Orthogonal Vector Coloring*

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Abstract

A vector coloring of a graph is an assignment of a vector to each vertex where the presence or absence of an edge between two vertices dictates the value of the inner product of the corresponding vectors. In this paper, we obtain results on orthogonal vector coloring, where adjacent vertices must be assigned orthogonal vectors. We introduce two vector analogues of list coloring along with their chromatic numbers and characterize all graphs that have (vector) chromatic number two in each case.

In this paper, we define and explore possible vector-space analogues of the list-chromatic number of a graph. The first section gives basic definitions and terminology related to graphs, vector representations, and coloring. Section 2 introduces vector coloring and the corresponding definitions of the list-vector and subspace chromatic numbers of a graph and presents some results and related problems. In the final section, we characterize all graphs that have chromatic number two in each case.

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1 Vector Coloring

We will assume that the reader is familiar with some of the more common definitions in graph theory and graph coloring. For a general introduction, the reader is encouraged to refer to Diestel's book [6] on graph theory or Jensen and Toft's book [11] on coloring problems.

Given a field $\mathbb{F}$, subsets S, A, B, and C of $\mathbb{F}$, a positive integer d, and a nondegenerate bilinear form $b(x,y)$ on $\mathbb{F}^d$, a *vector representation* [24] of a simple graph G with vertices $v_1,\ldots,v_n$ is a list of vectors $\vec{v}_1,\ldots,\vec{v}_n$ in $\mathbb{F}^d$ whose components are in S such that for all i and j, $b(\vec{v}_i,\vec{v}_i)\in A$, if v_i is adjacent to v_j in G then $b(\vec{v}_i,\vec{v}_j)\in B$, and if v_i is not adjacent to v_j in G then $b(\vec{v}_i,\vec{v}_j)\in C$.

Various choices of the parameters involved have led to many interesting questions and results using Euclidean spaces and inner products. For example, Lovász defines an *orthonormal representation* with $\mathbb{F}=\mathbb{R}=S=B$, $A=\{1\}$ and $C=\{0\}$ in his solution of the Shannon capacity of C_5 [20] and his characterization (with Saks and Schrijver) of k-connected graphs [17, 18]. See the survey by Lovász and Vesztergombi [19] for further information.

Given a particular type of vector representation and a graph G, one may ask what is the smallest dimension d that admits a vector representation of G. For example, the case where $\mathbb{F}=S=A$, $B=\{1\}$, and $C=\{0\}$ is treated by Alekseev and Lozin [1]. Such investigations have produced interesting results, such as when $\mathbb{F}=S=A$, $B\subseteq(-\infty,0)$, and $C=\{0\}$, it turns out [32] that the smallest dimension d that admits a vector representation of G depends on whether $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{Q}$. The *minimum semidefinite rank* of a simple graph [12] is the smallest d such that G admits a vector representation with $\mathbb{F}=\mathbb{C}=S=A$, $B=\mathbb{C}\setminus\{0\}$, and $C=\{0\}$.

Peeters [27] follows the lead of Lovász in noting connections between these "geometric dimensions" of a graph and the chromatic number of its complement. Others have explored connections between vector representations and coloring problems in minimum rank problems [29] and elsewhere: Karger, Motwani, and Sudan [13] define a *vector k-coloring* to be a vector representation with $\mathbb{F}=\mathbb{R}=S=A=B$ and $C=(-\infty,-1/(k-1)]$, a problem further studied by Feige, Langberg, and Schechtman [8].

Of these many different vector representations, we believe the orthogonal representations of Lovász and Peeters lend themselves best to an analogue of the list-chromatic number of a graph. However, orthogonal representations are traditionally defined in a manner opposite to graph coloring, and this can lead to confusion or the constant use of graph complements when trying to relate the two. To ameliorate this situation, we will adopt the coloring approach to the vector representation definition.

Definition 1.1. For a graph G, a *valid orthogonal k-vector coloring* over the field $\mathbb{F}$ of G is a vector representation of G with $\mathbb{F}=S=C$, $A=(0,\infty)$, $B=\{0\}$, and $d=k$. The *vector chromatic number* $\chi_v(G,\mathbb{F})$ is the least k so that G has a valid orthogonal k-vector coloring over $\mathbb{F}$.

From this point we will assume all vector colorings are orthogonal vector colorings as defined here and not those of Karger, Motwani, and Sudan. Note that replacing a vector in a valid vector coloring with a nonzero scalar multiple of that vector results in another valid vector coloring, so that χ_v would be unchanged if we took $A = \{1\}$ (although not having to normalize vectors in proofs and examples is convenient). However, we then claim unique choices of vector in proofs and examples when we really mean unique up to nonzero scalar multiple. Finally, we will only consider the fields $\mathbb{C}$ and $\mathbb{R}$ as choices for $\mathbb{F}$, and will often use $\chi_v(G)$ when results apply equally to $\chi_v(G, \mathbb{C})$ and $\chi_v(G, \mathbb{R})$.

A first attempt at finding a meaningful vector analogue for the list chromatic number might be to assign lists of vectors to each vertex. However, for any k, there exists a k-by-k unitary matrix with no zero entries (for example, the Fourier transform on the group $\mathbb{Z}/n\mathbb{Z}$ [31, 34]). Thus, for any k, by taking an orthonormal basis and its image under multiplication by such a unitary matrix it would always be possible to assign lists of size k to the complete graph on two vertices that would not admit a valid vector coloring. Similarly, two adjacent vertices of any graph could be used to create list assignments of arbitrary size that do not admit a valid vector coloring.

Instead we propose two definitions, the list-vector chromatic number and the subspace chromatic number. While it is not clear from the work in this paper which of these, if either, is the "right" analogue for χ_l, the results show that both are interesting invariants in their own right.

Definition 1.2. A graph G is *k-vector-choosable* over $\mathbb{F}$ if for every k-list assignment, where the elements in the lists are vectors from an orthonormal basis of $\mathbb{F}^n$ for some $n \geqslant k$, there exists a valid vector coloring using vectors from the span of each list. The smallest k such that G is k-vector-choosable is the *list-vector chromatic number*, $\chi_{lv}(G)$.

Definition 1.3. A *k-subspace assignment* for a graph G with $V(G) = \{v_1, \ldots, v_n\}$ is a list of subspaces $S_1, \ldots, S_n$ of $\mathbb{F}^d$ for some $d \geqslant n$ where each S_i has dimension k. Given a k-subspace assignment, a *valid vector coloring* of G is a valid coloring of G such that if vertex v_i is colored with $\vec{v}_i$ then $\vec{v}_i \in S_i$. A graph G is *k-subspace choosable* over $\mathbb{F}$ if every k-subspace assignment admits a valid vector coloring. The least such k for which G is k-subspace choosable is called the *subspace chromatic number*, $\chi_S(G, \mathbb{F})$.

Remark 1.4. A valid coloring of a graph G using colors $c_1, \ldots, c_d$ yields a valid vector coloring of G by replacing c_i with $\vec{e}_i$, where $\{\vec{e}_1, \ldots, \vec{e}_d\}$ is an orthonormal basis of $\mathbb{F}^d$. Thus $\chi_v(G) \leqslant \chi(G)$ and $\chi_{lv}(G) \leqslant \chi_l(G)$. If a graph G is k-vector choosable, then choosing a k-list assignment where all the lists are the same yields a valid vector-coloring of G. Thus $\chi_v(G) \leqslant \chi_{lv}(G)$. Further, if G is k-subspace choosable, then selecting a k-subspace assignment where all of the subspaces are the span of vectors from an orthonormal basis of $\mathbb{F}^d$ shows that G is also k-vector choosable, so that $\chi_{lv}(G) \leqslant \chi_S(G)$.

Many results from traditional graph coloring are equally applicable to vector coloring. In what follows, we will use χ_* to denote any of χ, χ_l, χ_v, χ_{lv}, or χ_S, although we will only provide proofs for the vector coloring invariants. Throughout the following, given a vector $\vec{v} \in \mathbb{F}^d$, we use $\vec{v}^\perp$ to denote the subspace of all vectors orthogonal to $\vec{v}$.

Proposition 1.5. *Let G be a graph and H a subgraph of G. Then $\chi_*(H) \leqslant \chi_*(G)$.*

Proof. The span of a valid vector coloring of G contains the span of a valid vector coloring of H. □

Proposition 1.6. *Let G be a graph and v a vertex of G. Then $\chi_*(G) \leqslant \chi_*(G-v)+1$. In particular, $\chi_*(G) \leqslant |G|$.*

Proof. Let $k = \chi_S(G-v)+1$. Given a k-subspace assignment for G, let S_w denote the subspace assigned to a vertex w. Choose a vector $\vec{v}$ in S_v and use the $\chi_S(G-v)$-subspace assignment for G obtained by replacing each S_w by $S_w \cap \vec{v}^\perp$. □

Corollary 1.7. *For any n, $\chi_*(K_n) = n$.*

Proof. A valid coloring (vector coloring) of K_n consists of n different colors (orthogonal vectors). □

Corollary 1.8. *For any graph G, $\omega(G) \leqslant \chi_*(G)$;*

Proposition 1.9. *For any graph G, $\chi_*(G) \leqslant \Delta(G)+1$;*

Proof. This is well-known for χ_l [10, pg. 345] as well as χ. Induct on $|G|$. The case $|G| = 1$ is trivial. Assume the statement is true for graphs with $k-1$ vertices, and let G have k vertices. Let $v \in V(G)$ and consider $G-v$. By the induction hypothesis, $\chi_S(G-v) \leqslant \Delta(G-v)+1 \leqslant \Delta(G)+1$. Assign to each vertex of G a subspace of dimension $\Delta(G)+1$ of $\mathbb{C}^d$ for some $d \geqslant \Delta(G)+1$. By definition, we can find a valid vector coloring for $G-v$ using these subspaces. Let the neighbors of v in G be colored with vectors $\vec{w}_1, \ldots, \vec{w}_j$. Let S be the subspace assigned to v. Since $\dim S = \Delta(G)+1$ and $j \leqslant \Delta(G)$,

$$\dim\left(S \cap \left(\bigcap_{i=1}^{j} \vec{w}_i^{\perp}\right)\right) \geqslant 1,$$

so there exists a valid choice of vector for v. □

The previous results are summarized in Figure 1. It is currently unknown whether any relationship exists between $\chi_{lv}(G)$ and $\chi(G)$, or between $\chi_S(G)$ and $\chi_l(G)$ or $\chi(G)$, although we conjecture that $\chi_{lv}(G) \geqslant \chi(G)$ and $\chi_S(G) \geqslant \chi_l(G)$.

Lemma 1.10. *Let v be a vertex of a graph G. If $\deg(v) < \chi_*(G)-1$, then $\chi_*(G-v) = \chi_*(G)$.*

Proof. Assume that $\chi_*(G-v) \leqslant \chi_*(G)-1$. Take a (minimal) valid (vector) coloring of $G-v$. To finish (vector) coloring G, we would need only to (vector) color v, which is adjacent to at most $\chi_*(G)-2$ vertices. However, we have $\chi_*(G)-1$ (dimensions) colors to pick a vector for v, showing that $\chi_*(G) \leqslant \chi_*(G)-1$, a contradiction. □

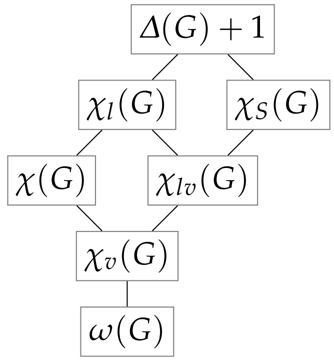


Figure 1: The chromatic numbers and their bounds

Proposition 1.11 (cf. Wallis [36, pg. 87]). *If a graph G satisfies $\chi_v(H) < \chi_v(G)$ for every proper subgraph H of G, then $\chi_v(G) \leqslant \delta(G) + 1$.*

Proof. Let $\chi_v(G) = n$ and let x be any vertex in G. Since $\chi_v(H) < \chi_v(G)$ for every proper subgraph H of G, $\chi_v(G - x) \leqslant n - 1$, so that there exists a valid vector coloring of $G - x$ in $\mathbb{F}^{n-1}$. Suppose by way of contradiction that $\deg(x) < n - 1$. Then Lemma 1.10 contradicts the fact that $\chi_v(G) = n$. Thus, $\deg(x) \geqslant n - 1$, but x was an arbitrary vertex, so that $\delta(G) \geqslant n - 1$ and $n = \chi_v(G) \leqslant \delta(G) + 1$. $\square$

Proposition 1.12 (cf. Thomassen [35]). *If G is a planar graph, then $\chi_S(G) \leqslant 5$.*

Proof. Because of Proposition 1.5, we can assume that adding any edge to G will result in a graph that is not planar (G is *plane triangulated*). Assign subspaces to each vertex, where S_v denotes the subspace assigned to vertex v assuming the following stricter conditions. Let B denote the boundary of G. Then

- If $v \in B$, then $\dim S_v \geqslant 3$.
- If v is not in B, then $\dim S_v = 5$.
- Assume we have already chosen 2 vectors for some 2 adjacent vertices on the boundary.

Following the third condition, say we assign $x \in B$ with vector α, and $y \in B$ is assigned vector β where $\langle \alpha, \beta \rangle = 0$.

We will now proceed by induction on $|G|$. We know that for $|G| = 3$, $\chi_S(G) \leqslant 5$. So assume that $\chi_S(G) \leqslant 5$ for $|G| < k$ and let $|G| = k$. Assume that the three additional conditions hold for G. We now have 2 cases, where G contains a chord and where it does not.

Suppose G has a chord, uv. Consider the 2 subgraphs G_1 and G_2 defined by this chord. Assume $x, y \in G_1$. Now by the induction hypothesis, we can find vectors for each vertex of G_1 to define a valid vector coloring. This assigns u and v valid vectors. Then G_2 now satisfies the three additional conditions, since u and v are on the boundary of G_2. By the induction hypothesis, there exists a valid vector coloring G_2 in $\mathbb{F}^5$. Then combining these colorings gives a valid vector coloring of G.

Assume then that G does not have a chord. Let $w - v_0 - x - y$ be a path on B. Without loss of generality, we can assume that $\alpha \in S_{v_0}$. Then $\operatorname{span}\{\alpha, \gamma, \delta\}$ is a subspace of S_{v_0} for some orthogonal γ, δ which are also orthogonal to α. Define $S'_{v_0} := \operatorname{span}\{\gamma, \delta\}$. Consider vertices $v_1, \ldots, v_t$ of G which are adjacent to v_0 but not on B. Then by the near-triangulation of G, we have that $w - v_1 - v_2 - \cdots - v_t - x$ is a path in G. Consider S_{v_i}, the subspace assigned to v_i. Note that $\dim S_{v_i} = 5$. Define $S'_{v_i} := S_{v_i} \cap S'^{\perp}_{v_0}$. Then anything in S'_{v_i} is orthogonal to δ and γ, and $S'_{v_i} \geqslant 3$. Consider $G - v_0$. This subgraph satisfies the additional conditions if we assign S'_{v_i} to each new boundary vertex v_i. Then by the induction hypothesis, this yields a valid 5-vector coloring of $G - v_0$. We are left now with only the task of assigning a vector to v_0. Let $\vec{w}$ be the vector assigned to w.

If $\langle \gamma, \vec{w} \rangle = 0$, we can assign γ to v_0. Similarly, if $\langle \delta, \vec{w} \rangle = 0$, we can assign δ to v_0. Otherwise, $\vec{w} = a_1\gamma + a_2\delta + \cdots$ for some scalars a_1, a_2. Then for v_0, we can pick a vector $b_1\gamma + b_2\delta$ such that $b_1a_1 + b_2a_2 = 0$ for nonzero b_1, b_2. Then we have found a valid coloring using subspaces of dimension 5, so $\chi_S(G) \leqslant 5$. □

1.1 Almost Orthogonal Vectors

We now give the vector equivalent of a well-known result of Gaddum and Nordhaus [22] that bounds the sum of the chromatic number of a graph and its complement.

Lemma 1.13. *Let G be a graph. Then $\chi_S(G) + \chi_S(\overline{G}) \leqslant |G| + 1$.*

Proof. We proceed by induction on $|G|$. If $|G| = 1$, $\chi_S(G) = 1 = \chi_S(\overline{G})$. Suppose that for any graph G on n vertices, we have that $\chi_S(G) + \chi_S(\overline{G}) \leqslant n + 1$. Let H be a graph on $n + 1$ vertices with complement $\overline{H}$. Consider the graph that remains when some vertex, v, is removed from H and $\overline{H}$. Let $G = H - v$. Then G is a graph on n vertices, and by the induction hypothesis, $\chi_S(G) + \chi_S(\overline{G}) \leqslant n + 1$. Also, by Proposition 1.6 we have that $\chi_S(H) \leqslant \chi_S(G) + 1$ and $\chi_S(\overline{H}) \leqslant \chi_S(\overline{G}) + 1$ so that

$$\chi_S(H) + \chi_S(\overline{H}) \leqslant \chi_S(G) + \chi_S(\overline{G}) + 2 \leqslant n + 3.$$

Suppose that H has q edges from v to G. Then there are $n - q$ edges from v to $\overline{G}$ in $\overline{H}$. If $\chi_S(H) < \chi_S(G) + 1$ or $\chi_S(\overline{H}) < \chi_S(\overline{G}) + 1$, we have that

$$\chi_S(H) + \chi_S(\overline{H}) < \chi_S(G) + \chi_S(\overline{G}) + 2 \leqslant n + 3$$

and thus $\chi_S(H) + \chi_S(\overline{H}) \leqslant n + 2$.

Otherwise, removing v strictly decreases the subspace chromatic number of the graph. Then $q \geqslant \chi_S(G)$ and $n - q \geqslant \chi_S(\overline{G})$ so that

$$\chi_S(G) + \chi_S(\overline{G}) \leqslant n - q + q = n$$

and $\chi_S(H) + \chi_S(\overline{H}) \leqslant n + 2$. □

Lemma 1.14. *Let G_1 and G_2 be graphs on the same vertex set. Then $\chi_v(G_1 \cup G_2) \leqslant \chi_v(G_1)\chi_v(G_2)$.*

Proof. Let $\chi_v(G_1) = n_1$ and $\chi_v(G_2) = n_2$. We begin by taking a n_1-vector coloring of G_1. Define V_1 to be the n_1-dimensional subspace spanned by the vectors in this coloring. Similarly, let V_2 denote the n_2-dimensional subspace spanned by the vectors in a n_2-vector coloring of G_2. If vertex v is represented by $\vec{v}_1 \in V_1$ and also represented by $\vec{v}_2 \in V_2$, then in the coloring of $G_1 \cup G_2$, represent v by $\vec{v}_1 \otimes \vec{v}_2$, where $\otimes$ is the tensor product. □

Corollary 1.15. *Let G be a graph with $|G| = n$. Then $n \leqslant \chi_v(G)\chi_v(\overline{G})$.*

Proof. By Lemma 1.14, $n = \chi_v(K_n) = \chi_v(G \cup \overline{G}) \leqslant \chi_v(G)\chi_v(\overline{G})$. □

Lemma 1.16. *For any graph G, $2\sqrt{|G|} \leqslant \chi_v(G) + \chi_v(\overline{G})$.*

Proof. Follows from Corollary 1.15 and the inequality of arithmetic and geometric means. □

Proposition 1.17. *For any graph G, $2\sqrt{|G|} \leqslant \chi_*(G) + \chi_*(\overline{G}) \leqslant |G| + 1$.*

Proof. That $\chi_l(G) + \chi_l(\overline{G}) \leqslant |G| + 1$ was proved by Erdős, Rubin and Taylor [7]. The rest follows from Lemma 1.16 and Lemma 1.13. □

Using Lemma 1.14 to show Corollary 1.15 is an idea found in Cameron et al. [5] and borrowed from a similar result for the original coloring problem found in Ore's book [23]. The proof of the lower bound, specifically the assertion that $n \leqslant \chi(G)\chi(\overline{G})$, in the original paper by Gaddum and Nordhaus, relies on the Pigeonhole Principle: if we let n_i be the number of vertices assigned the ith color, $n_1 + n_2 + \cdots + n_{\chi(G)} = |G|$ and so $\chi(\overline{G}) \geqslant \max_i n_i \geqslant |G|/k$. A first attempt to prove Proposition 1.17 for the vector chromatic numbers led to wondering whether there exists a vector space version of the Pigeonhole Principle. As it turns out, this question has already been asked by Erdős [21], and answered in the negative by Furedi and Stanley: for two positive integers d and k, define $\alpha(d,k)$ to be the maximum cardinality of a set of nonzero vectors in $\mathbb{R}^d$ such that every subset of $k+1$ vectors contains an orthogonal pair [2] (*almost orthogonal vectors*). In order to use Gaddum and Nordhaus' original argument, we would require that $\alpha(d,k) = dk$ for all d and k. While $\alpha(d,2) = 2d$ [30] and $\alpha(2,k) = 2k$, $\alpha(4,5) \geqslant 24$ [9], and little else seems to be known.

1.2 The Bell-Kochen-Specker Theorem

We have already seen that $\chi_v(G) \leqslant \chi(G)$ for any graph G. An example of a graph for which $\chi_v(G) < \chi(G)$ is surprisingly difficult to find. In fact, the first examples come from proofs of a well-known theorem in quantum theory.

Kochen and Specker [14–16, 33] (and Bell independently [3]) showed that in a Hilbert space $\mathcal{H}$ of dimension at least three there does not exist a function f from the set of projection operators on $\mathcal{H}$ to the set $\{0, 1\}$ such that for every subset of projections $\{P_i\}$ that commute and satisfy $\sum_i P_i = I$ (where I is the identity operator on $\mathcal{H}$), then $\sum_i f(P_i) = 1$. Note that, if $\vec{e}_1, \ldots, \vec{e}_n$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ of dimension n, and P_i is the orthogonal projection on the span of $\vec{e}_i$, then $\sum_i P_i = I$ and the P_i commute. Thus one way to prove the Kochen-Specker theorem is to provide a set of *Kochen-Specker vectors*, where it is impossible to assign either 1 or 0 to each vector in the set so that no two orthogonal vectors are both assigned 1 and in any subset of n mutually orthogonal vectors not all of the vectors are assigned zero.

If G is the graph of a set of Kochen-Specker vectors in $\mathbb{F}^n$ that does contain n mutually orthogonal vectors, then $\chi_v(G, \mathbb{F}) < \chi(G)$, since coloring G requires at least n colors, and if G could be colored using n colors, then assigning the value 1 to every vertex of a specified color and 0 to the others would contradict the Kochen-Specker property of the set of vectors. The original proof of the Kochen-Specker theorem consisted of 117 vectors in $\mathbb{R}^3$ whose graph has chromatic number 4. Successive papers have presented examples of sets of Kochen-Specker vectors of decreasing cardinality [4]. In 2005, using an algorithm for the exhaustive construction of sets of Kochen-Specker vectors, Pavičić et al. [25, 26] generated all sets of Kochen-Specker vectors with less than 25 vectors in $\mathbb{R}^4$, and with less than 31 vectors in $\mathbb{R}^3$. In dimension 3 and dimension 4, this approach has shown that a set of Kochen-Specker vectors must have at least 18 elements.

A Smaller Example

One of the first authors to give a smaller set of Kochen-Specker vectors than the original 117 was Peres [28], who provides sets of 33 vectors in $\mathbb{R}^3$ and 24 vectors in $\mathbb{R}^4$. We are able to exhibit a subset S of 17 of the later Kochen-Specker vectors of Peres with orthogonality graph G for which $\chi_v(G) = 4$ and $\chi(G) = 5$. As mentioned above, S cannot be a Kochen-Specker set.

We begin to construct this example by deleting only 6 vertices from Peres' graph. We describe the resulting 18-vertex graph $\mathcal{G}$ by the vectors assigned to each vertex. Following Peres, we use $\bar{1}$ to denote -1.

1000	0100	0010	0001	1100	$1\bar{1}00$
1111	$11\bar{1}\bar{1}$	$1\bar{1}1\bar{1}$	$1\bar{1}\bar{1}1$	1010	$10\bar{1}0$
$111\bar{1}$	$11\bar{1}1$	$1\bar{1}11$	$\bar{1}111$	1001	$100\bar{1}$

Note that the first four vectors of each row form a clique. We wish to show that $\chi(G) > 4$. Suppose that there is a coloring of $\mathcal{G}$ using four colors. Call the color associated with the vertex 1000 green. Then at the end of the first row, one of the vertices 1100 and $1\bar{1}00$ must be green, since both vectors are orthogonal to 0010 and 0001. Similar reasoning shows that at the right of the second row, we also must have a green vertex from the vectors 1010 and $10\bar{1}0$. This gives four possible options that are displayed below, each of which leads to a contradiction. Besides the original choice of

1100 or $1\bar{1}00$ and 1010 or $10\bar{1}0$, the resulting colorings, with green vectors indicated, are forced. In each case, by reasoning as above, we must also have one of the vectors 1001 and $100\bar{1}$ colored green, and it may be seen this is not possible.

$$\begin{array}{cccccc}
\boxed{1000} & 0100 & 0010 & 0001 & \boxed{1100} & 1\bar{1}00 \\
\boxed{1111} & 11\bar{1}\bar{1} & 1\bar{1}1\bar{1} & 1\bar{1}\bar{1}1 & \boxed{1010} & 10\bar{1}0 \\
\boxed{111\bar{1}} & 11\bar{1}1 & 1\bar{1}11 & \bar{1}111 & 1001 & 100\bar{1}
\end{array}$$

$$\begin{array}{cccccc}
\boxed{1000} & 0100 & 0010 & 0001 & \boxed{1100} & 1\bar{1}00 \\
1111 & \boxed{11\bar{1}\bar{1}} & 1\bar{1}1\bar{1} & 1\bar{1}\bar{1}1 & 1010 & \boxed{10\bar{1}0} \\
111\bar{1} & \boxed{11\bar{1}1} & 1\bar{1}11 & \bar{1}111 & 1001 & 100\bar{1}
\end{array}$$

$$\begin{array}{cccccc}
\boxed{1000} & 0100 & 0010 & 0001 & 1100 & \boxed{1\bar{1}00} \\
1111 & 11\bar{1}\bar{1} & \boxed{1\bar{1}1\bar{1}} & 1\bar{1}\bar{1}1 & \boxed{1010} & 10\bar{1}0 \\
111\bar{1} & 11\bar{1}1 & \boxed{1\bar{1}11} & \bar{1}111 & 1001 & 100\bar{1}
\end{array}$$

$$\begin{array}{cccccc}
\boxed{1000} & 0100 & 0010 & 0001 & 1100 & \boxed{1\bar{1}00} \\
1111 & 11\bar{1}\bar{1} & 1\bar{1}1\bar{1} & \boxed{1\bar{1}\bar{1}1} & 1010 & \boxed{10\bar{1}0} \\
111\bar{1} & 11\bar{1}1 & 1\bar{1}11 & \boxed{\bar{1}111} & 1001 & 100\bar{1}
\end{array}$$

Note that in the 18-vertex graph, the vertex assigned to vector 1000 has degree three. But since $\chi(G) = 5$, we know that by Lemma 1.10, we can remove this vertex to get a 17-vertex graph with the same chromatic number. Thus the graph described by the following vectors satisfies $\chi_v(G) < \chi(G)$, and can be seen in Figure 2.

$$\begin{array}{cccccc}
 & 0100 & 0010 & 0001 & 1100 & 1\bar{1}00 \\
1111 & 11\bar{1}\bar{1} & 1\bar{1}1\bar{1} & 1\bar{1}\bar{1}1 & 1010 & 10\bar{1}0 \\
111\bar{1} & 11\bar{1}1 & 1\bar{1}11 & \bar{1}111 & 1001 & 100\bar{1}
\end{array}$$

Quantum Chromatic Number

Another specific example of a graph on 18 elements for which $\chi_v < \chi$ is the orthogonality graph of the vectors

$$\begin{array}{cccccc}
001\bar{1} & 1000 & 0111 & 010\bar{1} & 0010 & 1101 \\
1\bar{1}00 & 0001 & 1110 & 10\bar{1}0 & 0100 & 1011 \\
01\bar{1}0 & 100\bar{1} & 1111 & 11\bar{1}\bar{1} & 1\bar{1}1\bar{1} & 1\bar{1}\bar{1}1,
\end{array}$$

which is given in a paper by Cameron et al. [5] that explores the quantum chromatic number χ_q of a graph. In general, $\omega(G) \leqslant \chi_q(G) \leqslant \chi(G)$. The *rank-one quantum chromatic number* of a graph $G = (V, E)$, $\chi_q^{(1)}(G)$, is the smallest positive integer c such

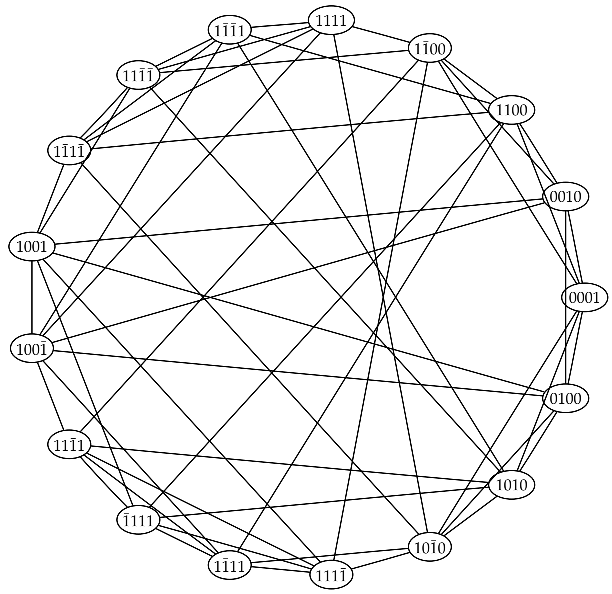


Figure 2:

that there exist unitary matrices $\{U_v\}_{v \in V} \in M_c(\mathbb{C})$ such that the diagonal entries of $U_v^* U_w$ are all zero whenever v and w are adjacent in G. In general, $\chi_v(G, \mathbb{C}) \leqslant \chi_q^{(1)}(G)$, since given the unitary matrices $\{U_v\}$, taking the first row of each matrix yields a valid vector coloring of G.

Lemma 1.18 ([5]). *Let G be a graph with a valid vector coloring in $\mathbb{R}^4$. Then $\chi_q^{(1)}(G) = 4$.*

By Lemma 1.18, we will also have that $\chi_q(\mathcal{G}) = 4$, yielding a smaller example that was previously known for both $\chi_v(G) < \chi(G)$ and $\chi_q(G) < \chi(G)$. The relationship between χ_v and χ_q is currently unknown.

2 2-Choosable Graphs

In 1979, Erdős, Rubin and Taylor characterized all 2-choosable graphs [7]. By repeatedly removing degree-one vertices, this characterization is given in terms of the *core* of the graph, which is what remains after repeatedly removing all vertices of degree one. The graph $\Theta_{a,b,c}$ is defined to be the graph where two vertices joined by three distinct paths of a, b, and c edges.

Theorem 2.1 ([7]). *A graph G is 2-choosable if and only if the core of G is K_1, an even cycle, or $\Theta_{2,2,2n}$ for $n \in \mathbb{N}$.*

We now proceed to characterize all 2-vector choosable and 2-subspace choosable graphs, and begin by considering the subspace-chromatic number of trees, even cycles, and $\Theta_{2,2,2n}$.

Lemma 2.2. *Let T be a tree. Assign a vector $\vec{v}$ to any vertex v of T, and assign subspaces of dimension two to the remaining vertices of T. Then T has a valid vector coloring.*

Proof. The proof is by induction on $|T|$. If $|T| = 1$, then $T = K_1$, and one vector suffices. Now suppose that $|T| = k$ and that the statement holds for all trees on $k-1$ vertices. $T - v$ is either a tree or a forest with two connected components. Orthogonally project the subspace(s) assigned to the neighbor(s) of v on $\vec{v}^{\perp}$, choose a vector for each neighbor from the result, and apply the induction hypothesis. $\square$

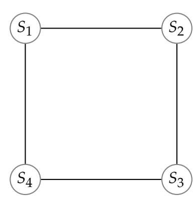
Proposition 2.3. *Let T be a tree. Then $\chi_S(T) = 1$ if $T = K_1$, and $\chi_S(T) = 2$ otherwise.*

Proof. Given a subspace assignment, choosing any vertex of T and any vector at that vertex results in the assumptions of Lemma 2.2. $\square$

We now turn our attention to even cycles and discover the surprising result that the choice of field makes a difference in the subspace-chromatic number, even in small examples. We first consider C_4.

Lemma 2.4. *The subspace-chromatic number $\chi_S(C_4, \mathbb{R}) = 3$.*

Proof. Using Proposition 1.9, we only need to show that there is a set of real subspace assignments of dimension 2 for C_4 which does not yield a valid coloring. Consider the following subspace assignments:



where

$$S_1 = \operatorname{span}\{\vec{e}_1, \vec{e}_2\}$$
$$S_2 = \operatorname{span}\{\vec{e}_2 + \vec{e}_3, \vec{e}_1 - \vec{e}_4 + \vec{e}_5\}$$
$$S_3 = \operatorname{span}\{\vec{e}_1 - 3\vec{e}_2 + 2\vec{e}_3 + 4\vec{e}_5, \vec{e}_4\}$$
$$S_4 = \operatorname{span}\{\vec{e}_1 + \vec{e}_3, \vec{e}_2 - \vec{e}_4 + 2\vec{e}_5\}$$

Suppose there exists a valid vector coloring. Then we have three choices of vector for $\vec{v}_1 \in S_1$, $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_1 + b \cdot \vec{e}_2$, where b is a nonzero scalar. Suppose for $\vec{v}_1$, we choose $\vec{e}_1$. Then for $\vec{v}_2$ we must choose $\vec{e}_2 + \vec{e}_3$ from S_2, and continuing in this way, we have $\vec{v}_3 = \vec{e}_4$ in S_3 and $\vec{v}_4 = \vec{e}_1 + \vec{e}_3$ in S_4, which is not a valid vector coloring.

If instead we choose $\vec{v}_1 = \vec{e}_2$, then we must choose $\vec{e}_1 + \vec{e}_3$ for $\vec{v}_4$, and continuing in this direction, from the respective subspaces, we have $\vec{v}_3 = \vec{e}_4$ and $\vec{v}_2 = \vec{e}_2 + \vec{e}_3$, which is also not a valid vector coloring.

Thus $\vec{v}_1 = \vec{e}_1 + b\vec{e}_2$, where b is a nonzero real number. This forces

$$\vec{v}_4 = b(\vec{e}_1 + \vec{e}_3) - (\vec{e}_2 - \vec{e}_4 + 2\vec{e}_5)$$
$$\vec{v}_2 = b(\vec{e}_1 - \vec{e}_4 + \vec{e}_5) - (\vec{e}_2 + \vec{e}_3)$$

and the choice of $\vec{v}_2$ gives

$$\vec{v}_3 = b(\vec{e}_1 - 3\vec{e}_2 + 2\vec{e}_3 + 4\vec{e}_5) + (5b+1)\vec{e}_4.$$

To be a valid coloring, we need

$$\langle b(\vec{e}_1 - 3\vec{e}_2 + 2\vec{e}_3 + 4\vec{e}_5) + (5b+1)\vec{e}_4, b(\vec{e}_1 + \vec{e}_3) - (\vec{e}_2 - \vec{e}_4 + 2\vec{e}_5)\rangle = 0.$$

This forces $3b^2 + 1 = 0$, contradicting that b is real. □

Unlike the real case, we will see that over the complex numbers, the subspace chromatic number of C_{2n} is 2 for all natural n. First, we extend this counterexample to find the subspace chromatic number of any even cycle over the real numbers.

Lemma 2.5. *Let G be a graph, let v be a vertex of G of degree two such that the two neighbors of v in G are not adjacent, and let e and f be the edges incident on v in G. If $\chi_S(G/e/f) > 2$ then $\chi_S(G) > 2$ as well.*

Proof. Let x and y be the neighbors of v in G. Since $\chi_S(G/e/f) > 2$, there exists a subspace assignment for $G/e/f$ where each subspace has dimension two and for which there is no valid vector coloring of $G/e/f$. Let S be the subspace assigned to the vertex v_{ef} that has replaced vertices v, x, and y in $G/e/f$. Create a subspace assignment for G from that for $G/e/f$ by assigning S to v, x, and y. Suppose there exists a valid vector coloring for G from this subspace assignment. Then, since S has dimension two, vertices x and y must be assigned the same vector, giving a valid vector coloring for the original subspace assignment on $G/e/f$, a contradiction. □

Proposition 2.6. *For any cycle C, $\chi_S(C, \mathbb{R}) = 3$.*

Proof. If C is an odd cycle, note that $\chi_v(C) = 3$ so that $\chi_S(C) = 3$ by Remark 1.4. Use Lemma 2.5 to extend the counterexample of Lemma 2.4 to any C_{2n}, giving that $\chi_S(C_{2n}, \mathbb{R}) \geqslant 3$, and apply Proposition 1.9. □

Corollary 2.7. *For any $n \in \mathbb{N}$, $\chi_S(\Theta_{2,2,2n}, \mathbb{R}) = 3$.*

Proof. Any graph of the form $\Theta_{2,2,2n}$ will have an induced even cycle and the result follows from Proposition 2.6. □

The following proposition, which completely characterizes all graphs with subspace chromatic number 2 over the real numbers, follows directly from Proposition 2.6.

Proposition 2.8. *Let G be a graph. Then $\chi_S(G, \mathbb{R}) \leqslant 2$ if and only if G is a tree.*

Now that we have characterized all graphs that are 2-vector choosable over the real numbers, we return to the case of the even cycle to find the subspace chromatic number over the complex numbers.

Lemma 2.9. *Let S and T be two-dimensional subspaces of $\mathbb{C}^n$ for some $n \geqslant 2$ such that for every nonzero vector $\vec{s} \in S$ there exists a nonzero vector $\vec{t} \in T$ such that $\langle \vec{s}, \vec{t} \rangle \neq 0$. Then given an orthonormal basis $\vec{s}_1, \vec{s}_2$ of S, there exists an orthonormal basis $\vec{t}_1, \vec{t}_2$ of T such that $\langle \vec{s}_i, \vec{t}_j \rangle = 0$ if and only if $i = j$.*

Proof. Extend the orthonormal basis $\vec{s}_1, \vec{s}_2$ of S to an orthonormal basis $\vec{s}_1, \dots, \vec{s}_n$ of $\mathbb{C}^n$. By assumption, there exists a nonzero vector $\vec{w}_1 \in T$ such that $\vec{w}_1$ is not orthogonal to $\vec{s}_1$. Extend $\vec{w}_1$ to an orthonormal basis $\vec{w}_1, \vec{w}_2$ of T. Also by assumption, $\vec{w}_2$ cannot be orthogonal to both $\vec{s}_1$ and $\vec{s}_2$, and $\vec{s}_2$ cannot be orthogonal to both $\vec{w}_1$ and $\vec{w}_2$ Thus we may write $\vec{w}_1$ and $\vec{w}_2$ uniquely as

$$\begin{aligned} \vec{w}_1 &= a_1\vec{s}_1 + a_2\vec{s}_2 + \cdots + a_n\vec{s}_n \\ \vec{w}_2 &= b_1\vec{s}_1 + b_2\vec{s}_2 + \cdots + b_n\vec{s}_n \end{aligned}$$

where $a_1 \neq 0$ and $a_1b_2 - b_1a_2 \neq 0$. Applying Gauss-Jordan elimination and normalizing the resulting vectors yields the desired basis of T. □

Proposition 2.10. *The subspace chromatic number $\chi_S(C_{2n}, \mathbb{C}) = 2$.*

Proof. Begin by labeling the vertices of C_{2n} as $v_1, \dots, v_{2n}$ (where v_i is adjacent to v_{i+1}) and assign these vertices corresponding subspaces $S_1, \dots, S_{2n}$ of dimension 2. Without loss of generality, assume that $S_1 = \operatorname{span}\{\vec{e}_1, \vec{e}_2\}$. If there are adjacent vertices u and v of C_{2n} that have been assigned subspaces S_u and S_v where there exists a nonzero vector $\vec{u}$ in S_u that is orthogonal to every vector in S_v, choose $\vec{u}$ for vertex u, and apply the reasoning of Lemma 2.2 to get a vector coloring of the vertices of C_{2n}. Since whatever vector is chosen for v is orthogonal to $\vec{u}$, we have a valid vector coloring.

Thus we may assume that for every pair of adjacent vertices, u and v of C_{2n} that have been assigned subspaces S_u and S_v, and for every nonzero vector $\vec{u} \in S_u$, there exists a nonzero vector $\vec{v} \in V$ such that $\vec{u}$ is not orthogonal to $\vec{v}$. Choose a basis $\vec{e}_{11}, \vec{e}_{12}$ of the subspace S_1 assigned to v_1. If there exists a valid vector coloring of C_{2n} where v_1 is assigned $\vec{e}_{11}$ or $\vec{e}_{12}$, we are done. Thus we may assume that we have exhausted these two cases and are now forced to choose $\vec{e}_{11} + b\vec{e}_{12}$ for vertex v_1, where b is some nonzero complex scalar.

By Lemma 2.9 there is an orthonormal basis $\vec{e}_{21}$ and $\vec{e}_{22}$ of S_2 such that $\langle \vec{e}_{1i}, \vec{e}_{2j} \rangle = 0$ if and only if $i = j$. Applying Lemma 2.9 repeatedly, for each i, there exists an orthonormal basis $\vec{e}_{k1}$ and $\vec{e}_{k2}$ of S_i such that $\langle \vec{e}_{(k-1)i}, \vec{e}_{kj} \rangle = 0$ if and only if $i = j$. Starting again with vector $\vec{v}_1 = \vec{e}_{11} + b\vec{e}_{12}$ for vertex v_1, assign the vectors

$$\vec{v}_k = \begin{cases} \vec{e}_{k1} + b\vec{e}_{k2} & \text{if } k \text{ is odd,} \\ \bar{b}\vec{e}_{k1} - \vec{e}_{k2} & \text{if } k \text{ is even,} \end{cases}$$

to the remaining vertices. Then, by construction, the inner product of the vectors representing any two adjacent vertices will be zero regardless of the choice of b, with the exception of

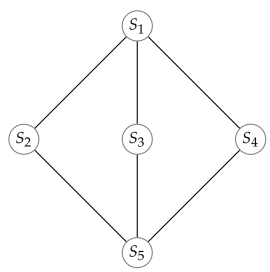
$$\begin{aligned}\langle \vec{v}_1, \vec{v}_{2n} \rangle &= \langle \vec{e}_{11} + b\vec{e}_{12}, \overline{b}\vec{e}_{(2n)1} - \vec{e}_{(2n)2} \rangle \\ &= b^2 \langle \vec{e}_{12}, \vec{e}_{(2n)1} \rangle + b(\langle \vec{e}_{11}, \vec{e}_{(2n)1} \rangle - \langle \vec{e}_{12}, \vec{e}_{(2n)2} \rangle) - \langle \vec{e}_{11}, \vec{e}_{(2n)2} \rangle \, .\end{aligned}$$

By the Fundamental Theorem of Algebra, there exists a complex number b for which $\langle \vec{v}_1, \vec{v}_{2n} \rangle = 0$, yielding a valid vector coloring of C_{2n} for any assignment of two-dimensional subspaces. $\square$

We can now turn our attention to the subspace chromatic number over the complex numbers of the third type of 2-choosable graph, $\Theta_{2,2,2n}$. We find here that unlike in the case with even cycles, the subspace chromatic number over the complex numbers of $\Theta_{2,2,2n}$ is the same as that over the reals.

Lemma 2.11. *The subspace chromatic number $\chi_S(\Theta_{2,2,2}, \mathbb{C}) \geqslant 3$.*

Proof. We show that $\Theta_{2,2,2}$ is not 2-vector colorable. Consider the following subspace assignments:



where

$$\begin{aligned} S_1 &= \text{span}\{\vec{e}_1, \vec{e}_2\} \\ S_2 &= \text{span}\{\vec{e}_1 + \vec{e}_3, \vec{e}_2\} \\ S_3 &= \text{span}\{\vec{e}_1, \vec{e}_2 + \vec{e}_3\} \\ S_4 &= \text{span}\{\vec{e}_1, \vec{e}_2 + \vec{e}_4\} \\ S_5 &= \text{span}\{\vec{e}_1 + \vec{e}_2 + \vec{e}_4, \vec{e}_3\}. \end{aligned}$$

If a scalar multiple of $\vec{e}_1$ is chosen for vertex v_1, then for vertex v_2 we require $\vec{e}_2$. Then for vertex v_5, we need $\vec{e}_3$, which leads us to choose $\vec{e}_1$ for vertex v_3, a contradiction.

On the other hand if we chose $\vec{e}_2$ for vertex v_1, then $\vec{e}_1$ must be chosen for vertex v_4, and then $\vec{e}_3$ for vertex v_5. This forces $\vec{e}_2$ to be assigned to vertex v_2, which is not orthogonal to $\vec{e}_2$.

Then we are left only with the choice of $\vec{e}_1 + b\vec{e}_2$ for vertex v_1, for some nonzero scalar b. This causes the choice of $\overline{b}(\vec{e}_1 + \vec{e}_3) - \vec{e}_2$ for vertex v_2, and $\vec{e}_2 + \vec{e}_3 - \overline{b}\vec{e}_1$ for vertex v_3. This assignment for vertex v_2 causes the choice of $b(\vec{e}_1 + \vec{e}_2 + \vec{e}_4) + (1-b)\vec{e}_3$ for vertex v_5. For this vector to be orthogonal to the choice for vertex v_3, we require that

$$0 = b - 1 + b^2 - b = b^2 - 1$$

which means that b must be ± 1.

If we had chosen $\vec{e}_1 - \vec{e}_2$ for the first vertex, we must choose $\vec{e}_1 + \vec{e}_2 + \vec{e}_4$ for vertex v_4, and the vector assigned to vertex v_5 is now $\vec{e}_1 + \vec{e}_2 - 2\vec{e}_3 + \vec{e}_4$, a contradiction.

If we had chosen instead $\vec{e}_1 + \vec{e}_2$ for vertex 1, we must choose $\vec{e}_1 - \vec{e}_2 - \vec{e}_4$ for vertex v_4, and the vector assigned to vertex v_5 is now $\vec{e}_1 + \vec{e}_2 + \vec{e}_4$, a contradiction. Thus there does not exist a valid vector coloring and $\chi_S(\Theta_{2,2,2}) \geqslant 3$. □

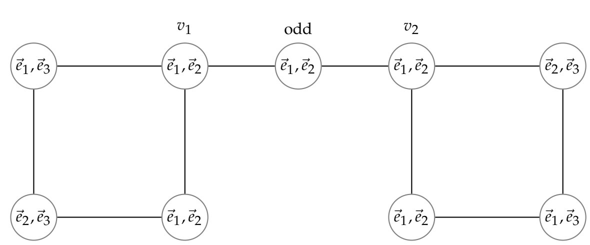
Proposition 2.12. *For any $l, m, n > 0$, the subspace chromatic number $\chi_S(\Theta_{2l,2m,2n}, \mathbb{C}) = 3$.*

Proof. That $\chi_S(\Theta_{2,2,2n}, \mathbb{C}) \geqslant 3$ follows from Lemma 2.11 by using Lemma 2.5. To show that $\Theta_{2,2,2}$ is 3-vector colorable, notice that we can vector color one of the induced even cycles by Proposition 2.10. If one vertex remains uncolored, it was assigned a subspace of dimension three and has at most two already colored neighbors. If more than one vertex remains uncolored, the remaining vertices induce a tree and have at most one already colored neighbor, a situation which exceeds the assumptions of Lemma 2.2. □

Proposition 2.13. *If a graph G contains more than one induced even cycle, then $\chi_S(G, \mathbb{C}) \geqslant 3$.*

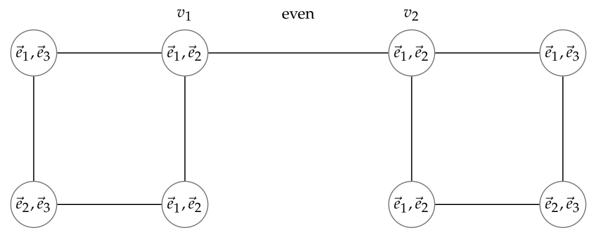
Proof. If a graph contains multiple even cycles, then it either has an induced $\Theta_{2n,2m,2k}$ and we are done by Proposition 2.12, or it has an induced subgraph consisting of two even cycles connected by a path. Using Lemma 2.5, we need only consider two copies of C_4 that share a vertex and two copies of C_4 that are connected by a path of one vertex not on either cycle.

For the latter case, consider the following subspace assignments, where the subspaces are the spans of the vectors found in each vertex.



Here a choice of some vector $\vec{v}$ for v_1 causes a nonzero scalar multiple of $\vec{v}$ to be assigned to v_2. This will then cause a contradiction in vector coloring one of the copies of C_4.

For the first case, consider the following subspace assignments.



Here a choice of some vector $\vec{v}$ for v_1 causes a scalar multiple of the orthogonal vector, $\vec{v}^{\perp}$ to be assigned to v_2. This will again cause a contradiction in vector coloring one of the copies of C_4. □

We can now classify all graphs with subspace chromatic number 2 over the complex numbers.

Theorem 2.14. *For any graph G, $\chi_S(G, \mathbb{C}) \leqslant 2$ if and only if G is a tree or G contains exactly one cycle and that cycle is even.*

Proof. Follows from Proposition 2.10 and Proposition 2.13. □

Knowing the characterization of all 2-choosable graphs and having completely characterized all 2-subspace choosable graphs we now ask: when is a graph 2-vector choosable? Can there be a graph which is not 2-choosable but is 2-vector choosable?

Proposition 2.15. *For any graph G, $\chi_{lv}(G) = 2$ if and only if $\chi_l(G) = 2$.*

Proof. If $\chi_l(G) = 2$, then we know that $\chi_{lv}(G) = 2$, so we just need to show that there are no graphs such that $\chi_{lv}(G) = 2$ but $\chi_l(G) \neq 2$. If a graph contains an odd cycle, then $\chi_{lv}(G) \geqslant 3$. So, again, we must show that if G includes multiple even cycles, then $\chi_{lv}(G) \geqslant 3$. Notice, however, that the subspace assignments in Proposition 2.13 are all spans of lists of basis elements. □

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