Potential-Based Strategies for Tic-Tac-Toe on the Integer Lattice with Numerous Directions

Klay Kruczek
Western Oregon University
kruczekk@wou.edu

Eric Sundberg
Occidental College
sundberg@oxy.edu

Submitted: May 29, 2009; Accepted: Dec 18, 2009; Published: Jan 5, 2010
Mathematics Subject Classification: 91A46

Abstract

We consider a tic-tac-toe game played on the d-dimensional integer lattice. The game that we investigate is a Maker–Breaker version of tic-tac-toe. In a Maker–Breaker game, the first player, Maker, only tries to occupy a winning line and the second player, Breaker, only tries to stop Maker from occupying a winning line. We consider the bounded number of directions game, in which we designate a finite set of direction-vectors $\mathcal{S} \subset \mathbb{Z}^d$ which determines the set of winning lines. We show, by using the Erdős–Selfridge theorem and a modification of a theorem by Beck about games played on almost-disjoint hypergraphs, that for the special case when the coordinates of each direction-vector are bounded, i.e., when $\mathcal{S} \subset \{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$, Breaker can win this game if the length of each winning line is on the order of $d^2 \lg(dk)$ and $d^2 \lg(k)$, respectively. In addition, we show that Maker can build winning lines of length up to $(1+o(1))d \lg k$ if $\mathcal{S}$ is the set of *all* direction-vectors with coordinates bounded by k. We also apply these methods to the n-consecutive lattice points game on the N^d board with (essentially) $\mathcal{S} = \mathbb{Z}^d$, and we show that the phase transition from a win for Maker to a win for Breaker occurs at $n = (d + o(1)) \lg N$.

1 Introduction

The traditional game of 3×3 tic-tac-toe is a type of *positional* game. In particular, 3×3 tic-tac-toe is an example of what we call a strong positional game. In general, a positional game is a two-person game with complete information played on a hypergraph $(V, \mathcal{H})$, where V is an arbitrary set, called the *board* of the game, and $\mathcal{H}$ is a family of subsets of V, called the *winning sets*. The two players, Player 1 and Player 2, alternately occupy previously unoccupied elements of V. In a *strong* positional game, the first player to occupy all points of some winning set wins. We say that Player 1 has a *winning strategy* if no matter what Player 2 does, Player 1 can follow that strategy to win the game. If neither player has a winning strategy, we say that the game is a *draw*.

The traditional game of 3×3 tic-tac-toe is an example of a strong positional game where the nine positions are the vertices and the eight lines (3 vertical, 3 horizontal, and 2 diagonal) are the winning sets. Most people are aware that 3×3 tic-tac-toe is a draw game.

By using a *strategy stealing* argument (see [1] for a description), it can be shown that in all strong positional games, either Player 1 has a winning strategy or the game is a draw. Thus, it is reasonable to consider an alternate positional game in which it is possible for Player 2 to have a winning-strategy. One such game is the Maker–Breaker game. A *Maker–Breaker* positional game is where the first player, Maker, only tries to occupy winning sets, and the second player, Breaker, only tries to stop Maker from doing so. Thus, in a Maker–Breaker positional game, Maker wins if she occupies all points of some winning set and Breaker wins if he prevents Maker from doing so. Therefore, by definition, someone always wins in a Maker–Breaker positional game (there are no draws). It is interesting to note that when 3×3 tic-tac-toe is played as a Maker–Breaker positional game, Maker has a winning strategy, as Maker does not need to block Breaker from obtaining a winning line.

Since we will be considering a *semi-infinite* game, i.e., a game where $|V| = \infty$, $|\mathcal{H}| = \infty$, yet $\forall A \in \mathcal{H}$, $|A| < \infty$, we should describe what constitutes a win for Breaker in such a game. If a Maker–Breaker game is played on a semi-infinite hypergraph $(V, \mathcal{H})$, then we say that Breaker has a winning strategy on $(V, \mathcal{H})$ if for all $j \in \mathbb{Z}^+$, Breaker can prevent Maker from completely occupying a winning set by turn j.

We consider the following Maker–Breaker game on $\mathbb{Z}^d$. The vertices of the board are all the points of $\mathbb{Z}^d$. Each winning line has length m, and the directions of the winning lines are determined by a set of vectors $\mathcal{S}$, which we call the set of *direction vectors*. We require that for each $\vec{v} \in \mathcal{S}$, the greatest common divisor of its coordinates is 1, which we denote by $\gcd(\vec{v}) = 1$. Also, for each vector $\vec{v} \in \mathcal{S}$, the vector $-\vec{v} \notin \mathcal{S}$, since $\vec{v}$ and $-\vec{v}$ determine the same set of winning lines. This way we can say that the set of winning lines is

$$\{\{\vec{p}, \vec{p}+\vec{v}, \vec{p}+2\vec{v}, \ldots \vec{p}+(m-1)\vec{v}\} : \vec{p} \in \mathbb{Z}^d, \vec{v} \in \mathcal{S}\},$$

We refer to this game as $MB^d_{\mathcal{S}}(m)$.

Our paper contains three main results which are found in Sections 2, 3, and 4. In Section 2, we focus on the *bounded-coordinates* version of $MB^d_{\mathcal{S}}(m)$, i.e., where $\mathcal{S} \subset \{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$, and we describe a strategy that allows Breaker to win if the length m of each winning line is $m = (2 + o(1))d^2 \lg(dk)$ (as $k \to \infty$ and $d \to \infty$), where lg is the binary logarithm. Similarly, Breaker has an explicit winning strategy if (i) d is fixed, $k \to \infty$, and $m = (2 + o(1))d(d + 1) \lg(k)$ or (ii) k is fixed, $d \to \infty$, and $m = (2 + o(1))d^2 \lg(d)$. This strategy is essentially a generalization of a Player 2's winning strategy given by Beck [1] (page 157), for the 40-*in-a-row* game played on $\mathbb{Z} \times \mathbb{Z}$. We also modify a theorem by Beck [1] (Thm. 34.1, pg. 464) which allows us to further improve our result by showing that Breaker can win if the length of each winning line is (i) $m = (1 + o(1))d^2 \lg(k)$ ($k \to \infty$ and $d \to \infty$), (ii) $m = (1 + o(1))d(d + 1) \lg k$ (d is fixed and $k \to \infty$), or (iii) $m = d^2(\lg(k) + 3c\sqrt{\lg k} + 1 + \frac{1}{k} + o(1))$ ($k \geqslant 2$ is fixed, $d \to \infty$, and $0 < c < \frac{1}{2}$ is an arbitrary constant).

In Section 3 we focus on the *full* bounded-coordinates version of $MB^d_{\mathcal{S}}(m)$, i.e., we require that $\mathcal{S}$ consists of *all* direction-vectors $\vec{v}$ such that $\|\vec{v}\|_\infty \leqslant k$, with the exception that if $\vec{v} \in \mathcal{S}$, then $-\vec{v} \notin \mathcal{S}$. For the full bounded-coordinates game, we are able to show that Maker can build winning lines of length up to $(1 + o(1))d \lg k$. Thus, for the full bounded-coordinates version of $MB^d_{\mathcal{S}}(m)$, by combining our results from Sections 2 and 3 we see that when d is fixed and $k \to \infty$, there is only a factor of $(d + 1)$ that separates what Maker can achieve and what Breaker can prevent.

In Section 4 we consider the *n-consecutive lattice points game on N^d*, which is a finite game. The board is the set of points of the N^d hypercube, and each winning line is a set of n consecutive lattice points lying on a straight line, where we allow *any* direction-vector for determining the direction of a winning line. Using similar strategies to those used in Sections 2 and 3, we show that the phase transition from a win for Maker to a win for Breaker occurs at $n = (d + o(1)) \lg N$.

2 Potential-Based Breaker's Strategies for the Bounded Coordinates Game

Consider the classic 3×3 tic-tac-toe game and the N^d tic-tac-toe games. Each direction-vector $\vec{v}$ in those games has the property that the magnitude of each of its coordinates is at most 1, i.e., if $\vec{v} = (v_1, \ldots, v_d)$ is a direction-vector, then $|v_i| \leqslant 1$ for $1 \leqslant i \leqslant d$. A logical generalization of this game is to consider a tic-tac-toe game played on $\mathbb{Z}^d$ where the set of direction-vectors $\mathcal{S}$ contains direction-vectors (i.e., vectors $\vec{v}$ with $\gcd(\vec{v}) = 1$) whose coordinates satisfy $|v_i| \leqslant k$ for $1 \leqslant i \leqslant d$. In this section we prove two theorems that give criteria for Breaker to have an explicit winning strategy in the *bounded coordinates game*, i.e., $MB^d_{\mathcal{S}}(m)$ where $\mathcal{S} \subset \{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$. Both theorems generalize techniques used by Beck [1].

The first theorem states that Breaker can block lines whose lengths are on the order of $d^2 \lg(kd)$ (as $k, d \to \infty$) in the bounded coordinates game, and relies directly on the Erdős–Selfridge theorem [2]. The Erdős–Selfridge theorem says that for a finite hypergraph $\mathcal{H}$, if $\sum_{A \in \mathcal{H}} 2^{-|A|} < \frac{1}{2}$, then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{H}$. During each turn, the Erdős–Selfridge strategy assigns a "potential" (based on a power-of-2 scoring function) to the current position of the board. This potential measures Maker's ability to win from that position, and the Erdős–Selfridge strategy requires Breaker to occupy a point that will destroy the most potential.

Our second theorem, which states that Breaker can block lines whose lengths are on the order of $d^2 \lg(k)$ (as $k, d \to \infty$) in the bounded coordinates game, uses a more complicated approach developed by Beck and it also makes use of a potential-based technique. Both theorems only apply to finite games, so first we describe how Breaker can win the semi-infinite game $MB^d_{\mathcal{S}}(m)$ where $\mathcal{S} \subset \{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$ by essentially winning an infinite number of finite games.

We begin by partitioning the board $\mathbb{Z}^d$ into sub-boards which are d-dimensional hypercubes. Each hypercube has size $(mk)^d$, where m, the length of each winning line, will be determined later. On each sub-board B, we create a finite game whose set of winning lines $\mathcal{F}_B$ is defined so that $\mathcal{F}_B$ is $\left(\frac{m}{d+1}\right)$-uniform, and so that if Breaker wins on each sub-board, then he wins in the whole semi-infinite game. Each $A \in \mathcal{F}_B$ will be determined by a point–direction-vector pair $(\vec{p}, \vec{v})$ as follows: for each point–direction-vector pair $(\vec{p}, \vec{v}) \in B \times \mathcal{S}$, we let $A = \{\vec{p}, \vec{p}+\vec{v}, \vec{p}+2\vec{v}, \ldots, \vec{p}+\left(\frac{m}{d+1}-1\right)\vec{v}\}$ be an edge in $\mathcal{F}_B$ if $A \subset B$. Then during each turn, in whichever sub-board Maker occupies a point, Breaker responds in that same sub-board according to his blocking-strategy for the finite game in that sub-board. We will show that if m is large enough, then Breaker can win in every sub-board, and by extension he will win in the *entire* semi-infinite game $MB^d_{\mathcal{S}}(m)$.

First notice that a winning line can intersect at most $d+1$ different sub-boards. This is because each "switch" from one sub-board to the next is due to a progression in at least one of the dimensions. There are only d dimensions, so there can be at most d switches. If there were $d+1$ switches, then the line would have two switches in the same dimension, which would imply it covered enough distance in that dimension to completely cross a sub-board. However, each winning line has m points, and therefore only takes $m-1$ "steps" from the first point in the line to the last point. If a line has direction-vector $\vec{v} = (v_1, \ldots, v_d)$, then in the dimension corresponding to the j^{th} coordinate, the distance *in that dimension* between the first point of the line and the last point of the line is $(m-1)|v_j|$. However, $(m-1)|v_j| < mk$ for $1 \leqslant j \leqslant d$; thus a winning line cannot have two switches in the same coordinate.

Since a winning line A can intersect at most $d+1$ different sub-boards, it must intersect some sub-board in at least $|A|/(d+1)$ points. If A intersects sub-board B in at least $\frac{m}{d+1}$ points, then $A \cap B \supseteq A'$ for some $A' \in \mathcal{F}_B$, and Breaker will eventually occupy a point of A', via his blocking-strategy for the finite game played on B. Thus, Breaker will eventually occupy a point of A. Therefore, in the proofs of Theorems 1 and 2, we determine the length m of the winning-lines which allows Breaker to win on each of the individual sub-boards by using his finite-game blocking-strategy.

Theorem 1 *In the bounded coordinates version of $MB^d_{\mathcal{S}}(m)$, i.e., when $\mathcal{S} \subset \{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$, if $k, d \to \infty$ or if $d \to \infty$ and k is fixed, then Breaker has an explicit winning strategy if $m = (2+o(1))d^2 \lg(dk)$. If d is fixed and $k \to \infty$, Breaker has an explicit winning strategy if $m = (2+o(1))d(d+1)\lg(k)$.*

Proof: As described above, we partition the board $\mathbb{Z}^d$ into sub-boards of size $(mk)^d$. The Erdős–Selfridge theorem says that for a finite hypergraph $\mathcal{H}$, if $\sum_{A \in \mathcal{H}} 2^{-|A|} < \frac{1}{2}$, then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{H}$. For an arbitrary sub-board B, consider the game played on the finite hypergraph $(B, \mathcal{F}_B)$. Each $A \in \mathcal{F}_B$ satisfies $|A| = \frac{m}{d+1}$. We bound $|\mathcal{F}_B|$ from above by $(mk)^d \frac{(2k+1)^d}{2}$, which is an upper bound on $|B \times \mathcal{S}|$. (In this over-count, we have counted vectors $\vec{v}$ such that $\gcd(\vec{v}) > 1$, but we have avoided counting both $\vec{v}$ and $-\vec{v}$.) So if we find a value of m so that

$$\sum_{A \in \mathcal{F}_B} 2^{-|A|} \leqslant (mk)^d \cdot \frac{(2k+1)^d}{2} \cdot 2^{-m/(d+1)} < 1/2,$$

then Breaker will have an explicit winning strategy for the game played on the sub-board B. We see that the right-hand inequality is equivalent to

$$d(d+1)[\lg(m)+\lg(2k^2+k)] < m.$$

which will be satisfied if either

$$m > (2+o(1))d^2\lg(kd) \text{ as } k, d \to \infty, \text{or as } d \to \infty \text{ and } k \text{ is fixed,}$$

or

$$m > (2+o(1))d(d+1)\lg(k) \text{ as } k \to \infty, \text{ and } d \text{ is fixed.}$$

■

If we use a more sophisticated blocking-strategy for Breaker, then we can improve our results from Theorem 1 by a factor of 2 when d is fixed and $k \to \infty$, and we have an even bigger improvement in the cases where $d \to \infty$ since we eliminate the $\lg(d)$ term.

Theorem 2 *In the bounded coordinates version of $MB^d_{\mathcal{S}}(m)$, i.e., when $\mathcal{S} \subset \{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$, if $k \to \infty$ and $d \to \infty$, then Breaker has an explicit winning strategy if $m = (1+o(1))d^2\lg(k)$. If d is fixed and $k \to \infty$, Breaker has an explicit winning strategy if $m = (1+o(1))d(d+1)\lg(k)$. If $k \geqslant 2$ is fixed and $d \to \infty$, Breaker has an explicit winning strategy if $m = d^2(\lg k + 3c\sqrt{\lg k} + 1 + \frac{1}{k} + o(1))$, where $0 < c < \frac{1}{2}$ is an arbitrary constant.*

Proof: Our proof proceeds exactly like the proof of Theorem 1, except instead of using the Erdős–Selfridge theorem on each sub-board of size $(mk)^d$, we use a modified form of Theorem 34.1 from Beck [1] on these sub-boards.

Theorem 3 (Beck [1]) *Let $\mathcal{F}$ be an n-uniform Almost Disjoint hypergraph. Assume that the Maximum Degree of $\mathcal{F}$ is at most D, that is, every point of the board is contained in at most D hyperedges of $\mathcal{F}$. Moreover, assume that the total number of winning sets is $|\mathcal{F}| = M$. If there is an integer ℓ with $2 \leqslant \ell \leqslant n/2$, such that*

$$M\binom{n(D-1)}{\ell} < 2^{n\ell-\ell(\ell+1)-\binom{\ell}{2}-1}, \tag{1}$$

then Breaker has an explicit winning strategy for the game played on $\mathcal{F}$.

Notice that Theorem 3 is about almost disjoint hypergraphs, where a hypergraph $\mathcal{F}$ is called *almost disjoint* if $|A \cap B| \leqslant 1$ for all distinct $A, B \in \mathcal{F}$. Each hypergraph from a sub-game of $MB^d_{\mathcal{S}}(m)$ played on an $(mk)^d$ sub-board is close to being almost disjoint. However, two intersecting hyperedges with the same direction-vector will often intersect in more than one point. To handle this complication, we modify the proof of Theorem 3 and determine that Breaker can win on such a game if

$$M\binom{n(D-1)}{\ell} < 2^{n\ell-\ell(\ell+3)-\binom{\ell}{2}-1}. \tag{2}$$

Since the proof of Theorem 3 is rather difficult, we will describe how to appropriately modify the proof in order to obtain our result. In the following discussion, the reader should think of $\mathcal{F}$ as the set of winning sets $\mathcal{F}_B$ of a particular sub-game on a particular sub-board B where $n = \frac{m}{d+1}$.

Theorem 3 follows a *BigGame–SmallGame decomposition* where, in Breaker's mind, during each turn, the game is partitioned into two games, so that the BigGame shrinks and the SmallGame grows as the whole game progresses. Initially, the vertex set of the BigGame is all of $V(\mathcal{F})$ and the vertex set of the SmallGame is empty. We let $V_{\text{BIG}}(j)$ and $V_{\text{small}}(j)$ denote the set of vertices of the BigGame and SmallGame, respectively, for the time occurring immediately after Maker's jth move up until her $(j+1)$th move. We let V_{BIG} and V_{small} (with no index) be the vertices in the BigGame and SmallGame, respectively, at the end of the game. Breaker follows the self-imposed rule that whenever Maker occupies a point from the BigGame, Breaker responds in the BigGame; however, whenever Maker occupies a point from the SmallGame, Breaker allows himself to respond in either the BigGame or the SmallGame. We let $X(j)$ and $Y(j)$ be the sets of points that Maker has occupied and Breaker has occupied, respectively, by the end of turn j. We let $X_{\text{BIG}}(j), X_{\text{small}}(j), Y_{\text{BIG}}(j), Y_{\text{small}}(j)$ denote the sets of points that Maker has occupied from the BigGame, Maker has occupied from the SmallGame, Breaker has occupied from the BigGame, and Breaker has occupied from the SmallGame, respectively, by the end of turn j. (We use the phrasing, "by the end of turn j," to emphasize that the sets of Maker's points are updated immediately after her move during turn j, which occurs halfway through turn j.) Breaker essentially uses the BigGame as a device for positioning himself to win the SmallGame. The SmallGame is essentially where Breaker focuses on blocking any surviving sets which Maker is close to completely occupying.

A set $A \in \mathcal{F}$ becomes *dangerous* during turn j if the following three criteria occur:

1. A is a survivor set at the end of turn $j-1$, i.e., $A \cap Y(j-1) = \emptyset$.

2. By the end of turn j, Maker occupies all except for $\ell+3$ points of A in the BigGame, i.e., $|A \setminus X_{\text{BIG}}(j)| = \ell + 3$.

3. At the end of turn $j-1$, Maker did not occupy all except for $\ell+3$ points of A in the BigGame, i.e., $|A \setminus X_{\text{BIG}}(j-1)| > \ell + 3$.

If a set $A \in \mathcal{F}$ becomes dangerous during turn j, then immediately after Maker's move during turn j (i.e., before Breaker's move during turn j), the set $E = A \setminus X_{\text{BIG}}(j)$ is classified as an *emergency set* (in the SmallGame), the set A is labeled a *dangerous ancestor* of E, and the vertices of E are moved to the SmallBoard, i.e., $V_{\text{small}}(j) := V_{\text{small}}(j-1) \cup E$ and $V_{\text{BIG}}(j) := V_{\text{BIG}}(j-1) \setminus E$. We stress that this is the only way that points move to the SmallBoard (and points never move back to the BigBoard). Thus, every point $p \in V_{\text{small}}$ is brought into the SmallBoard by some emergency set E, which we call a *small-parent* of p. And just as every point $p \in V_{\text{small}}$ has at least one small-parent, we note that every emergency set E has at least one dangerous ancestor. Now that we have described how points are moved to the SmallBoard, we describe small sets in general.

Let $A \in \mathcal{F}$. Suppose there exists a $j \in \mathbb{Z}^+$ such that $A \cap Y(j-1) = \emptyset$ and $|A \cap V_{\text{small}}(j)| \geqslant \ell + 3$. Let i be the minimum such j. Then the set $S = A \cap V_{\text{small}}(i)$ is classified as a *small set* immediately after Maker's move during turn i and remains a small set for the duration of the game, and we call A an *ancestor* of S. We note that a small set may have many ancestors, but we will stipulate that there are not repeated edges in the SmallGame. Now we describe the BigGame and what constitutes a win for Maker in each of the two games.

An ℓ-element subfamily $\mathcal{G} = \{A_1, \ldots, A_\ell\} \subseteq \mathcal{F}$ is called *almost-disjointly-linked* if $|A_i \cap A_j| \leqslant 1$ for $1 \leqslant i < j \leqslant \ell$ and there exists a set $A \in \mathcal{F} \setminus \mathcal{G}$ such that $|A \cap A_i| = 1$ for $1 \leqslant i \leqslant \ell$. A *big set* $B = \bigcup_{A \in \mathcal{G}} A$ is a set where $\mathcal{G}$ is an almost-disjointly-linked ℓ-element subfamily of $\mathcal{F}$. Maker wins the BigGame if at some turn j, there is some big set B such that $B \cap X_{\text{BIG}}(j)$ contains all but $\ell(\ell+3)$ points of B and $B \cap Y_{\text{BIG}}(j-1) = \emptyset$. Maker wins the SmallGame if at some turn j, there is a small set S such that $S \subseteq X_{\text{small}}(j)$. We will show that if Breaker wins the BigGame, then he can follow a simple strategy to win the SmallGame. Moreover, if Breaker wins the SmallGame, then he wins the overall game.

In order to describe how winning the BigGame allows Breaker to win the SmallGame, we need to make some more observations and definitions. For a set $A \in \mathcal{F}$ with direction-vector $\vec{v}$, we call $D(A) = \vec{v}$ the *direction-vector* of A. Likewise, for a small set S with direction-vector $\vec{v}$, we call $D(S) = \vec{v}$ the *direction-vector* of S. We note that for a point $p \in V_{\text{small}}$, every small-parent of p has the same direction-vector. Indeed, let E_1 and E_2 both be small-parents of p, and consider the turn j when p joins the SmallBoard. Let $x_j \in V_{\text{BIG}}$ be the point occupied by Maker during turn j. Since E_1 and E_2 are small-parents of p, not only do they both contain p, but they both became emergency sets during turn j; thus their respective dangerous ancestors A_1 and A_2 both contain the points x_j and p. Since A_1 and A_2 share at least two points, we must have $D(A_1) = D(A_2)$. But since $E_i \subseteq A_i$ and $|E_i| = \ell + 3$ for $i \in \{1, 2\}$, we also have that each emergency set shares more than two points with its dangerous ancestor and therefore $D(E_1) = D(A_1) = D(A_2) = D(E_2)$.

Since every small-parent of $p \in V_{\text{small}}$ has the same direction-vector, we define the *direction-vector* of p to be the direction-vector of any of the small-parents of p, i.e., if $p \in V_{\text{small}}$ and E is a small-parent of p with $D(E) = \vec{v}$, then $D(p) = \vec{v}$ also. For each small set S, we call $p \in S$ a *same-direction-vector point* for S if $D(p) = D(S)$. We let $C_S = \{p \in S : D(p) = D(S)\}$ be the set of same-direction-vector points of S and we let $N_S = \{p \in S : D(p) \neq D(S)\}$ be the points of S whose direction-vector is different from that of S, i.e., $N_S = S \setminus C_S$. We now give an important lemma which states that if Breaker wins the BigGame, then $|N_S|$ is not too large. The proof hinges on the idea that if Breaker wins the BigGame, then there cannot be ℓ dangerous sets that are "linked" by a single set.

Lemma 1 *If Breaker wins the BigGame, then each small set S contains at most $\ell - 1$ points whose direction-vector does not match that of S, i.e., for each small set S, $|N_S| \leqslant \ell - 1$.*

Proof of Lemma 1: Assume towards a contradiction that $|N_S| \geqslant \ell$, and let $p_1, \ldots, p_\ell \in$

N_S. Let E_i be a small-parent of p_i and A_i a dangerous ancestor of E_i for $1 \leqslant i \leqslant \ell$. Let A be an ancestor of S. Since E_i is a small-parent of p_i and $p_i \in N_S$, we know that

$$D(A_i) = D(E_i) \neq D(S) = D(A).$$

Since $D(A_i) \neq D(A)$, then $|A_i \cap A| \leqslant 1$ for $1 \leqslant i \leqslant \ell$. Moreover, since $p_i \in A_i \cap A$, we have $|A_i \cap A| = 1$ for $1 \leqslant i \leqslant \ell$. It should also be rather obvious that for $i \neq j$, $|A_i \cap A_j| \leqslant 1$. For suppose $|A_i \cap A_j| \geqslant 2$ with $i \neq j$. Since each $A' \in \mathcal{F}$ is a subset of a line segment in $\mathbb{R}^d$, then we must have that A_i, A_j, and $A_i \cup A_j$ are all subsets of the same line segment. Since A and $A_i \cup A_j$ both contain the two points p_i and p_j, this implies that $D(A) = D(A_i \cup A_j)$, which is a contradiction since $D(A_i \cup A_j) = D(A_i) = D(A_j)$. Therefore, $|A_i \cap A_j| \leqslant 1$ for $i \neq j$.

So we have established that $\mathcal{G} = \{A_1, \ldots, A_\ell\}$ is a family of sets that is almost-disjointly-linked by A. Thus, $B = \bigcup_{i=1}^{\ell} A_i$ is a big set. Since $E_i \subseteq A_i$ is an emergency set for $1 \leqslant i \leqslant \ell$, we know that $|A_i \setminus X_{\mathrm{BIG}}| = \ell + 3$ for $1 \leqslant i \leqslant \ell$. The set $B \setminus X_{\mathrm{BIG}}$ contains those points of B that are not occupied by Maker, and

$$|B \setminus X_{\mathrm{BIG}}| \leqslant \sum_{i=1}^{\ell} |A_i \setminus X_{\mathrm{BIG}}| = \ell(\ell + 3),$$

i.e., Maker occupied all except for at most $\ell(\ell + 3)$ points of B and therefore won the BigGame, which is a contradiction to the fact that Breaker wins the BigGame. Therefore by contradiction, it must be the case that each small set S contains at most $\ell - 1$ points whose direction-vector does not match that of S. $\square$

Since each small set S satisfies the property that $|S| \geqslant \ell + 3$, an obvious corollary of Lemma 1 is the following:

Corollary 1 *If Breaker wins the BigGame, then each small set S contains at least four same-direction-vector points, i.e., for each small set S, $|C_S| \geqslant 4$.*

To help describe Breaker's strategy for stopping Maker from fully occupying a small set, we impose an ordering on the points of each geometric line with direction-vector $\vec{v} \in \mathcal{S}$ that passes through the game board as follows. Fix a geometric line with direction-vector $\vec{v} = (v_1, \ldots, v_d) \in \mathcal{S}$ that passes through the game board. Let v_i be the first non-zero coordinate of $\vec{v}$. If p and q are points on this line, then we say that $p < q$ if the i^{th} coordinate of p is less than the i^{th} coordinate of q. Moreover, if $p < q$ according to this ordering, we say that p is *to the left* of q, and q is *to the right* of p. In a similar vein, if p and q are vertices of the game board such that $p < q$, then we define the *interval* $[p, q]$ to be the intersection of the geometric line segment connecting p and q and the vertices of the game board. Naturally, $(p, q] = [p, q] \setminus \{p\}$, and $[p, q) = [p, q] \setminus \{q\}$, and $(p, q) = [p, q] \setminus \{p, q\}$.

We now describe Breaker's strategy, which we call the *Nearest-Neighbor-Strategy*, for stopping Maker from winning in the SmallGame. Suppose that Maker occupies a point p from the SmallBoard. Consider the geometric line with direction-vector $D(p)$ that contains p and consider the largest Breaker-free interval I (as defined above) that contains

p and is a subset of that line. Breaker responds by taking an unoccupied point $q \in I$ (from the board) whose distance to p is minimum, i.e., the closest unoccupied point in I. (In the case of two points with minimum distance from p, Breaker will take the point to the left of p. If there are no unoccupied points in I, then Breaker takes a random point.) We should again emphasize that q may be in either the BigBoard or the SmallBoard. If q is in the SmallBoard, the idea is to block small sets that contain both p and q, whereas if q is in the BigBoard, the idea is to block the ancestors of the small sets that contain both p and q, thus preventing such small sets from ever existing. We say that Breaker *stops* a small set S if he either directly occupies a vertex of S or he prevents S from existing by occupying a point contained in every ancestor of S. We claim that if Breaker wins the BigGame and follows this strategy, then he prevents Maker from fully occupying a small set.

Lemma 2 *If Breaker wins the BigGame and he uses the Nearest-Neighbor-Strategy described above, then he also wins the SmallGame.*

Proof of Lemma 2: Suppose that Breaker follows a strategy that allows him to win the BigGame and that he follows the Nearest-Neighbor-Strategy described above. Assume towards a contradiction that Maker fully occupies a small set S. By Corollary 1, every small set contains at least four same-direction-vector points. Suppose that the same-direction-vector points of S are labeled $p_1, p_2, \dots, p_s$ (where $s \geqslant 4$) and appear from left to right in that order. Let us consider the first "interior" same-direction-vector point in S that Maker occupies, i.e., the first p_i that Maker occupies where $2 \leqslant i \leqslant s-1$. We may assume that when Maker occupies her first interior same-direction-vector point, Breaker has not yet occupied any points that would have already stopped S (i.e., blocked S or prevented it from ever existing).

<u>**Case 1:**</u> *When Maker occupies p_i, both p_{i-1} and p_{i+1} are unoccupied.*

For this case, the unoccupied point closest to p_i is certainly contained in $[p_{i-1}, p_{i+1}]$, and therefore Breaker will respond with a point in the interval $[p_{i-1}, p_{i+1}]$ and will stop S at the end of this turn.

<u>**Case 2:**</u> *When Maker occupies p_i, one of p_{i-1} or p_{i+1} is already occupied by Maker.*

Notice that for this case to hold, since p_i is the first interior point that Maker occupies, either $i = 2$ or $i = s-1$. Without a loss of generality, we may assume that $i = 2$ and Maker already occupies p_1. Let q be the point Breaker occupied in response to p_1. If $q \in (p_1, p_2)$, then $q \in S$ (or q is in every ancestor of S) and Breaker has stopped S. Thus, we may assume that q is to the left of p_1 and $q \notin S$ (or q does not stop S). However, based on how Breaker chose q, it is the closest unoccupied point to the left of p_1, and therefore when Maker occupies p_2, there are no unoccupied points in the interval $[q, p_1]$. Thus, when Breaker goes to choose his response to p_2, he is forced to pick a point in the interval $(p_1, p_3]$ because he must pick an unoccupied point from a Breaker-free interval that contains p_2. Since p_2 is the first interior same-direction-vector point that Maker occupied (and we assumed that Breaker has not occupied any points that would stop S yet) we know that p_3 is unoccupied (because $s \geqslant 4$ implies p_3 is interior); therefore Breaker will have a point in S (or every ancestor of S) at the end of this turn.

In both Case 1 and Case 2 we reach a contradiction; therefore it must be the case that Maker cannot fully occupy a small set. □

It should be clear that if Maker were to fully occupy all of the vertices of a set $A \in \mathcal{F}$, then she would have to occupy all of the vertices of a small set $S \subseteq A$. This gives the following obvious corollary of Lemma 2:

Corollary 2 *If Breaker can win the BigGame, then he can block every winning set $A \in \mathcal{F}$ in either the BigGame or in the SmallGame.*

Thus, we are left to show that Breaker has a winning strategy in the BigGame. We do so via a lemma that is similar to the Erdős–Selfridge theorem.

Lemma 3 *(Under the assumption that inequality (2) holds) Breaker has a winning strategy in the BigGame.*

Proof: We use the following version of a potential-based lemma used in Beck's proof of Theorem 3. This lemma can be found as Lemma 1 in Section 35 of [1].

> **Beck's Lemma**: Breaker has a winning strategy in the BigGame if the number of big sets is less than 2^{b-1}, where b is a lower bound on the number of points Maker must occupy from a big set B before Breaker occupies his first point in B.

Let us find a value for b as well as an upper bound on the number of big sets.

Recall that Breaker wins the BigGame if for each big set B, he can occupy a point from B before Maker occupies all but $\ell(\ell+3)$ points of B. Let us find a lower bound on the size of each big set B. Let $B = \bigcup_{i=1}^{\ell} A_i$ be a big set, where $\mathcal{G} = \{A_1, \ldots, A_\ell\}$ is an almost-disjointly-linked ℓ-element subfamily of $\mathcal{F}$ and each $A_i \in \mathcal{F}$. Then we have

$$|B| = \left|\bigcup_{i=1}^{\ell} A_i\right| \geqslant \sum_{i=1}^{\ell} |A_i| - \sum_{1 \leqslant i < j \leqslant \ell} |A_i \cap A_j| \geqslant n\ell - \binom{\ell}{2},$$

since $|A_i \cap A_j| \leqslant 1$ for each pair of elements in the almost-disjointly-linked ℓ-element subfamily. Since $|B| \geqslant n\ell - \binom{\ell}{2}$ for each big set B, we know that in order for Maker to win the BigGame, she must occupy at least $n\ell - \binom{\ell}{2} - \ell(\ell+3)$ points of some big set B before Breaker occupies his first point in B. Thus, in our case, we have $b = n\ell - \binom{\ell}{2} - \ell(\ell+3)$.

Now let us get an upper bound on the number of big sets in the BigGame. We can do this by first selecting a set $A \in \mathcal{F}$, then choosing an ℓ-element family almost-disjointly-linked by A. There are $M = |\mathcal{F}|$ choices for A. Since each point of $\mathcal{F}$ is in at most D hyperedges, each point of A will be in at most $D-1$ other hyperedges. Thus, there are at most $n(D-1)$ hyperedges which intersect A, giving at most $\binom{n(D-1)}{\ell}$ choices for an ℓ-element family almost-disjointly-linked by A. Therefore we have the following inequality:

$$\#\text{ of Big Sets } \leqslant M\binom{n(D-1)}{\ell}. \tag{3}$$

So using our bound from inequality (3) and our initial assumption in inequality (2), we have

$$\# \text{ of Big Sets } \leqslant M\binom{n(D-1)}{\ell} < 2^{n\ell-\ell(\ell+3)-\binom{\ell}{2}-1} = 2^{b-1},$$

so the hypothesis of Beck's Lemma is satisfied and Breaker has a winning-strategy for the BigGame. □

Let us now show what values of m allow inequality (2) to hold, depending on whether $k \to \infty$ and/or $d \to \infty$.

As stated in the proof of Theorem 1, an upper bound on $|B\times\mathcal{S}|$ is $(mk)^d\frac{(2k+1)^d}{2}$, which in turn is an upper bound on $M = |\mathcal{F}_B|$.

Since $\mathcal{F}_B$ is $\left(\frac{m}{d+1}\right)$-uniform, we have $n = \frac{m}{d+1}$. We also note that the maximum-degree D satisfies $D \leqslant n|\mathcal{S}| \leqslant \frac{m}{d+1}\frac{(2k+1)^d}{2}$, since for each point and each slope, the number of lines of length n going through that point with that slope is at most n. Therefore we can bound the binomial coefficient in inequality (2) as follows:

$$\binom{n(D-1)}{\ell} \leqslant \left(\frac{\left(\frac{m}{d+1}\right)^2\frac{(2k+1)^d}{2}e}{\ell}\right)^\ell \leqslant \left(\frac{m^2(2k+1)^d e}{2d^2\ell}\right)^\ell.$$

Thus, it is enough to show that

$$(mk)^d\frac{(2k+1)^d}{2}\left(\frac{m^2(2k+1)^d e}{2d^2\ell}\right)^\ell < 2^{(m\ell)/(d+1)-\ell(\ell+3)-\binom{\ell}{2}-1}, \tag{4}$$

for some integer ℓ with $2 \leqslant \ell \leqslant \frac{m}{2(d+1)}$ in order to ensure that Breaker will win on $\mathcal{F}_B$. By applying the binary logarithm to both sides of inequality (4) and simplifying, we get

$$\begin{aligned} d&\left[\lg m + 1 + 2\lg(k) + \lg\left(1+\frac{1}{2k}\right)\right] \\ &+ \ell\left[2\lg m + d + d\lg k + d\lg\left(1+\frac{1}{2k}\right) - 1 + \lg e - 2\lg d - \lg \ell\right] \\ &< \frac{\ell}{d+1}m - \frac{3}{2}\ell^2 - \frac{5}{2}\ell, \end{aligned} \tag{5}$$

Using the fact that $\lg(1+\frac{1}{2k}) < \frac{1}{k}$ and further simplification of (5) yields,

$$\begin{aligned} (d+1)\Bigg(&d\lg k + d + \frac{d}{k} + \frac{d\lg m}{\ell} + \frac{2d\lg k}{\ell} + \frac{d}{\ell} + \frac{d}{k\ell} + 2\lg m - \lg\ell - 2\lg d \\ &+\frac{3}{2}\ell + \lg e + \frac{3}{2}\Bigg) < m. \end{aligned} \tag{6}$$

If we let $\ell = \lfloor c\sqrt{m}\rfloor$ (where $0 < c < \frac{1}{2}$ is constant), then inequality (6) is satisfied when

$$m > d^2\lg k(1+o(1)) \quad \text{as } d, k \to \infty,$$

or

$$m > (d+1)d \lg k(1+o(1)) \quad \text{as } k \to \infty, \text{ and } d \text{ is fixed,}$$

or

$$m > d^2(\lg k + 3c\sqrt{\lg k} + 1 + \frac{1}{k} + o(1)) \quad \text{as } d \to \infty, \text{ and } k \geqslant 2 \text{ is fixed.}$$

■

3 Maker's Strategy for the Full Bounded Coordinates Game

The game we consider in this section is Maker–Breaker tic-tac-toe on $\mathbb{Z}^d$ with direction-vector set $\mathcal{S} = \langle\{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}\rangle$. (The $\langle\cdot\rangle$ notation will mean start with the set $\{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}$, then throw out any vectors $\vec{v}$ with $\gcd(\vec{v}) > 1$, then from what is left throw out one of $-\vec{v}$ or $\vec{v}$.) Thus, we are considering $MB^d_{\mathcal{S}}(m)$ with $\mathcal{S}$ as defined above, i.e., the *full bounded coordinates* game.

Theorem 4 *In the full bounded coordinates version of $MB^d_{\mathcal{S}}(m)$, i.e., when $\mathcal{S} = \langle\{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}\rangle$, if $k \to \infty$, then Maker has a winning strategy if $m = (1+o(1))d\lg(k)$. If k is fixed and d $\to \infty$, Maker has a winning strategy if $m = (1+o(1))d\lg(\frac{2}{3}k)$.*

Proof: We use Theorem 1.2 from Beck [1], which states that Maker has a winning strategy for the Maker–Breaker game played on an m-uniform, finite hypergraph $(V, \mathcal{H})$ if

$$|\mathcal{H}| > 2^{m-3} \cdot \Delta_2(\mathcal{H}) \cdot |V|, \tag{7}$$

where $\Delta_2(\mathcal{H})$ is the max-pair degree. The max-pair degree is defined as follows:

> assume that, fixing any two distinct points of the board V, there are at most $\Delta_2(\mathcal{H})$ winning sets $A \in \mathcal{H}$ containing both points, and equality occurs for some point pair, then we call $\Delta_2(\mathcal{H})$ the *max-pair degree*.

However, in order to use this theorem, we must restrict Maker's moves (and strategy) to a finite hypergraph $\mathcal{F}$ which is defined so that a win in $\mathcal{F}$ constitutes a win in the original semi-infinte game. Since Maker doesn't have to block any of Breaker's lines, whenever Breaker occupies a point outside of $\mathcal{F}$, Maker just takes a random point in $\mathcal{F}$.

Let us define our hypergraph $(V, \mathcal{F})$. The set of vertices is a $(3mk)^d$ hypercube in $\mathbb{Z}^d$, where m (to be determined later) is the length of a winning line in $\mathcal{F}$. A winning line in $\mathcal{F}$ is determined by taking a point $\vec{p}$ in the $(mk)^d$ hypercube that is centered in the middle of V, and appending $m-1$ points in the direction of one of the direction-vectors $\vec{v} \in \mathcal{S}$ to get a winning line $\{\vec{p}, \vec{p}+\vec{v}, \ldots, \vec{p}+(m-1)\vec{v}\}$.

Now that we have defined $(V, \mathcal{F})$, let us find an upper bound on m which, if satisfied, will imply that inequality (7) is also satisfied and thus Maker has a winning strategy on $\mathcal{F}$. We will start by finding a lower bound on $|\mathcal{F}|$. As described above, $|\mathcal{F}| = |\mathcal{S}|(mk)^d$.

Since $|\mathcal{S}| = \frac{1}{2}|\{\vec{v} \in \mathbb{Z}^d : \|\vec{v}\| \leqslant k, \text{and } \gcd(\vec{v}) = 1\}|$, we can get a lower bound on $|\mathcal{S}|$ by first counting the number of vectors $\vec{v} = (v_1, v_2, \dots, v_d) \in \mathbb{Z}^d$ with $|v_i| \leqslant k$ satisfying all of the following: (i) $v_1, v_2 > 0$, (ii) $v_2 \leqslant v_1$, and (iii) $\gcd(v_1, v_2) = 1$. Since this only accounts for when v_1 and v_2 have the same sign, we can double this quantity to account for when v_1 and v_2 have opposite signs. Thus, $|\mathcal{S}| \geqslant 2 \cdot \Phi(k) \cdot (2k+1)^{d-2}$, where $\Phi(k) = \Sigma_{i=1}^{k} \phi(i)$ and ϕ is Euler's phi-function. By modifying the proof of Theorem 330 (regarding the average order of $\phi(k)$) in [3], it can be shown that $|\Phi(k) - \frac{3}{\pi^2}k^2| \leqslant 2k\ln(k) + 2k$, which easily yields $\Phi(k) \geqslant \frac{1}{5}k^2$ for all $k \in \mathbb{Z}^+$. So we may conclude that

$$|\mathcal{F}| \geqslant \frac{1}{10}(2k)^d \cdot (mk)^d.$$

Notice that $\Delta_2(\mathcal{F}) = m - 1$ because two lines can intersect in at least two points only if they are collinear. Therefore, it will be enough for us to find a value of m that satisfies the following inequality

$$\frac{1}{10}(2k)^d \cdot (mk)^d > 2^{m-3}m(3mk)^d. \tag{8}$$

By manipulating inequality (8) we get

$$m < d\lg k + d\lg\left(\frac{2}{3}\right) - \lg m + 3 + \lg\frac{1}{10}. \tag{9}$$

Inequality (9) is satisfied when

$$m < (1 + o(1))d\lg k \text{ as } k \to \infty$$

or

$$m < (1 + o(1))d\lg\left(\frac{2}{3}k\right) \text{ as } d \to \infty, \text{ and } k \text{ is fixed.}$$

■

4 The Maker–Breaker n-Consecutive Lattice Points Game on N^d.

In Chapter 14 of Beck [1], he introduces the n-consecutive lattice points game played on the $N \times N$ board $\{(a, b) \in \mathbb{Z}^2 : 0 \leqslant a, b \leqslant N - 1\}$. In the Maker–Breaker version of this game, Maker's goal is to occupy n consecutive lattice points on any straight line, where every rational slope is allowed. Beck is able to show in Theorem 14.1 of [1] that the phase transition from a win for Maker to a win for Breaker happens at $n = (2 + o(1))\lg N$. In order to do this, he uses Theorem 1.2 and a modification of Theorem 34.1 of [1]. Using a similar approach, we will prove the following theorem with respect to the n-consecutive lattice points game played on the d-dimensional analog of the $N \times N$ board:

Theorem 5 *In the Maker–Breaker game on the N^d board where Maker's goal is to occupy n consecutive lattice points on a line ("n-consecutive"), the phase transition from a win for Maker to a win for Breaker happens at $n = (d + o(1)) \lg N$.*

Proof: Our proof proceeds like the proofs of Theorems 2 and 4, except this time we are not tiling $\mathbb{Z}^d$ with hypercubes, rather we are considering a single d-dimensional hypercube $B = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : 0 \leqslant a_i \leqslant N-1 \text{ for } 1 \leqslant i \leqslant d\}$ and the set of direction-vectors $\mathcal{S} = \{\vec{v} \in \mathbb{Z}^d : \gcd(\vec{v}) = 1 \text{ and } \|\vec{v}\|_\infty \leqslant \frac{N}{n-1}\}$. Each $A \in \mathcal{F}$ will be of the form $\{\vec{p}, \vec{p}+\vec{v}, \vec{p}+2\vec{v}, \ldots, \vec{p}+(n-1)\vec{v}\}$ where $(\vec{p}, \vec{v})$ is a point–direction-vector pair $(\vec{p}, \vec{v})$. Note that we specify $\|\vec{v}\|_\infty \leqslant \frac{N}{n-1}$, because otherwise if there is a component v_i of $\vec{v}$, such that $|v_i| > \frac{N}{n-1}$, then for each $\vec{p} \in B$, the point $\vec{p} + (n-1)\vec{v}$ will not be in B. To help draw a parallel to our other proofs, we will let $k = \frac{N}{n-1}$.

We first show that Breaker has a winning strategy for the n-consecutive points game when $n = (d + o(1)) \lg N$. In Theorem 2, the hypergraph $\mathcal{F}$ was $\binom{m}{d+1}$-uniform and the size of the board was $(mk)^d$, but for this game, we have that $\mathcal{F}$ is n-uniform and the size of the board is N^d. Therefore, inequality (4) in the proof of Theorem 2 becomes

$$N^d \frac{(2k+1)^d}{2} \left(\frac{n^2 (2k+1)^d e}{2\ell} \right)^\ell < 2^{n\ell - \ell(\ell+3) - \binom{\ell}{2} - 1}, \tag{10}$$

where $2 \leqslant \ell \leqslant \frac{n}{2}$. By applying the binary logarithm to both sides of inequality (10) and simplifying, we get that Breaker wins the n-consecutive points game when

$$\frac{d}{\ell}\big(\lg N + \lg(2k+1) \big) + 2 \lg n + d \lg(2k+1) - \lg \ell + \frac{3}{2}\ell + \lg e + \frac{3}{2} < n. \tag{11}$$

If we let $\ell = \lfloor c\sqrt{n} \rfloor$ (with some positive constant c) and substitute in $k = \frac{N}{n-1}$, we get

$$\frac{d}{c\sqrt{n}} \left[\lg N + (c\sqrt{n} + 1) \lg \left(\frac{2N}{n-1} + 1 \right) \right] + 2 \lg n - \lg(c\sqrt{n}) + \frac{3}{2} c\sqrt{n} + \lg e + \frac{3}{2} < n.$$

This inequality is satisfied for $n = (d + o(1)) \lg N$ as $N \to \infty$.

To show that Maker has a winning strategy for the n-consecutive points game when $n = (d + o(1)) \lg N$, we will slightly modify the proof of Theorem 4. In particular, $|\mathcal{F}| \geqslant \frac{1}{10} (\frac{2N}{n-1})^d N^d$ and $|V| = N^d$ in inequality (7). Therefore, Maker wins the n-consecutive points game if n satisfies the inequality

$$\frac{1}{10} \left(\frac{2N}{n-1} \right)^d N^d > 2^{n-3} (n-1) N^d. \tag{12}$$

Inequality (12) is satisfied when $n = (d + o(1)) \lg N$ as $N \to \infty$. ∎

5 Open Problems

While we showed in Theorem 5 that the phase transition from a win for Maker to a win for Breaker happens at $n = (d + o(1)) \lg N$ for the n-consecutive lattice points game on N^d, we were unable to do so for the *full* bounded coordinates version of $MB^d_{\mathcal{S}}(m)$. Theorem 2 says that Breaker wins if $m = (1 + o(1))d(d + 1)\lg(k)$ when d is fixed and $k \to \infty$, while Theorem 4 states that Maker wins if $m = (1 + o(1))d\lg(k)$ when $k \to \infty$. This leaves a factor of $d + 1$ gap between the two results. We pose the following open problem.

Open Problem 1 *Determine the phase transition from a Maker's win to a Breaker's win for the full bounded coordinates version of $MB^d_{\mathcal{S}}(m)$, i.e., when $\mathcal{S} = \langle\{\vec{v} : \|\vec{v}\|_\infty \leqslant k\}\rangle$.*

In a previous paper [4], the authors proved that there is a pairing-strategy that allows Breaker to win $MB^d_{\mathcal{S}}(m)$ as long as $m \geqslant 3|\mathcal{S}|$, with no restrictions on the elements of $\mathcal{S}$. While this pairing-strategy performs much worse on the bounded-coordinates version of $MB^d_{\mathcal{S}}(m)$ than the potential-based techniques from Section 2 (we would need m on the order of $(2k)^d$ for the pairing-strategy to work), the pairing-strategy could outperform those potential-based techniques if $|\mathcal{S}|$ is very small, yet one of the direction-vectors $\vec{v} \in \mathcal{S}$ has huge coordinates, i.e., $\|\vec{v}\|_\infty$ is huge for some $\vec{v} \in \mathcal{S}$. Therefore, we pose the following open problem.

Open Problem 2 *Can the potential-based techniques from Section 2 be modified to produce a Breaker's winning-strategy for $MB^d_{\mathcal{S}}(m)$ where the bounds on the length m of the winning lines are a function of $|\mathcal{S}|$, rather than a function of* $\max\{\|\vec{v}\|_\infty : \vec{v} \in \mathcal{S}\}$*?*

Acknowledgements

The authors would like to thank József Beck for bringing the problems studied in this paper to our attention, and for numerous valuable suggestions and continued support. We also wish to thank Tamás Lengyel and the referee for making helpful suggestions and comments that improved the presentation of this paper.

References

[1] József Beck. Combinatorial Games: Tic-Tac-Toe Theory. Cambridge University Press, 2008.

[2] P. Erdős and J. L. Selfridge. On a combinatorial game. J. Combinatorial Theory Ser. A, 14:298–301, 1973.

[3] G. H. Hardy and E. M. Wright. An introduction to the theory of numbers. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman.

[4] Klay Kruczek and Eric Sundberg. A pairing strategy for tic-tac-toe on the integer lattice with numerous directions. Electron. J. Combin., 15(1):Note 42, 6, 2008.