1 + s, \ldots, j_t + s) = \langle d_1, \ldots, d_s, d'_1, \ldots, d'_t\rangle.$$

If h is a positive integer, we also use the notation hRL to mean the sum of h copies of RL.

In the next section we shall outline our main results.

2 Realizable lists

We shall focus our attention on multisets $L = \{1^a, 2^b, 3^c\}$ where a, b, c, are the number of times that 1,2,3 occur in L.

We can immediately observe that if the multiset has only one symbol d then it admits no linear realization unless $d = 1$, in which case there is the trivial perfect realization $R\{1^a\} = \langle \underbrace{+1, +1, \ldots, +1}_{a} \rangle = (0, 1, \ldots, a)$, whatever the multiplicity a may be.

The main results of our paper are the following statements.

Theorem 2.1 *A multiset $L = \{1^a, 2^b, 3^c\}$ is linearly realizable if and only if the integers a, b, c satisfy one of the following conditions*

(i) $a = 0, b \geqslant 4, c \geqslant 3$
(ii) $a = 0, b = 3$ *and* $1 \leqslant c \leqslant 8$ *or* $c \not\equiv 0$ *mod 3*
(iii) $(a, b, c) \in \{(0, 2, 2), (0, 2, 3), (0, 4, 1), (0, 4, 2), (0, 7, 2), (0, 8, 2)\}$
(iv) $a \geqslant 2, b = 0, c \in \mathbf{N}$
(v) $a \geqslant 1, b \in \mathbf{N}, c = 0$
(vi) $a \geqslant 1, b \geqslant 1, c \geqslant 1, (a, b, c) \neq (1, 1, 3k + 2), k \in \mathbf{N}$

Corollary 2.2 *The realizations in the previous theorem are also cyclic realizations of L when $a + b + c \geqslant 6$.*

Theorem 2.3 *The multiset $L = \{1^a, 2^b, 3^c\}$ with $(a, b, c) = (1, 1, 3k + 2), k \in \mathbf{N}$ is cyclically realizable.*

Theorem 2.4 *The multisets with at most two elements chosen among 1, 2, 3 are all cyclically realizable except in the following cases:*
$\{2^{2k+1}\}$, $\{3^{3k+2}\}$, $\{1^1, 3^{3k+1}\}$, $\{2^1, 3^{3k+1}\}$, $k \in \mathbf{N}$.

The previous theorems imply the following

Theorem 2.5 *Any multiset $L = \{1^a, 2^b, 3^c\}$ with $a \geqslant 1, b \geqslant 1, c \geqslant 1$ and $a + b + c \geqslant 6$, is cyclically realizable.*

Notice that $a + b + c + 1$ in the theorem is not required to be prime. In the case when $a + b + c + 1$ is prime we have

Theorem 2.6 *Buratti's conjecture is true for any multiset $L = \{1^a, 2^b, 3^c\}$ with $a \geqslant 1, b \geqslant 1, c \geqslant 1$ ($a + b + c \geqslant 6$ and $a + b + c + 1$ a prime number).*

The proof of Theorem 2.1 is split in a series of Lemmas in Sections 3 and 4.

3 Linear realizations of multisets on two symbols chosen among 1, 2, 3

We state beforehand the following result.

Proposition 3.1 *If a multiset $\{1^a, 2^b, 3^c\}$ admits a perfect linear realization then b is even.*

Proof Let $\langle d_1, \cdots, d_{n-1}\rangle$ be the sequence of the signed differences of the perfect realization of $\{1^a, 2^b, 3^c\}$, $n = a + b + c + 1$. We have, because of the perfection,

$$d_1 + \cdots + d_{n-1} = a + b + c$$

and then, since any d_i equals $\pm 1, \pm 2, \pm 3$,

$$|d_1| + \cdots + |d_{n-1}| \equiv a + b + c \mod 2$$

On the other hand,
$$|d_1| + \cdots + |d_{n-1}| = a + 2b + 3c$$

and then
$$|d_1| + \cdots + |d_{n-1}| \equiv a + c \mod 2.$$

It follows that $a + b + c \equiv a + c \mod 2$. Thus we get the statement. □

3.1 Small cases

Lemma 3.2 *The multiset* $\{2^1, 3^c\}$, $c \in \mathbf{N}$, *is not linearly realizable.*

Proof First, observe that the sequence of differences $\langle \ldots + d, -d, \ldots \rangle$ does not give a Hamiltonian path because it gives rise to a repetition. The sequence $\langle +2, +3, +3, \ldots \rangle$ can only reach the integers congruent to 2 modulo 3, while $\langle +3, +3, \ldots, +3, \pm 2, -3, -3, \ldots, -3 \rangle$ can only reach two congruence classes modulo 3. □

Lemma 3.3 *For the multiset* $\{2^2, 3^c\}$, $c \in \mathbf{N}$, *we have a perfect linear realization when* $c = 3$, *an imperfect linear realization when* $c = 2$ *and no other.*

Proof A perfect linear realization $R\{2^2, 3^3\}$ is the following

0 —(+3)— 3 —(−2)— 1 —(+3)— 4 —(−2)— 2 —(+3)— 5

It is immediate to see that there is only this (imperfect) linear realization of $\{2^2, 3^2\}$

0 —(+3)— 3 —(−2)— 1 —(+3)— 4 —(−2)— 2

It is easy to see that $\{2^2, 3^1\}$ cannot be linearly realizable. When $c \geqslant 4$ the multiset $\{2^2, 3^c\}$ is not linearly realizable. To show this we argue as follows. The basic observation is that we use the two 2's to switch from one congruence class modulo 3 to another. This means that we are forced to start with +3 and continue until we obtain the whole 0-class.

After which we may either continue with $-2, -3, \ldots$ or with $+2, -3, \ldots$. In the first case we continue with -3 until we exhaust the 1-class, namely reaching down to 1. Now, any choice $\pm 2, \pm 3$ will either give a repetition, or a number outside the interval $[0, c+2]$. In the second case (which is only possible when $c \equiv 2 \bmod 3$) we continue with -3 until we exhaust the 2-class, reaching down to 2, after which we must use $+2, +3, \ldots$ up to $c + 1$, thus skipping 1. □

Lemma 3.4 *There exist imperfect linear realizations of the multisets* $\{2^3, 3^{3k+1}\}$, $\{2^3, 3^{3k+2}\}$, $\{2^3, 3^3\}$, $\{2^3, 3^6\}$.

Proof A linear realization $r\{2^3, 3^{3k+1}\}$ is the following:

$$0 \xrightarrow{+2} 2 \underbrace{\xrightarrow{+3} 5 \cdots 3k-1 \xrightarrow{+3}}_{k} 3k+2 \xrightarrow{+2} 3k+4 \underbrace{\xrightarrow{-3} 3k+1 \cdots 4 \xrightarrow{-3}}_{k+1} 1 \xrightarrow{+2} 3 \underbrace{\xrightarrow{+3} 6 \cdots 3k \xrightarrow{+3}}_{k} 3k+3$$

A linear realization $r\{2^3, 3^{3k+2}\}$ is the following:

$$0 \xrightarrow{+2} 2 \underbrace{\xrightarrow{+3} 5 \cdots 3k+2 \xrightarrow{+3}}_{k+1} 3k+5 \xrightarrow{-2} 3k+3 \underbrace{\xrightarrow{-3} 3k \cdots 6 \xrightarrow{-3}}_{k} 3 \xrightarrow{-2} 1 \underbrace{\xrightarrow{+3} 4 \cdots 3k+1 \xrightarrow{+3}}_{k+1} 3k+4$$

A linear realization $r\{2^3, 3^3\}$ is the following:

$$0 \xrightarrow{+2} 2 \xrightarrow{+3} 5 \xrightarrow{-2} 3 \xrightarrow{+3} 6 \xrightarrow{-2} 4 \xrightarrow{-3} 1$$

A linear realization $r\{2^3, 3^6\}$ is the following:

$$0 \xrightarrow{+3} 3 \xrightarrow{-2} 1 \xrightarrow{+3} 4 \xrightarrow{+3} 7 \xrightarrow{+2} 9 \xrightarrow{-3} 6 \xrightarrow{+2} 8 \xrightarrow{-3} 5 \xrightarrow{-3} 2$$

□

Lemma 3.5 *The multiset* $\{2^3, 3^{3k}\}$*,* $k \geqslant 3$*, is not linearly realizable.*

Proof Suppose we start with $+2$ and follow up with the string of $+3$'s thus getting to $3k+2$. In this case, once we start with a congruence class we must complete it, either going from the smallest to the largest value or vice versa. Now, we cannot use $+2$ because we would get the value $3k+4$ which is larger than the maximum $3k+3$. So we must use -2 and follow with a string of -3's. However we would miss $3k+3$, the largest element in the 0-class.

The remaining cases are done analogously. □

Lemma 3.6 *There exist linear realizations of* $\{2^4, 3^c\}$*,* $c \geqslant 1$*. We have perfect linear realizations when* $c \equiv 1, 2$ *mod* 3*, and* $c = 6$*.*

Proof A linear realization $R\{2^4, 3^{3k+1}\}$ is the following:

$$0 \xrightarrow{+2} 2 \underbrace{\xrightarrow{+3} 5 \cdots 3k-1 \xrightarrow{+3}}_{k} 3k+2 \xrightarrow{+2} 3k+4 \underbrace{\xrightarrow{-3} 3k+1 \cdots 4 \xrightarrow{-3}}_{k+1} 1 \xrightarrow{+2} 3 \underbrace{\xrightarrow{+3} 6 \cdots 3k \xrightarrow{+3}}_{k} 3k+3 \xrightarrow{+2} 3k+5$$

Notice that this amounts to the realization $r\{2^3, 3^{3k+1}\}$ given in Lemma 3.4 with an added $+2$. Similarly, we get a perfect realization $R\{2^4, 3^{3k+2}\}$ by adding a final $+2$ to the imperfect realization $r\{2^3, 3^{3k+2}\}$ given in the same lemma.

For the case $\{2^4, 3^{3k}\}$ we can use the following formula

$$0 \xrightarrow{+2} 2 \underbrace{\xrightarrow{+3} 5 \cdots 3k-1 \xrightarrow{+3}}_{k} 3k+2 \xrightarrow{+2} 3k+4 \xrightarrow{-3} 3k+1 \xrightarrow{+2} 3k+3 \underbrace{\xrightarrow{-3} 3k \cdots 6 \xrightarrow{-3}}_{k} 3 \xrightarrow{-2} 1 \underbrace{\xrightarrow{+3} 4 \cdots 3k-5 \xrightarrow{+3}}_{k-1} 3k-2$$

Finally, the case $\{2^4, 3^6\}$ is a particular case of $\{2^{2k}, 3^{3k}\}$ which can be shown to be all perfectly realizable as $R\{2^{2k}, 3^{3k}\} = kR\{2^2, 3^3\}$, where $R\{2^2, 3^3\}$ is the perfect linear realization described in Lemma 3.3.

□

Lemma 3.7 *There exist linear realizations $r\{2^7, 3^2\}$, $R\{2^8, 3^2\}$.*

Proof We have $R\{2^8, 3^2\} = 2R\{2^4, 3^1\}$, $R\{2^4, 3^1\}$ a particular case of Lemma 3.6 (all the multisets of the form $\{2^{4k}, 3^k\}$ have a similar realization). The diagram is the following:

$$0 \xrightarrow{+2} 2 \xrightarrow{+2} 4 \xrightarrow{-3} 1 \xrightarrow{+2} 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \xrightarrow{-3} 6 \xrightarrow{+2} 8 \xrightarrow{+2} 10$$

Removing the final node in this realization gives an imperfect realization $r\{2^7, 3^2\}$.

□

Lemma 3.8 *There exist linear realizations $r\{2^b, 3^3\}$, $b \geqslant 2$.*

Proof A realization $r\{2^{2k+2}, 3^3\}$, $k \geqslant 0$, is the following:

$$\underbrace{0 \xrightarrow{+2} 2 \cdots 2k{-}2 \xrightarrow{+2}}_{k} 2k \xrightarrow{+3} 2k{+}3 \xrightarrow{+2} 2k{+}5 \xrightarrow{-3} 2k{+}2 \xrightarrow{+2} 2k{+}4 \xrightarrow{-3} 2k{+}1 \underbrace{\xrightarrow{-2} 2k{-}1 \cdots 3 \xrightarrow{-2}}_{k} 1$$

A realization $r\{2^{2k+1}, 3^3\}$, $k \geqslant 1$, is the following:

$$\underbrace{0 \xrightarrow{+2} 2 \cdots 2k{-}2 \xrightarrow{+2}}_{k} 2k \xrightarrow{+3} 2k{+}3 \xrightarrow{-2} 2k{+}1 \xrightarrow{+3} 2k{+}4 \xrightarrow{-2} 2k{+}2 \xrightarrow{-3} 2k{-}1 \underbrace{\xrightarrow{-2} 2k{-}3 \cdots 3 \xrightarrow{-2}}_{k-1} 1$$

□

Lemma 3.9 *The multiset $\{2^b, 3^2\}$, $b \geqslant 2$, is linearly realizable if and only if $b = 2, 3, 4, 7, 8$.*

Proof In constructing a linear realization of $\{2^b, 3^2\}$, one could start with

$$0 \xrightarrow{+3} 3 \xrightarrow{+2} 5$$

However, after this, one could never obtain a vertex labeled 1, so this path would not be Hamiltonian.

Another possibility is to start with

$$0 \xrightarrow{+3} 3 \xrightarrow{-2} 1$$

One certainly could not follow up with +2, because that would produce a repetition, nor with a −2, hence one would have to choose a +3:

$$0 \xrightarrow{+3} 3 \xrightarrow{-2} 1 \xrightarrow{+3} 4$$

Now, continuing with +2 would miss 5. On the other hand, continuing with −2 would give us an imperfect realization of $\{2^2, 3^2\}$.

Another possibility is to start with a string of $+2$, after which there could follow either a $+3$ or a -3 and a new sequence of ± 2's. This would produce a string of odd numbers. Since, after using the second available $+3$ we would get back to even numbers, we had better make sure that between the two 3's there is the complete string of odd numbers. Moreover, since we will be using either a string of -2's or a string of $+2$'s, we must make sure that, respectively, $2h+3$ is the maximum of the odd numbers, or $2h-3$ is the minimum of the odd numbers, namely 1. If $2h-3=1$ then $h=2$ and so we have

$$0 \xrightarrow{+2} 2 \xrightarrow{+2} 4 \xrightarrow{-3} 1 \xrightarrow{+2} 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \xrightarrow{-3} 6 \xrightarrow{+2} 8 \xrightarrow{(+2)} (10)$$

and these are $r\{2^7, 3^2\}$ ($R\{2^8, 3^2\}$).

In the case where $2h+3$ is the maximum of the odd numbers, it is necessary that, at the end of the string of odd numbers, namely after the vertex labeled 1, there is the label $2h+2$. In other words, we must have $1+3=2h+2$, i.e., $h=1$. This gives the only possibilities as

$$0 \xrightarrow{+2} 2 \xrightarrow{+3} 5 \xrightarrow{-2} 3 \xrightarrow{-2} 1 \xrightarrow{+3} 4 \xrightarrow{(+2)} (6)$$

and these are $r\{2^3, 3^2\}$ ($R\{2^4, 3^2\}$).

□

3.2 The line $c = \frac{b}{4} + \frac{3}{2}$

Lemma 3.10 *There exist linear realizations of* $\{2^5, 3^c\}$*, for any* $c \geqslant 3$.

Proof A linear realization $r\{2^5, 3^{3k}\}$ is the following:

$$0 \xrightarrow{+2} 2 \xrightarrow{+2} 4 \xrightarrow{-3} 1 \xrightarrow{+2} 3 \underbrace{\xrightarrow{+3} 6 \cdots 3k \xrightarrow{+3}}_{k} 3k+3 \xrightarrow{+2} 3k+5 \underbrace{\xrightarrow{-3} 3k+2 \; 8 \cdots \xrightarrow{-3}}_{k} 5 \xrightarrow{+2} 7 \underbrace{\xrightarrow{+3} 10 \cdots 3k+1 \xrightarrow{+3}}_{k-1} 3k+4$$

A linear realization $r\{2^5, 3^{3k+1}\}$ is the following:

$$0 \xrightarrow{+2} 2 \xrightarrow{+2} 4 \xrightarrow{-3} 1 \xrightarrow{+2} 3 \underbrace{\xrightarrow{+3} 6 \cdots 3k+3 \xrightarrow{+3}}_{k+1} 3k+6 \xrightarrow{-2} 3k+4 \underbrace{\xrightarrow{-3} 3k+1 \; 10 \cdots \xrightarrow{-3}}_{k-1} 7 \xrightarrow{-2} 5 \underbrace{\xrightarrow{+3} 8 \cdots 3k+2 \xrightarrow{+3}}_{k} 3k+5$$

Finally, a linear realization $r\{2^5, 3^{3k+2}\}$ is the following:

$$0 \xrightarrow{+2} 2 \xrightarrow{+3} 5 \xrightarrow{-2} 3 \xrightarrow{-2} 1 \underbrace{\xrightarrow{+3} 4 \cdots 3k+4 \xrightarrow{+3}}_{k+2} 3k+7 \xrightarrow{-2} 3k+5 \underbrace{\xrightarrow{-3} 3k+2 \; 11 \cdots \xrightarrow{-3}}_{k-1} 8 \xrightarrow{-2} 6 \underbrace{\xrightarrow{+3} 9 \cdots 3k+3 \xrightarrow{+3}}_{k} 3k+6$$

□

Lemma 3.11 *There exist perfect linear realizations of* $\{2^6, 3^c\}$*, for any* $c \geqslant 3$ *and of* $\{2^8, 3^c\}$*, for any* $c \geqslant 2$*, and (imperfect) linear realization of* $\{2^7, 3^c\}$*, for any* $c \geqslant 2$.

Proof For the multiset $\{2^6, 3^c\}$, $c \geqslant 3$, it is enough to add a final vertex $(6+c)$ with a difference $+2$ in the realizations of the previous lemma. For the multiset $\{2^8, 3^c\}$, $c \geqslant 2$, we have the following perfect realizations:

$$R\{2^8, 3^c\} = \begin{cases} R\{2^4, 3^{c-1}\} + R\{2^4, 3^1\} & \text{if } c \not\equiv 1 \bmod 3 \\ R\{2^4, 3^{c-2}\} + R\{2^4, 3^2\} & \text{if } c \equiv 1 \bmod 3 \end{cases}$$

By removing the final vertex from the realization $R\{2^8, 3^c\}$ we obtain a realization $r\{2^7, 3^c\}$, for any $c \geqslant 2$. □

Lemma 3.12 *We have linear realizations of all the multisets $\{2^b, 3^c\}$, when $b \geqslant 4$ and $c \geqslant \frac{b}{4} + \frac{3}{2}$.*

Proof We proceed by induction on $b \geqslant 4$. We have already shown that the statement is true for $b = 4, 5, 6, 7, 8$. Assume the statement is true for $\{2^{b_0}, 3^{c_0}\}$, when $b_0 \geqslant 8$ and $c_0 \geqslant \frac{b_0}{4} + \frac{3}{2}$. Consider $\{2^{b_0+1}, 3^c\}$ with $c \geqslant \frac{b_0+1}{4} + \frac{3}{2}$. We can write the decomposition

$$r\{2^{b_0+1}, 3^c\} = R\{2^4, 3^1\} + r\{2^{b_0-3}, 3^{c-1}\}$$

Indeed,

$$b_0 - 3 \geqslant 4 \text{ and } c \geqslant \frac{b_0+1}{4} + \frac{3}{2} \iff c - 1 \geqslant \frac{b_0-3}{4} + \frac{3}{2}$$

□

Remark 3.13 The linear realizations of $\{2^6, 3^c\}$, $\{2^8, 3^c\}$ presented in Lemma 3.11 are all perfect. Hence it can be seen in the proof by induction in the previous Lemma that all the linear realizations of $\{2^b, 3^c\}$ for even b and $c \geqslant \frac{b}{4} + \frac{3}{2}$ are perfect.

Lemma 3.14 *We have (imperfect) linear realizations of all the multisets $\{2^b, 3^c\}$, when $4 \leqslant c \leqslant \frac{b}{4} + 2$ (which implies $b \geqslant 8$).*

Proof Let $c = 3 + h$ with $h \geqslant 1$ then $b \geqslant 4c - 8 = 4 + 4h$. Hence

$$r\{2^b, 3^{3+h}\} = hR\{2^4, 3^1\} + r\{2^{b-4h}, 3^3\}$$

□

Lemma 3.15 *We have (imperfect) linear realizations of all the multisets $\{1^a, 2^b\}$, when $a \geqslant 1, b \geqslant 0$.*

Proof It is enough to give the following list of differences:

$$\langle \underbrace{+1, \ldots, +1}_{a-1}, \underbrace{+2, \ldots, +2}_{[\frac{b+1}{2}]}, (-1)^b, \underbrace{-2, \ldots, -2}_{b-[\frac{b+1}{2}]} \rangle$$

where $[\frac{b+1}{2}]$ denotes the integer part of $\frac{b+1}{2}$. □

Lemma 3.16 *We have linear realizations of all the multisets $\{1^a, 3^c\}$, when $a \geqslant 2, c \geqslant 0$ and $a = 1, c = 0$. They are perfect when $c \not\equiv 1$ mod 3.*

Proof It is enough to give the following list of differences:

$$\langle \underbrace{+1, \ldots, +1}_{a-2}, \underbrace{+3, \ldots, +3}_{[\frac{c+2}{3}]}, (-1)^{\bar{c}}, \underbrace{-3, \ldots, -3}_{[\frac{c}{3}]}, (-1)^{\bar{c}}, \underbrace{+3, \ldots, +3}_{[\frac{c+1}{3}]} \rangle$$

where $[x]$ denotes the integer part of x and $c = 3q + \bar{c}$, $-1 \leqslant \bar{c} \leqslant 1$. When $c \not\equiv 1$ mod 3 it is easy to see that the last vertex is $c + a$, so the realizations are perfect. The case $a = 1$, $c = 0$ is trivial. □

The lemmas in the present section complete the proof of the items from (i) to (v) of Theorem 2.1.

4 Linear realizations of multisets on three symbols 1, 2, 3

Lemma 4.1 *We have (imperfect) linear realizations of all the multisets $\{1^1, 2^1, 3^c\}$, when $c > 0, c \equiv 0, 1$ mod 3. There are no linear realizations of $\{1^1, 2^1, 3^c\}$ when $c \equiv 2$ mod 3.*

Proof We have

i) $c = 3k, k \geqslant 1$

$$0 \underbrace{\xrightarrow{+3} 3 \cdots 3k-3 \xrightarrow{+3}}_{k} 3k \xrightarrow{+2} 3k+2 \underbrace{\xrightarrow{-3} 3k-1 \cdots 5 \xrightarrow{-3}}_{k} 2 \xrightarrow{-1} 1 \underbrace{\xrightarrow{+3} 4 \cdots 3k-2 \xrightarrow{+3}}_{k} 3k+1$$

ii) $c = 3k + 1, k \geqslant 0$

$$0 \underbrace{\xrightarrow{+3} 3 \cdots 3k \xrightarrow{+3}}_{k+1} 3k+3 \xrightarrow{-2} 3k+1 \underbrace{\xrightarrow{-3} 3k-2 \cdots 4 \xrightarrow{-3}}_{k} 1 \xrightarrow{+1} 2 \underbrace{\xrightarrow{+3} 5 \cdots 3k-1 \xrightarrow{+3}}_{k} 3k+2$$

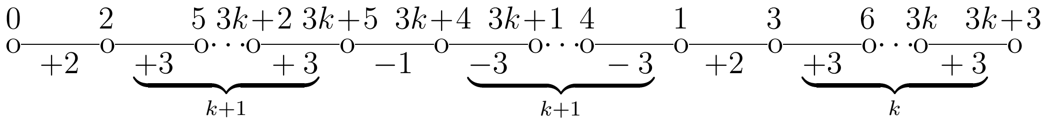
To see that there are no realizations when $c = 3k + 2$, we argue as follows. The only hope to obtain the full set of congruence classes modulo 3 is to use a list of differences such as $\langle +3, \ldots, +3, \pm x, -3, \ldots, -3, \pm y, +3, \ldots, +3 \rangle$, where $\{x, y\} = \{1, 2\}$.

The first string of $+3$'s must exhaust the 0-congruence class and therefore we reach up to the vertex $3k + 3$. The second string of -3's will describe another congruence class in decreasing order. For this class to be complete it can only begin with $3k + 4$ or $3k + 2$. So x must be equal to 1. If we choose $+1$, then the smallest element in the next class is 1. After which it is impossible to describe the remaining class, using ± 2. If we choose -1 then the smallest element in the next class is 2, and again we find it impossible to describe the remaining class. In fact, if we add -2 we get 0 which is not permissible, while if we add $+2$ we will never be able to get 1. □

Lemma 4.2 *We have linear realizations of all the multisets* $\{1^1, 2^2, 3^c\}$*, when* $c > 0$*. They are perfect if* $c \equiv 0, 1$ *mod 3.*

Proof For $c = 3k$, $k \geqslant 1$ and $c = 3k+1$, $k \geqslant 0$, if we add $3k+3$ and, respectively, $3k+4$ as final vertex to the realizations i) and ii) of Lemma 4.1 we obtain the desired perfect realizations.

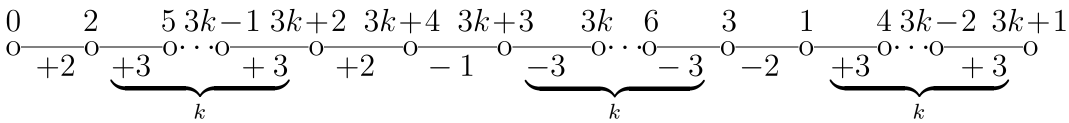
For $c = 3k+2, k \geqslant 0$, we have



□

Lemma 4.3 *There are (imperfect) linear realizations of all the multisets* $\{1^1, 2^3, 3^c\}$*, when* $c > 0$*.*

Proof For $c = 3k, k \geqslant 1$ we have



For $c = 3k+1$ or $c = 3k+2, k \geqslant 0$, we get

$$r\{1^1, 2^3, 3^c\} = R\{1\} + r\{2^3, 3^c\}$$

using Lemma 3.4. □

Lemma 4.4 *There are (imperfect) linear realizations of all the multisets* $\{1^1, 2^b, 3^c\}$*, when* $b \geqslant 4$ *and* $c > 0$*.*

Proof We have

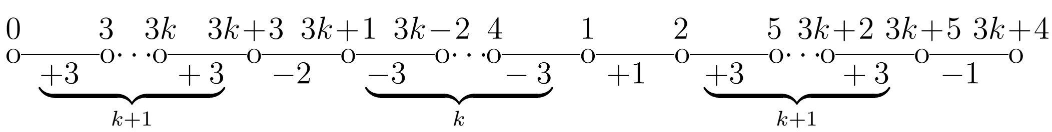
i) $b = 4$: $r\{1^1, 2^4, 3^c\} = R\{1\} + r\{2^4, 3^c\}$

ii) $b = 5$: $r\{1^1, 2^5, 3^c\} = \begin{cases} R\{2^4, 3^c\} + r\{1^1, 2^1\} & \text{if } c = 1, 2 \\ R\{1^1\} + r\{2^5, 3^c\} & \text{if } c > 2 \end{cases}$

iii) $b \geqslant 6$: $r\{1^1, 2^b, 3^c\} = \begin{cases} R\{2^4, 3^c\} + r\{1^1, 2^{b-4}\} & \text{if } c = 1, 2 \\ R\{2^6, 3^c\} + r\{1^1, 2^{b-6}\} & \text{if } c > 2 \end{cases}$ □

Lemma 4.5 *We have (imperfect) linear realizations of all the multisets* $\{1^2, 2^1, 3^{3k+2}\}$*, when* $k \geqslant 0$*.*

Proof We have



□

Lemma 4.6 *We have (imperfect) linear realizations of all the multisets* $\{1^a, 2^b, 3^c\}$, *when* $a \geqslant 1, b \geqslant 1, c \geqslant 1, (a,b,c) \neq (1,1,3k+2)$.

Proof We get

$$r\{1^a, 2^b, 3^c\} = \begin{cases} R\{1^{a-1}\} + r\{1^1, 2^b, 3^c\} & \text{if } (b,c) \neq (1, 3k+2) \\ R\{1^{a-2}\} + r\{1^2, 2^1, 3^{3k+2}\} & \text{if } (b,c) = (1, 3k+2) \text{ and } a > 1 \end{cases}$$

where the first line is a consequence of Lemmas 4.1, 4.2, 4.3, 4.4 while the second line follows from Lemma 4.5. □

Lemma 4.6 proves item (vi) of Theorem 2.1 thereby completing the proof of the whole theorem.

Given the great importance for linear realizations to be perfect in view of possible proofs by induction, we think it appropriate to emphasize the following results.

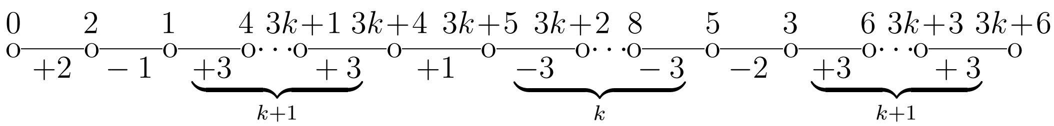
Proposition 4.7 *There exist perfect linear realizations of all the multisets* $\{1^a, 2^2, 3^c\}$, *when* $a \geqslant 1$ *for* $c \not\equiv 2 \bmod 3$ $(c > 0)$ *and* $a \geqslant 2$ *for* $c \equiv 2 \bmod 3$.

Proof If $c \not\equiv 2 \bmod 3$, then

$$R\{1^a, 2^2, 3^c\} = R\{1^{a-1}\} + R\{1^1, 2^2, 3^c\}$$

using Lemma 4.2.

If $c = 3k+2, k \geqslant 0$, then



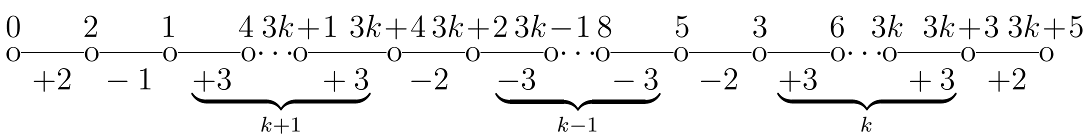
□

Proposition 4.8 *There exist perfect linear realizations of all the multisets* $\{1^a, 2^b, 3^c\}$, *when* $a \geqslant 1$, $b = 4, 6$ *and* $c > 0$.

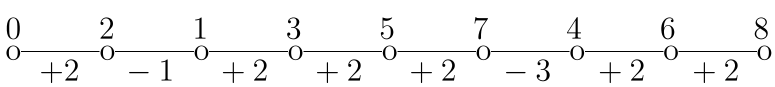
Proof

i) $b = 4$. If $c \not\equiv 0 \bmod 3$, $\quad R\{1^a, 2^4, 3^c\} = R\{1^a\} + R\{2^4, 3^c\}$.

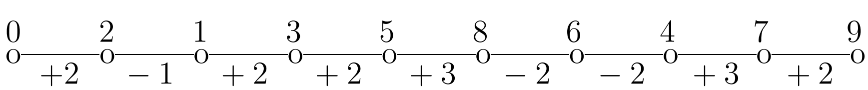
If $c = 3k$, $k > 0$, we have



ii) b=6. If $c = 1$:



If $c = 2$:



If $c > 2$, $\quad R\{1^a, 2^6, 3^c\} = R\{1^a\} + R\{2^6, 3^c\}$. □

5 Cyclic realizations

5.1 Proof of Theorem 2.3

A possible cyclic realization of the multiset $\{1^1, 2^1, 3^{3k+2}\}$ is

$$0 \xrightarrow{+3} 3 \xrightarrow{-1} 2 \underbrace{\xrightarrow{+3} 5 \cdots 3k{-}1 \xrightarrow{+3}}_{k} 3k{+}2 \xrightarrow{+2} 3k{+}4 \underbrace{\xrightarrow{-3} 3k{+}1 \cdots 4 \xrightarrow{-3}}_{k+1} 1 \underbrace{\xrightarrow{-3} 3k{+}3 \xrightarrow{-3} 3k \cdots 9 \xrightarrow{-3}}_{k} 6$$

where the computations are done modulo $3k+5$. □

5.2 Proof of Theorem 2.4

This is an easy consequence of Theorem 4.1 in [1]. □

We note that the cases that are not cyclically realizable in Theorem 2.4 do not contradict the conjecture because in those cases $|L|+1$ is not a prime number.

References

[1] P. Horak, A. Rosa, On a problem of Marco Buratti, *Electron. J. Combin.*, 16 (2009), # R20.

[2] J. H. Dinitz, S. R. Janiszewski, On Hamiltonian Paths with Prescribed edge lengths in the Complete Graph, *Bull. Inst. Combin. Appl.*, to appear.