Some inequalities in functional analysis, combinatorics, and probability theory

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Abstract

The main purpose of this paper is to show that many inequalities in functional analysis, probability theory and combinatorics are immediate corollaries of the best approximation theorem in inner product spaces. Besides, as applications of the de Caen-Selberg inequality, the finite field Kakeya and Nikodym problems are also studied.

Keywords: inner product space, orthogonal projection, Kakeya set, Nikodym set

1 Brief Introduction

Let $(H, <\cdot,\cdot>)$ be an inner product space over $\mathbb{R}$ throughout. Given $x \in H$ and a finite dimensional subspace M, denote by x_M the orthogonal projection of x onto M. It is geometrically evident that (we always assume $\frac{0}{0} = 0$ in this paper)

$$\|x\|^2 \geqslant \|x_M\|^2 = \max_{y\in M} \frac{<x_M, y>^2}{\|y\|^2} = \max_{y\in M} \frac{<x, y>^2}{\|y\|^2}. \tag{1}$$

Particularly, if $M = \text{span}\{y_i\}_{i=1}^n$ for some given set of elements $y_1, \ldots, y_n$, then

$$\|x\|^2 \geqslant \max_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n} \frac{<x, \sum_{i=1}^n \alpha_i y_i>^2}{\|\sum_{i=1}^n \alpha_i y_i\|^2}. \tag{2}$$

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The main purpose of this paper is to show that many inequalities in functional analysis, probability theory and combinatorics are immediate corollaries of (2). For the sake of completeness we determine the unique orthogonal projection x_M (many authors of textbooks on functional analysis only dealt the case when $\{y_i\}_{i=1}^n$ are linear independent). Write $x_M = \sum_{i=1}^n \beta_i y_i$ for some $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$. Since the smooth function

$$\Psi(\alpha_1, \ldots, \alpha_n) \doteq \|x - \sum_{i=1}^n \alpha_i y_i\|^2 = \|x\|^2 - 2\sum_{i=1}^n \alpha_i < x, y_i > + \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j < y_i, y_j >$$

attains its minimum $d(x, M)^2$ at $(\beta_1, \ldots, \beta_n)$,

$$\frac{\partial \Psi}{\partial \alpha_i}(\beta_1, \ldots, \beta_n) = 0 \quad (i = 1, 2, \ldots, n).$$

Equivalently,

$$\begin{pmatrix} < y_1, y_1 > & < y_1, y_2 > & \cdots & < y_1, y_n > \\ < y_2, y_1 > & < y_2, y_2 > & \cdots & < y_2, y_n > \\ \vdots & \vdots & \ddots & \vdots \\ < y_n, y_1 > & < y_n, y_2 > & \cdots & < y_n, y_n > \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} < x, y_1 > \\ < x, y_2 > \\ \vdots \\ < x, y_n > \end{pmatrix}. \tag{3}$$

If $(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ is another solution to (3), then

$$\begin{aligned} \Big\| \sum_{i=1}^n (\beta_i - \gamma_i) y_i \Big\|^2 &= (\beta_1 - \gamma_1, \cdots, \beta_n - \gamma_n)(< y_i, y_j >)_{n \times n} \begin{pmatrix} \beta_1 - \gamma_1 \\ \vdots \\ \beta_n - \gamma_n \end{pmatrix} \\ &= (\beta_1 - \gamma_1, \cdots, \beta_n - \gamma_n) \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0. \end{aligned}$$

Consequently $x_M = \sum_{i=1}^n \beta_i y_i = \sum_{i=1}^n \gamma_i y_i$.

Among many inequalities will be discussed later, we show particular interest in the de Caen-Selberg inequality [1, 2]:

$$\Big| \bigcup_{i=1}^n A_i \Big| \geqslant \sum_{i=1}^n \frac{|A_i|^2}{\sum_{j=1}^n |A_i \cap A_j|}, \tag{4}$$

where $\{A_i\}_{i=1}^n$ are finite sets. In Section 5 we will present some applications of the de Caen-Selberg inequality to the study of the finite field Kakeya and Nikodym problems in classical analysis.

2 Inequalities in Functional Analysis

2.1 Known inequalities

For any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, by (2) and the Cauchy-Schwarz inequality ($|\alpha_i \alpha_j| \leqslant \frac{\alpha_i^2 + \alpha_j^2}{2}$) one obtains the Pečarić inequality [13]

$$\|x\|^2 \geqslant \frac{\Big(\sum_{i=1}^{n} \alpha_i < x, y_i > \Big)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^2 | < y_i, y_j > |}. \tag{5}$$

(The following arguments are standard [13]) Substituting $\alpha_i = \frac{< x, y_i >}{\sum_{k=1}^{n} |< y_i, y_k >|}$ into (5) yields the Selberg inequality [1]

$$\|x\|^2 \geqslant \sum_{i=1}^{n} \frac{< x, y_i >^2}{\sum_{j=1}^{n} | < y_i, y_j > |}. \tag{6}$$

Substituting $\alpha_i = \text{sgn}(< x, y_i >)$ into (5) or applying the Cauchy-Schwarz inequality from (6) yields the Heilbronn inequality [10]

$$\|x\|^2 \geqslant \frac{\Big(\sum_{i=1}^{n} | < x, y_i > |\Big)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} | < y_i, y_j > |}. \tag{7}$$

The Selberg inequality (6) is certainly stronger than the Bombieri inequality [1]

$$\|x\|^2 \geqslant \frac{\sum_{i=1}^{n} < x, y_i >^2}{\max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | < y_i, y_j > |}. \tag{8}$$

If $\{y_i\}_{i=1}^n$ are orthogonal, then the Selberg inequality (6) turns out to be the classical Bessel inequality

$$\|x\|^2 \geqslant \sum_{i=1}^{n} \frac{< x, y_i >^2}{< y_i, y_i >}. \tag{9}$$

Substituting $\alpha_i = 1$ into (2) yields the Chung-Erdős inequality [3]

$$\|x\|^2 \geqslant \frac{\Big(\sum_{i=1}^{n} < x, y_i > \Big)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} < y_i, y_j >}. \tag{10}$$

In a partial summary,

$$(2) \succ (5) \succ (6) \succ (7),$$

where $(\bullet) \succ (\bullet\bullet)$ means Estimate $(\bullet)$ is stronger than Estimate $(\bullet\bullet)$.

3 From Functional Analysis to Combinatorics

3.1 Immediate corollaries

In this section we always choose $H = l^2$. Let A, B be finite subsets of $\mathbb{N}$ and χ_A, χ_B be the corresponding indictor functions. Then

$$< \chi_A, \chi_B >= |A \cap B|,$$

and χ_A, χ_B are orthogonal means A, B are disjoint sets. Given finite subsets $\{A_i\}_{i=1}^n$ of $\mathbb{N}$, define $y_i = \chi_{A_i}$ $(i \in [n])$ and $x = \chi_{\cup_i A_i}$. Then $< x, y_i >= |(\cup_j A_j) \cap A_i| = |A_i|$. By (2) and (3), we obtain

Theorem 3.1.

$$\Big|\bigcup_{i=1}^n A_i\Big| \geqslant \max_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n} \frac{\big(\sum_{i=1}^n \alpha_i |A_i|\big)^2}{\sum_{i=1}^n \sum_{j=1}^n \alpha_i\alpha_j |A_i \cap A_j|} = \sum_{i=1}^n \sum_{j=1}^n \beta_i\beta_j |A_i \cap A_j|, \tag{11}$$

where $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ *is any solution to*

$$\begin{pmatrix} |A_1 \cap A_1| & |A_1 \cap A_2| & \cdots & |A_1 \cap A_n| \\ |A_2 \cap A_1| & |A_2 \cap A_2| & \cdots & |A_2 \cap A_n| \\ \vdots & \vdots & \ddots & \vdots \\ |A_n \cap A_1| & |A_n \cap A_2| & \cdots & |A_n \cap A_n| \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} |A_1| \\ |A_2| \\ \vdots \\ |A_n| \end{pmatrix}. \tag{12}$$

Note in this context the Selberg inequality (6) turns out to be the de Caen inequality (4) and the Bessel inequality (9) turns out to be a trivial equality. Also note that

$$\sup_{\alpha_i>0} \frac{\big(\sum_{i=1}^n \alpha_i |A_i|\big)^2}{\sum_{i=1}^n \sum_{j=1}^n \alpha_i\alpha_j |A_i \cap A_j|} = \sup_{\alpha_i>0} \frac{\big(\sum_{i=1}^n \alpha_i |A_i|\big)^2}{\sum_{i=1}^n \sum_{j=1}^n \alpha_i^2 |A_i \cap A_j|} = \sup_{\alpha_i>0} \sum_{i=1}^n \frac{\alpha_i \left|A_i\right|^2}{\sum_{j=1}^n \alpha_j \left|A_i \cap A_j\right|}.$$

3.2 A slightly different variant

In this subsection, we provide a slightly different variant of (12).

Theorem 3.2. *The following matrix equation always has a solution*

$$\Big(\frac{|A_i \cap A_j|}{|A_i||A_j|}\Big)_{n\times n} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}; \tag{13}$$

any solution to (13) *satisfies*

$$\sum_{i=1}^n q_i = \max_{(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n} \frac{\big(\sum_{i=1}^n \alpha_i |A_i|\big)^2}{\sum_{i=1}^n \sum_{j=1}^n \alpha_i\alpha_j |A_i \cap A_j|}. \tag{14}$$

Proof. Write $P = \big(\frac{|A_i\cap A_j|}{|A_i||A_j|}\big)_{n\times n}$, $Q = \big(|A_i \cap A_j|\big)_{n\times n}$ and $R = \mathrm{diag}(1/|A_1|, \dots, 1/|A_n|)$. Obviously, $P = RQR$, $Q = R^{-1}PR^{-1}$. Let $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ be a solution to (12). Then

$$P \begin{pmatrix} \beta_1|A_1| \\ \beta_2|A_2| \\ \vdots \\ \beta_n|A_n| \end{pmatrix} = RR^{-1}PR^{-1} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = RQ \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = R \begin{pmatrix} |A_1| \\ |A_2| \\ \vdots \\ |A_n| \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

This solves the existence. Suppose $(q_1, q_2, \cdots, q_n)^T$ is a solution to (13), that is,

$$RQR \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \Leftrightarrow Q \begin{pmatrix} q_1/|A_1| \\ q_2/|A_2| \\ \vdots \\ q_n/|A_n| \end{pmatrix} = \begin{pmatrix} |A_1| \\ |A_2| \\ \vdots \\ |A_n| \end{pmatrix}.$$

By (11), (12) and (13),

$$\begin{aligned} \max_{(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n} \frac{\big(\sum_{i=1}^n \alpha_i |A_i|\big)^2}{\sum_{i=1}^n \sum_{j=1}^n \alpha_i\alpha_j |A_i \cap A_j|} &= \sum_{i=1}^n \sum_{j=1}^n \frac{q_i}{|A_i|} \cdot \frac{q_j}{|A_j|} \cdot |A_i \cap A_j| \\ &= (q_1, q_2, \cdots, q_n) P \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = (q_1, q_2, \cdots, q_n) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^n q_i. \end{aligned}$$

So we get (14). This concludes the whole proof. □

3.3 A combinatorial proof

In this subsection, we provide a combinatorial proof for the inequality in (11) to help understand the equality case. To achieve the goal we need only prove

$$\Big|\bigcup_{i=1}^{n} A_i\Big| \geqslant \frac{\big(\sum\limits_{i=1}^{n} \alpha_i |A_i|\big)^2}{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \alpha_i\alpha_j |A_i \cap A_j|}.$$

holds for all integral weights $\alpha_i \in \mathbb{Z}$ such that $\sum_{i=1}^{n} \alpha_i |A_i| > 0$. Suppose this is the case. Let $U = \cup_{i=1}^{n} A_i$ and χ_i be the indicator function of A_i. Define $f(x) = \sum_{i=1}^{n} \alpha_i \chi_i(x)$ and for all $k \in \mathbb{Z}$,

$$U^k \doteq \{x \in U : f(x) = k\}, \quad A_i^k \doteq A_i \cap U^k.$$

Obviously, $f = \sum_{k\in\mathbb{Z}} k\chi_{U^k}$. Note

$$\sum_{i=1}^{n} \alpha_i |A_i^k| = \sum_{i=1}^{n} \alpha_i \int_U \chi_{A_i \cap U^k} = \sum_{i=1}^{n} \alpha_i \int_U \chi_i \cdot \chi_{U^k} = \int_U f \cdot \chi_{U^k} = k \cdot |U^k|, \tag{15}$$

and

$$\sum_{k\in\mathbb{Z}} k|A_i^k| = \sum_{k\in\mathbb{Z}} k \int_U \chi_i \cdot \chi_{U^k} = \int_{A_i} \sum_{k\in\mathbb{Z}} k\chi_{U^k} = \int_{A_i} \sum_{j=1}^{n} \alpha_j \chi_j = \sum_{j=1}^{n} \alpha_j |A_i \cap A_j|, \tag{16}$$

here the integration means $\int_U g = \sum_{x\in U} g(x)$. By (15),

$$|U| = \sum_{k\in\mathbb{Z}} |U^k| \geqslant \sum_{k\neq 0} \frac{\sum_{i=1}^{n} \alpha_i |X_i^k|}{k}.$$

Now we need an inequality: for all $r, s > 0$ one has

$$\frac{1}{s} \geqslant \frac{2}{r} - \frac{s}{r^2} \quad \Big(\Leftrightarrow (\frac{1}{s} - \frac{1}{r})^2 \geqslant 0 \Big).$$

By (15) again, $\sum_{i=1}^{n} \alpha_i |A_i^k|$ and k have the same sign, and consequently for $r > 0$,

$$\begin{aligned} \frac{\sum_{i=1}^{n} \alpha_i |A_i^k|}{k} &\geqslant \begin{cases} \frac{2}{r}\sum_{i=1}^{n} \alpha_i |A_i^k| - \frac{k}{r^2}\sum_{i=1}^{n} \alpha_i |A_i^k| & \text{if } k > 0 \\ -\frac{2}{r}\sum_{i=1}^{n} \alpha_i |A_i^k| - \frac{k}{r^2}\sum_{i=1}^{n} \alpha_i |A_i^k| & \text{if } k < 0 \end{cases} \\ &\geqslant \frac{2}{r}\sum_{i=1}^{n} \alpha_i |A_i^k| - \frac{k}{r^2}\sum_{i=1}^{n} \alpha_i |A_i^k| \qquad\qquad \text{if } k \neq 0. \end{aligned}$$

Recall that $\frac{2}{r}\sum_{i=1}^{n} \alpha_i |A_i^k| - \frac{k}{r^2}\sum_{i=1}^{n} \alpha_i |A_i^k| = 0$ when $k = 0$. By (16),

$$|U| \geqslant \sum_{k\in\mathbb{Z}} \left(\frac{2}{r}\sum_{i=1}^{n} \alpha_i |A_i^k| - \frac{k}{r^2}\sum_{i=1}^{n} \alpha_i |A_i^k| \right) = \frac{2}{r}\sum_{i=1}^{n} \alpha_i |A_i| - \frac{1}{r^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j |A_i \cap A_j| \doteq W(r).$$

Finally,

$$|U| \geqslant \max_{r>0} W(r) = W(r^*) = \frac{\big(\sum_{i=1}^{n} \alpha_i |A_i|\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j |A_i \cap A_j|},$$

where $r^* = (\sum_{i=1}^n \alpha_i|A_i|)/(\sum_{i=1}^n \sum_{j=1}^n \alpha_i\alpha_j|A_i \cap A_j|)$. This concludes the whole proof. A byproduct of this proof is the following characterization of the equality case:

$$\Big|\bigcup_{i=1}^{n} A_i\Big| = \frac{\big(\sum_{i=1}^{n} \alpha_i |A_i|\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j |A_i \cap A_j|} \Leftrightarrow \sum_{i=1}^{n} \alpha_i \chi_i(x)\Big|_{\bigcup_{i=1}^n A_i} \text{is a non-zero constant function.}$$

4 From Functional Analysis to Probability Theory

4.1 Finitely many events

In this section we choose H to be the L^2 space of the given probability space $(\Omega, \mathcal{F}, P)$. Let E, F be two events and χ_E, χ_F be the corresponding indicator functions. It is well-known that Hilbert space theory and probability theory are intimately connected by

$$< \chi_E, \chi_F >= P(E \cap F).$$

Note χ_E, χ_F are orthogonal means E, F are disjoint. Given events $\{E_i\}_{i=1}^n$, define $y_i = \chi_{E_i}$ $(i \in [n])$ and $x = \chi_{\cup_i E_i}$. By (2) and (3), we extend the Gallot-Kounias inequality [9, 11] to its full generality in the following form.

Theorem 4.1 (Gallot-Kounias)**.**

$$P(\bigcup_{i=1}^{n} E_i) \geqslant \max_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n} \frac{\big(\sum_{i=1}^{n} \alpha_i P(E_i)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j P(E_i \cap E_j)} = \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i\gamma_j P(E_i \cap E_j), \tag{17}$$

where $(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ *is any solution to*

$$\begin{pmatrix} P(E_1 \cap E_1) & P(E_1 \cap E_2) & \cdots & P(E_1 \cap E_n) \\ P(E_2 \cap E_1) & P(E_2 \cap E_2) & \cdots & P(E_2 \cap E_n) \\ \vdots & \vdots & \ddots & \vdots \\ P(E_n \cap E_1) & P(E_n \cap E_2) & \cdots & P(E_n \cap E_n) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} = \begin{pmatrix} P(E_1) \\ P(E_2) \\ \vdots \\ P(E_n) \end{pmatrix}. \tag{18}$$

To the authors' knowledge, it seems that the Gallot-Kounias inequality, being discovered 40 years ago, was almost forgotten by Mathematicians. Gallot and Kounias originally expressed their results in terms of generalized inverse of matrices, and this may prevent their results from being appreciated by others. So we restate their results in a more natural way in Theorem 4.1. Note in this context (10) turns out to be the original Chung-Erdős inequality [3]

$$P(\bigcup_{i=1}^{n} E_i) \geqslant \frac{\big(\sum_{i=1}^{n} P(E_i)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i \cap E_j)}, \tag{19}$$

and the Bessel inequality (9) turns out to be a trivial equality. Also note that

$$\sup_{\alpha_i>0} \frac{\big(\sum_{i=1}^{n} \alpha_i P(E_i)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j P(E_i \cap E_j)} = \sup_{\alpha_i>0} \frac{\big(\sum_{i=1}^{n} \alpha_i P(E_i)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i^2 P(E_i \cap E_j)} = \sup_{\alpha_i>0} \sum_{i=1}^{n} \frac{\alpha_i P(E_i)^2}{\sum_{j=1}^{n} \alpha_j P(E_i \cap E_j)}.$$

Similar to Theorem 3.2 one can establish the following theorem.

Theorem 4.2. *The following matrix equation always has a solution*

$$\Big(\frac{P(E_i \cap E_j)}{P(E_i)P(E_j)}\Big)_{n\times n} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}; \tag{20}$$

any solution to (20) *satisfies*

$$\sum_{i=1}^{n} q_i = \max_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n} \frac{\big(\sum_{i=1}^{n} \alpha_i P(E_i)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j P(E_i \cap E_j)}. \tag{21}$$

4.2 Borel-Cantelli lemma

Let $\{E_i\}_{i=1}^{\infty}$ be infinitely many events on the probability space $(\Omega, \mathcal{F}, P)$. The Borel-Cantelli lemma states that: (a) if $\sum_{i=1}^{\infty} P(E_i) < \infty$, then $P(\limsup E_i) = 0$; (b) if $\sum_{i=1}^{\infty} P(E_i) = \infty$ and $\{E_i\}_{i=1}^{\infty}$ are mutually independent, then $P(\limsup E_i) = 1$. Here $\limsup E_i = \cap_{i=1}^{\infty} \cup_{k=i}^{\infty} E_k$. The Borel-Cantelli lemma played an exceptionally important role in probability theory, and many investigations were devoted to the second part of the Borel-Cantelli lemma in the attempt to weaken the independence condition on $\{E_i\}_{i=1}^{\infty}$.

Towards this question, Erdős and Rényi [6, 14] obtained a nice result closely related to (19): if $\sum_{i=1}^{\infty} P(E_i) = \infty$, then

$$P(\limsup E_i) \geqslant \limsup_{n\to\infty} \frac{\big(\sum_{k=1}^{n} P(E_k)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i \cap E_j)}. \tag{22}$$

Recently, by carefully studying the effect of the denominator in the right hand of (22), the authors [8] established a weighted version of the Erdős-Rényi theorem which states:

Theorem 4.3 (Feng-Li-Shen). *If $\sum_{i=1}^{\infty} \alpha_i P(E_i) = \infty$, then*

$$P(\limsup E_i) \geqslant \limsup_{n\to\infty} \frac{\big(\sum_{k=1}^{n} \alpha_k P(E_k)\big)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j P(E_i \cap E_j)}. \tag{23}$$

5 Applications of the de Caen-Selberg Inequality

5.1 The finite field Kakeya set

Let $\mathbb{F}_q$ denote a finite field of q elements. Define a set $K \subset \mathbb{F}_q^n$ to be Kakeya if it contains a translate of any given line. The finite field Kakeya problem, posed by Wolff in his influential survey [17], conjectured that $|K| \geqslant C_n q^n$ holds for some constant C_n. Recently, using the polynomial method in algebraic extremal combinatorics, Dvir [4] completely confirmed this conjecture by proving

$$|K| \geqslant \binom{n+q-1}{n}. \tag{24}$$

If $n = 2$, it is well-known that (24) is sharp [7] and can be established by a simple counting argument [15]. For $n \geqslant 3$, see [16] for further improvement.

Similarly, we say a subset $E \subset \mathbb{F}_q^n$ is an (n, k)-set if it contains a translate of any given k-plane. Ellenberg, Oberlin and Tao [5] proved that if $2 \leqslant k < n$, then

$$|E| \geqslant q^n - \binom{n}{2} q^{n-k+1} + o(q^{n-k+1}) \quad (q \to \infty). \tag{25}$$

Using the de Caen-Selberg inequality we can slightly improve (25) when $k = n - 1 \geqslant 2$.

Theorem 5.1. *Any $(n, n-1)$-set $E \subset \mathbb{F}_q^n$ $(n \geqslant 3)$ satisfies*

$$|E| \geqslant q^n - q^2 + o(q^2) \quad (q \to \infty),$$

where $\mathbb{F}_q$ denotes a finite field of q elements.

Proof. Since the total number s of $(n-1)$-dimensional hyperplanes passing through the origin equals the total number of lines passing through the origin,

$$s = \frac{q^n - 1}{q - 1}.$$

Let $\{P_i\}_{i=1}^s$ be such hyperplanes. By the de Caen-Selberg inequality (4),

$$\begin{aligned}
|E| &\geqslant \sum_{i=1}^{s} \frac{|P_i|^2}{\sum_{j=1}^{s} |P_i \cap P_j|} \geqslant \frac{s \cdot q^{2n-2}}{q^{n-1} + (s-1)q^{n-2}} \\
&= \frac{s \cdot q^{2n-2} + q^n(q^{n-1} - q^{n-2}) - q^n(q^{n-1} - q^{n-2})}{(q^{n-1} - q^{n-2}) + s \cdot q^{n-2}} \\
&= q^n - \frac{q^n(q^{n-1} - q^{n-2})}{q^{n-1} + (s-1)q^{n-2}} \\
&= q^n - q^2 + o(q^2) \quad (q \to \infty).
\end{aligned}$$

□

5.2 The finite field Nikodym set

Define a set $B \subset \mathbb{F}_q^n$ to be Nikodym if for each $z \in B^c$ there exists a line L_z passing through z such that $L_z \backslash \{z\} \subset B$. Obviously, all such lines $\{L_z\}_{z \in B^c}$ are different from each other. Similar to (24) Li [12] proved (i)

$$|B| \geqslant \binom{n+q-2}{n}; \tag{26}$$

(ii) any two-dimensional Nikodym set $B \subset \mathbb{F}_q^2$ satisfies

$$|B| \geqslant \frac{2q^2}{3} + O(q) \quad (q \to \infty). \tag{27}$$

Using the de Caen-Selberg inequality we can improve (27) substantially as follows, which shows some difference between the two-dimensional Kakeya sets and Nikodym sets.

Theorem 5.2. *Any Nikodym set $B \subset \mathbb{F}_q^2$ satisfies*

$$|B| \geqslant q^2 - q^{3/2} - q,$$

where $\mathbb{F}_q$ denotes a finite field of q elements.

Proof. Let $s = |B^c|$. By the de Caen-Selberg inequality (4),

$$q^2 - s = |B| \geqslant \Big| \bigcup_{z \in B^c} L_z \backslash \{z\} \Big| \geqslant \sum_{i=1}^{s} \frac{(q-1)^2}{(q-1) + s - 1} = \frac{s(q-1)^2}{s + q - 2}.$$

Equivalently,

$$s^2 - (q+1)s - q^2(q-2) \leqslant 0.$$

Hence

$$|B| = q^2 - s \geqslant q^2 - \frac{q+1+\sqrt{(q+1)^2+4q^2(q-2)}}{2} \geqslant q^2 - q^{3/2} - q.$$

□

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