

# Positivity of three-term recurrence sequences \*

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## Abstract

In this paper, we give the sufficient conditions for the positivity of recurrence sequences defined by

$$a_n u_n = b_n u_{n-1} - c_n u_{n-2}$$

for  $n \geq 2$ , where  $a_n, b_n, c_n$  are all nonnegative and linear in  $n$ . As applications, we show the positivity of many famous combinatorial sequences.

## 1 Introduction

The significance of the positivity to combinatorics stems from the fact that only the nonnegative integer can have a combinatorial interpretation. There has been an amount of research devoted to this topic in recent years (see [1, 2, 5, 9, 10, 14, 15] for instance). The purpose of this paper is to present some sufficient conditions for the positivity of recurrence sequences.

Let  $u_0, u_1, u_2, \dots$  be a sequence of integer numbers. The sequence is called a (*linear*) *recurrence sequence* if it satisfies a homogeneous linear recurrence relation

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \dots + a_k u_{n-k} \quad (1)$$

for  $n \geq k$ , where  $a_1, a_2, \dots, a_k \in \mathbb{Z}$ . The linear recurrence relation (1) defines a unique integer sequence  $\{u_n\}_{n \geq 0}$  after the first  $k$  initial terms  $u_0, u_1, \dots, u_{k-1}$  are given. Let  $p(x) = x^k - a_1 x^{k-1} - \dots - a^k$  be its characteristic polynomial with discriminant  $D$ . Following [7], the positivity problem is stated as follows.

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**Positivity Problem.** *Let a linear recurrence relation (1) be given together with the initial terms  $u_i$  for  $i = 0, 1, \dots, k - 1$ . Is the recurrence sequence  $\{u_n\}_{n \geq 0}$  nonnegative, i.e., does it hold that  $u_n \geq 0$  for all  $n$ ?*

So far there have been some results on the positivity problem. For example, Halava *et al* [7] presented that the positivity problem is decidable for three-term recurrence sequences defined by

$$u_n = au_{n-1} + bu_{n-2} \tag{2}$$

for  $a, b \in \mathbb{Z}$ . More precisely, we can conclude the following result from [7] when  $ab \neq 0$ .

**Theorem 1.1.** *Suppose that the sequence  $\{u_n\}_{n \geq 0}$  satisfying the three-term recurrence relation (2) with the discriminant  $D = a^2 + 4b \geq 0$ . Let  $\lambda$  and  $\Lambda$  be the smaller and larger characteristic roots respectively. Then the full sequence  $\{u_n\}_{n \geq 0}$  is nonnegative if and only if one of the following conditions hold.*

- (i)  $a > 0, b < 0$  and  $u_1 \geq u_0\lambda \geq 0$ .
- (ii)  $a < 0, b > 0$  and  $u_1 = u_0\Lambda \geq 0$ .

In this paper, we are mainly interested in the positivity problem of sequences satisfying the following more general recurrence

$$a_n u_n = b_n u_{n-1} - c_n u_{n-2}, \tag{3}$$

where  $a_n, b_n, c_n$  are all nonnegative and linear in  $n$ . There are many combinatorial sequences satisfying this recurrence. For example, the central Delannoy sequence  $\{D(n)\}$  satisfies the recurrence

$$nD(n) = 3(2n - 1)D(n - 1) - (n - 1)D(n - 2) \tag{4}$$

with  $D(0) = 1, D(1) = 3$  and  $D(2) = 14$  (see [12] for instance). However, we cannot get that the sequence  $\{D(n)\}$  is nonnegative directly from the recurrence (4).

The paper is organized as follows. In Section 2, we present the sufficient conditions used frequently for the positivity of sequences satisfying the recurrence (3). In Section 3, we apply these results to derive the positivity of several combinatorial sequences, including the central Delannoy numbers, the Schröder numbers, and some orthogonal polynomials. Finally in Appendix, we prove Proposition 2.1.

## 2 Sufficient conditions for the positivity

In this section, we give the sufficient conditions for the positivity of  $\{u_n\}$  satisfying the recurrence

$$a_n u_n = b_n u_{n-1} - c_n u_{n-2},$$

where  $u_0, u_1 \geq 0$  and  $a_n, b_n, c_n$  are all nonnegative. Let  $x_n = \frac{u_n}{u_{n-1}}$  for  $n \geq 1$ . In order to establish the positivity of the sequence  $\{u_n\}$ , it suffices to check that  $\{x_n\}_{n \geq 1}$  satisfies

$$x_n \geq \frac{c_{n+1}}{b_{n+1}}.$$

By (3), the sequence  $\{x_n\}_{n \geq 1}$  satisfies the recurrence

$$a_n x_n = b_n - \frac{c_n}{x_{n-1}}.$$

Let  $p_n(x) = a_n x^2 - b_n x + c_n$  denote the  $n$ -th characteristic polynomial of the sequence satisfying the recurrence (3). Assume that  $b_n^2 - 4a_n c_n \geq 0$  for each  $n \geq 1$ . Then the  $n$ -th characteristic roots are

$$\lambda_n = \frac{b_n - \sqrt{b_n^2 - 4a_n c_n}}{2a_n} \quad \text{and} \quad \Lambda_n = \frac{b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}$$

respectively. Denote the limit of the sequence  $\{\lambda_n\}_{n \geq 1}$  by  $\lambda_\infty$ . By a simple calculation and  $b_n^2 \geq 4a_n c_n$ , we have

$$\lambda_n \geq \frac{c_n}{b_n}.$$

Hence we can conclude that if  $u_0, u_1 \geq 0$  and  $x_n \geq \lambda_{n+1}$  for  $n \geq 1$ , then the sequence  $\{u_n\}_{n \geq 0}$  is nonnegative.

In the following, we suppose that  $a_n, b_n, c_n$  are all linear in  $n$ . More precisely, let

$$a_n = \alpha_1 n + \alpha_0, \quad b_n = \beta_1 n + \beta_0, \quad c_n = \gamma_1 n + \gamma_0$$

and denote

$$A = \begin{vmatrix} \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{vmatrix}, \quad B = \begin{vmatrix} \gamma_0 & \gamma_1 \\ \alpha_0 & \alpha_1 \end{vmatrix}, \quad C = \begin{vmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{vmatrix}.$$

We can obtain the monotonicity of the  $n$ -th characteristic roots  $\{\lambda_n\}_{n \geq 1}$  and  $\{\Lambda_n\}_{n \geq 1}$ , which is only related to discriminants  $A, B, C$ .

**Proposition 2.1.** *Suppose that  $B^2 \leq AC$ . Then the following hold.*

- (i) *If  $C \leq 0$ , then sequences  $\{\lambda_n\}_{n \geq 1}$  and  $\{\Lambda_n\}_{n \geq 1}$  are nonincreasing in  $n$ .*
- (ii) *If  $C > 0$ , then sequences  $\{\lambda_n\}_{n \geq 1}$  and  $\{\Lambda_n\}_{n \geq 1}$  are nondecreasing in  $n$ .*

For the sake of the flow, the proof of Proposition 2.1 is given as an Appendix.

We can now give the following sufficient conditions for the positivity of recurrence sequences satisfying (3).

**Theorem 2.2.** *Let  $\{u_n\}_{n \geq 0}$  be a sequence of integer numbers and satisfy the three-term recurrence (3). Suppose that  $C \leq 0, B^2 \leq AC$  and  $u_1 \geq u_0 \lambda_1 \geq 0$ . Then the positivity problem of the sequence  $\{u_n\}_{n \geq 0}$  can be solved.*

*Proof.* Let  $x_n = \frac{u_n}{u_{n-1}}$  for  $n \geq 1$ . We need to prove that  $x_n \geq \lambda_{n+1}$  for all  $n \geq 1$ . From Proposition 2.1 (i), we have  $\lambda_n \geq \lambda_{n+1}$ . Hence it suffices to show  $x_n \geq \lambda_n$ . We proceed by induction on  $n$ . Clearly,  $x_1 \geq \lambda_1$  by the condition  $u_1 \geq u_0\lambda_1 \geq 0$ . Now assume that  $x_{n-1} \geq \lambda_{n-1}$  for  $n \geq 2$ . Note that  $p_n(\lambda_n) = 0$ , i.e.  $b_n - \frac{c_n}{\lambda_n} = a_n\lambda_n$ . So we have

$$a_n x_n = b_n - \frac{c_n}{x_{n-1}} \geq b_n - \frac{c_n}{\lambda_{n-1}} \geq a_n \lambda_n.$$

by induction hypothesis and Proposition 2.1 (i). Thus  $x_n \geq \lambda_n$  for all  $n \geq 1$ . This completes the proof.  $\square$

**Theorem 2.3.** *Let  $\{u_n\}_{n \geq 0}$  be a sequence of integer numbers and satisfy the three-term recurrence (3). Suppose that  $C > 0, B^2 \leq AC, \Lambda_1 \geq \lambda_\infty$  and  $u_1 \geq u_0\Lambda_1 \geq 0$ . Then the positivity problem of the sequence  $\{u_n\}_{n \geq 0}$  can be solved.*

*Proof.* Let  $x_n = \frac{u_n}{u_{n-1}}$ . In order to prove the positivity of  $\{u_n\}$ , it suffices to check that  $x_n \geq \lambda_{n+1}$  for all  $n \geq 1$ . From Proposition 2.1, we have  $\Lambda_1 \geq \lambda_{n+1}$ . So we only need to show that  $x_n \geq \Lambda_1$ . We proceed by induction on  $n$ . Clearly,  $x_1 \geq \Lambda_1$  by the condition  $u_1 \geq u_0\Lambda_1 \geq 0$ . Now assume that  $x_{n-1} \geq \Lambda_1$  for  $n \geq 2$ . Note that  $\lambda_n \leq \Lambda_1 \leq \Lambda_n$  following from Proposition 2.1 and the condition  $\Lambda_1 \geq \lambda_\infty$ . Hence  $p_n(\Lambda_1) = a_n\Lambda_1^2 - b_n\Lambda_1 - c_n \leq 0$ . Furthermore,

$$a_n x_n = b_n - \frac{c_n}{x_{n-1}} \geq b_n - \frac{c_n}{\Lambda_1} \geq a_n \Lambda_1$$

by the induction hypothesis. Then  $x_n \geq \Lambda_1$  for all  $n \geq 1$ . The proof is complete.  $\square$

When  $a_n, b_n, c_n$  are all constants, we have  $A = B = C = 0$  by the definition. So the sufficiency of Theorem 1.1 (i) is a special case of Theorem 2.2. In particular, if  $B^2 = AC$ , then we can obtain the following corollary which is interesting and useful from Proposition 2.1, Theorems 2.2 and 2.3.

**Corollary 2.4.** *Let  $\{u_n\}_{n \geq 0}$  be a sequence of integer numbers and satisfy the three-term recurrence (3). Suppose that  $B^2 = AC$ . Then the following results hold.*

- (i) *If  $b_n C + 2a_n B$  has the same sign as  $C$  for all  $n \geq 1$ , then the sequence  $\{\lambda_n\}_{n \geq 1}$  is constant. In addition, if  $u_1 \geq u_0\lambda_1 \geq 0$ , then the positivity problem of the sequence  $\{u_n\}_{n \geq 0}$  can be solved.*
- (ii) *If  $b_n C + 2a_n B$  has opposite sign of  $C$  for all  $n \geq 1$ , then the sequence  $\{\Lambda_n\}_{n \geq 1}$  is constant. In addition, if  $u_1 \geq u_0\Lambda_1 \geq 0$ , then the positivity problem of the sequence  $\{u_n\}_{n \geq 0}$  can be solved.*

### 3 Applications

In this section, we apply results obtained in the previous section to derive the positivity of several recurrence sequences in a unified manner.

Let  $\nu > -\frac{1}{2}$  be a parameter. The Gegenbauer polynomials sequence  $\{C_n^{(\nu)}(t)\}_{n \geq 0}$  satisfies the recurrence relation

$$nC_n^{(\nu)}(t) = 2t(\nu + n - 1)C_{n-1}^{(\nu)}(t) - (2\nu + n - 2)C_{n-2}^{(\nu)}(t) \quad (5)$$

with  $C_0^{(\nu)}(t) = 1$  and  $C_1^{(\nu)}(t) = 2t\nu$ . Then we have the following corollary.

**Corollary 3.1.** *The positivity problem of the Gegenbauer polynomials sequence  $\{C_n^{(\nu)}(t)\}$  can be solved for  $t \geq 1, \nu \geq \frac{1}{2}$ .*

*Proof.* From the recurrence (5), we have  $A = -2t(\nu - 1), B = 2(\nu - 1), C = -2t(\nu - 1)$ . Clearly,  $b_n^2 - 4a_n c_n = 4[(t^2 - 1)n^2 + 2(\nu - 1)(t^2 - 1)n + t^2(\nu - 1)^2] \geq 0$  for  $t \geq 1$  by direct calculation.

First consider the case  $t = 1$ . We have  $B^2 = AC$  and  $b_n C + 2a_n B = -4(\nu - 1)^2 \leq 0$ . If  $\nu > 1$ , then  $C < 0$ . By Corollary 2.4 (i), we have  $\lambda_n = 1$  for  $n \geq 1$ . And if  $\frac{1}{2} \leq \nu \leq 1$ , then  $C \geq 0$ . By Corollary 2.4 (ii), we have  $\Lambda_n = 1$  for  $n \geq 1$ . Thus the positivity of  $\{C_n^{(\nu)}(t)\}_{n \geq 0}$  follows from Corollary 2.4.

For  $t > 1$ , we have  $B^2 \leq AC$ . If  $\nu > 1$ , then  $C < 0$  and if  $\frac{1}{2} \leq \nu \leq 1$ , then  $C > 0$ . Also,  $\Lambda_1 = t\nu + \sqrt{t^2\nu^2 - 2\nu + 1}$  and  $\lambda_\infty = t - \sqrt{t^2 - 1}$ . Thus the sequence  $\{C_n^{(\nu)}(t)\}_{n \geq 0}$  is nonnegative from Theorems 2.2 and 2.3 respectively.  $\square$

In particular, for  $\nu = \frac{1}{2}$ ,  $C_n^{(\frac{1}{2})}(t)$  reduces to the Legendre polynomials  $P_n(t)$  and for  $\nu = 1$ , we have  $C_n^{(1)}(t) = U_n(t)$  is the Chebyshev polynomials of the second kind. So the Legendre polynomials sequence  $\{P_n(t)\}_{n \geq 0}$  and the Chebyshev polynomials sequence  $\{U_n(t)\}_{n \geq 0}$  are nonnegative for  $t \geq 1$ .

The derivative sequence of Gegenbauer polynomials  $\{\frac{d}{dt}C_n^{(\nu)}(t)\}_{n \geq 0}$  satisfies the following recurrence relation

$$(n - 1)\frac{d}{dt}C_n^{(\nu)}(t) = 2t(\nu + n - 1)\frac{d}{dt}C_{n-1}^{(\nu)}(t) - (2\nu + n - 1)\frac{d}{dt}C_{n-2}^{(\nu)}(t) \quad (6)$$

with  $\frac{d}{dt}C_n^{(\nu)}(0) = 0, \frac{d}{dt}C_n^{(\nu)}(1) = 2\nu$  and  $\frac{d}{dt}C_n^{(\nu)}(2) = 4\nu(\nu + 1)t$ . Then we have the following.

**Corollary 3.2.** *The positivity problem of the derivative sequence of Gegenbauer polynomials  $\{\frac{d}{dt}C_n^{(\nu)}(t)\}_{n \geq 0}$  can be solved for  $t \geq 1, \nu > 0$ .*

*Proof.* From the recurrence (6), we have  $A = -2t\nu, B = 2\nu, C = -2t\nu$ . And  $b_n^2 - 4a_n c_n = 4[(t^2 - 1)n^2 + 2(\nu - 1)(t^2 - 1)n + t^2(\nu - 1)^2 + 2\nu - 1] \geq 0$  for  $t \geq 1$ .

For  $t \geq 1, \nu > 0$ , we have  $C < 0$  and  $B^2 \leq AC$ . Also,  $x_2 = 2t(\nu + 1)$  and  $\Lambda_2 = t(\nu + 1) + \sqrt{t^2(\nu + 1)^2 - (2\nu + 1)}$ . Thus the positivity of the sequence  $\{\frac{d}{dt}C_n^{(\nu)}(t)\}_{n \geq 0}$  follows from Theorem 2.2.  $\square$

In what follows we list some more examples of recurrence sequences which are easy seen to satisfy the assumption of Theorem 2.3 or Corollary 2.4. Thus the positivity of these sequences is an immediate consequence of Theorem 2.3 or Corollary 2.4.

**Example 3.3.** The central Delannoy number  $D(n)$  is the number of lattice paths, *king walks*, from  $(0, 0)$  to  $(n, n)$  with steps  $(1, 0)$ ,  $(0, 1)$  and  $(1, 1)$  in the first quadrant. It is known that the central Delannoy numbers satisfy the recurrence

$$nD(n) = 3(2n - 1)D(n - 1) - (n - 1)D(n - 2)$$

with  $D(0) = 1$ ,  $D(1) = 3$  and  $D(2) = 14$  (see [12] for a bijective proof). By the recurrence, we have  $A = 3$ ,  $B = -1$ ,  $C = 3$ . Also,  $\Lambda_1 = 3$  and  $\lambda_\infty = 3 - 2\sqrt{2}$ . Hence the positivity of  $\{D(n)\}_{n \geq 0}$  follows from Theorem 2.3.

**Example 3.4.** The (large) Schröder number  $r_n$  is the number of king walks, *Schröder paths*, from  $(0, 0)$  to  $(n, n)$ , and never rising above the line  $y = x$ . The Schröder paths consist of two classes: those with steps on the main diagonal and those without. These two classes are equinumerous, and the number of paths in either class is the little Schröder number  $s_n$  (half the large Schröder number). It is known that the Schröder numbers of two kinds satisfy the recurrence

$$(n + 2)z_{n+1} = 3(2n + 1)z_n - (n - 1)z_{n-1}$$

with  $s_0 = s_1 = r_0 = 1$ ,  $r_1 = 2$ ,  $s_2 = 3$  and  $r_2 = 6$  (see Foata and Zeilberger [4] and Sulanke [16]). By the recurrence, we have  $A = 9$ ,  $B = -3$ ,  $C = 9$ . Also,  $\Lambda_2 = 3$  and  $\lambda_\infty = 3 - 2\sqrt{2}$ . Hence the positivity of  $\{r_n\}_{n \geq 0}$  follows from Theorem 2.3.

**Example 3.5.** Let  $h_n$  be the number of the set of all tree-like polyhexes with  $n + 1$  hexagons (Harary and Read [8]). It is known that  $h_n$  counts the number of lattice paths, from  $(0, 0)$  to  $(2n, 0)$  with steps  $(1, 1)$ ,  $(1, -1)$  and  $(2, 0)$ , never falling below the  $x$ -axis and with no peaks at odd level. The sequence  $\{h_n\}_{n \geq 0}$  is Sloane's A002212 and satisfies the recurrence

$$(n + 1)h_n = 3(2n - 1)h_{n-1} - 5(n - 2)h_{n-2}$$

with  $h_0 = h_1 = 1$  and  $h_2 = 3$ . By the recurrence, we have  $A = 45$ ,  $B = -15$ ,  $C = 9$ . Also,  $\Lambda_2 = 3$  and  $\lambda_\infty = 1$ . So  $\{h_n\}_{n \geq 0}$  is nonnegative by Theorem 2.3.

**Example 3.6.** Let  $w_n$  be the number of walks on cubic lattice with  $n$  steps, starting and finishing on the  $xy$  plane and never going below it (Guy [6]). The sequence  $\{w_n\}_{n \geq 0}$  is Sloane's A005572 and satisfies the recurrence

$$(n + 2)w_n = 4(2n + 1)w_{n-1} - 12(n - 1)w_{n-2}$$

with  $w_0 = 1$ ,  $w_1 = 4$  and  $w_2 = 17$ . By the recurrence, we have  $A = 144$ ,  $B = -36$ ,  $C = 12$ . Also,  $\Lambda_1 = 4$  and  $\lambda_\infty = 2$ . So  $\{w_n\}_{n \geq 0}$  is nonnegative by Corollary 2.4.

## 4 Concluding Remarks

Let  $u_0, u_1, u_2, \dots$  be a sequence of nonnegative numbers. The sequence is called *log-convex* (resp. *log-concave*) if for all  $k \geq 1$ ,  $u_{k-1}u_{k+1} \geq u_k^2$  (resp.  $u_{k-1}u_{k+1} \leq u_k^2$ ). Clearly,

a sequence  $\{u_k\}_{k \geq 0}$  of positive numbers is log-convex (resp. log-concave) if and only if the sequence  $\{u_{k+1}/u_k\}_{k \geq 0}$  is increasing (resp. decreasing). For the positive sequence satisfying the recurrence (3), we have recently established the following result for the log-convexity and log-concavity (see [11] for instance).

**Theorem 4.1** ([11]). *Let  $\{u_n\}_{n \geq 0}$  be a sequence of positive numbers and satisfy the three-term recurrence*

$$(\alpha_1 n + \alpha_0)u_{n+1} = (\beta_1 n + \beta_0)u_n - (\gamma_1 n + \gamma_0)u_{n-1} \quad (7)$$

for  $n \geq 1$ , where  $\alpha_1 n + \alpha_0, \beta_1 n + \beta_0, \gamma_1 n + \gamma_0$  are positive for  $n \geq 1$ . Suppose that  $AC \geq B^2$ . Then the following results hold.

- (i) If  $B < 0, C > 0, u_0 B + u_1 C \geq 0$  and  $u_1^2 \leq u_0 u_2$ , then the sequence  $\{u_n\}$  is log-convex.
- (ii) If  $B > 0, C < 0, z_0 B + z_1 C \leq 0$  and  $u_1^2 \geq u_0 u_2$ , then the sequence  $\{u_n\}$  is log-concave.

Using Theorem 4.1, we can get that sequences appeared in this paper are either log-concave or log-convex. For example, the central Delannoy sequence  $\{D(n)\}_{n \geq 0}$  is log-convex [11] and the sequence  $\{C_n^{(t)}(t)\}_{n \geq 0}$  is log-concave for  $\nu \geq 1, t \geq 1$  [3].

By the same technique used in the proof of Proposition 2.1, Theorems 2.2 and 2.3, we can also give more sufficient conditions for the positivity of sequences satisfying the recurrence (3) when  $B^2 > AC$ . As consequences, we can obtain the positivity problem of the Laguerre polynomials sequence  $\{L_n(t)\}_{n \geq 0}$  can be solved for  $t \leq 0$ .

## 5 Appendix. Proof of Proposition 2.1

The purpose of this Appendix is to prove Proposition 2.1.

*Proof.* We prove the result only for the case  $\lambda_n$  of (i) since the case (ii) is similar. In order to prove that  $\{\lambda_n\}_{n \geq 1}$  is nonincreasing, it suffices to show  $\lambda'_n \leq 0$  for  $n \geq 1$ . It is easy to get that the derivative of  $\lambda_n$  with respect to  $n$  is

$$\begin{aligned} \lambda'_n &= \left( \frac{b_n - \sqrt{b_n^2 - 4a_n c_n}}{2a_n} \right)' \\ &= \frac{(a_n b'_n - a'_n b_n)(\sqrt{b_n^2 - 4a_n c_n} - b_n) + 2a_n(a_n c'_n - a'_n c_n)}{2a_n^2 \sqrt{b_n^2 - 4a_n c_n}} \\ &= \frac{(\alpha_0 \beta_1 - \alpha_1 \beta_0)(\sqrt{b_n^2 - 4a_n c_n} - b_n) + 2a_n(\alpha_0 \gamma_1 - \alpha_1 \gamma_0)}{2a_n^2 \sqrt{b_n^2 - 4a_n c_n}} \\ &= -\frac{b_n C + 2a_n B - C \sqrt{b_n^2 - 4a_n c_n}}{2a_n^2 \sqrt{b_n^2 - 4a_n c_n}}. \end{aligned} \quad (8)$$

After rationalizing numerator, we have

$$\begin{aligned}\lambda'_n &= -\frac{(b_n C + 2a_n B)^2 - C^2(b_n^2 - 4a_n c_n)}{2a_n^2 \sqrt{b_n^2 - 4a_n c_n} (b_n C + 2a_n B + C \sqrt{b_n^2 - 4a_n c_n})} \\ &= -\frac{2(a_n B^2 + b_n B C + c_n C^2)}{a_n \sqrt{b_n^2 - 4a_n c_n} (b_n C + 2a_n B + C \sqrt{b_n^2 - 4a_n c_n})}.\end{aligned}$$

Note that  $b_n B + c_n C = -a_n A$  since

$$a_n A + b_n B + c_n C = \begin{vmatrix} a_n & \alpha_1 & \alpha_0 \\ b_n & \beta_1 & \beta_0 \\ c_n & \gamma_1 & \gamma_0 \end{vmatrix} = \begin{vmatrix} \alpha_1 n + \alpha_0 & \alpha_1 & \alpha_0 \\ \beta_1 n + \beta_0 & \beta_1 & \beta_0 \\ \gamma_1 n + \gamma_0 & \gamma_1 & \gamma_0 \end{vmatrix} = 0.$$

Hence

$$\lambda'_n = -\frac{2(B^2 - AC)}{\sqrt{b_n^2 - 4a_n c_n} (b_n C + 2a_n B + C \sqrt{b_n^2 - 4a_n c_n})}. \quad (9)$$

If  $C = 0$ , then  $B = 0$  since  $B^2 \leq AC = 0$ . Hence we have  $A = 0$  by the definition. It follows that  $\lambda'_n = 0$  from the recurrence (8).

Suppose now that  $C < 0$ . Then  $b_n C + 2a_n B$  is linear in  $n$ . Note that it changes sign at most once. Without loss of generality, we assume that it changes from nonnegative to nonpositive. Thus we can get  $\lambda'_n \leq 0$  first from (8), and then (9). This completes our proof of Proposition 2.1.  $\square$

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