On cross-intersecting uniform sub-families of hereditary families

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Abstract

A family $\mathcal{H}$ of sets is *hereditary* if any subset of any set in $\mathcal{H}$ is in $\mathcal{H}$. If two families $\mathcal{A}$ and $\mathcal{B}$ are such that any set in $\mathcal{A}$ intersects any set in $\mathcal{B}$, then we say that $(\mathcal{A}, \mathcal{B})$ is a *cross-intersection pair* (*cip*). We say that a cip $(\mathcal{A}, \mathcal{B})$ is *simple* if at least one of $\mathcal{A}$ and $\mathcal{B}$ contains only one set. For a family $\mathcal{F}$, let $\mu(\mathcal{F})$ denote the size of a smallest set in $\mathcal{F}$ that is not a subset of any other set in $\mathcal{F}$. For any positive integer r, let $[r] := \{1, 2, ..., r\}$, $2^{[r]} := \{A \colon A \subseteq [r]\}$, $\mathcal{F}^{(r)} := \{F \in \mathcal{F} \colon |F| = r\}$.

We show that if a hereditary family $\mathcal{H} \subseteq 2^{[n]}$ is *compressed*, $\mu(\mathcal{H}) \geqslant r + s$ with $r \leqslant s$, and $(\mathcal{A}, \mathcal{B})$ is a cip with $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(r)}$ and $\emptyset \neq \mathcal{B} \subset \mathcal{H}^{(s)}$, then $|\mathcal{A}| + |\mathcal{B}|$ is a maximum if $(\mathcal{A}, \mathcal{B})$ is the simple cip $\big(\{[r]\}, \{B \in \mathcal{H}^{(s)} \colon B \cap [r] \neq \emptyset\}\big)$; Frankl and Tokushige proved this for $\mathcal{H} = 2^{[n]}$. We also show that for any $2 \leqslant r \leqslant s$ and $m \geqslant r + s$ there exist (non-compressed) hereditary families $\mathcal{H}$ with $\mu(\mathcal{H}) = m$ such that no cip $(\mathcal{A}, \mathcal{B})$ with a maximum value of $|\mathcal{A}| + |\mathcal{B}|$ under the condition that $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(r)}$ and $\emptyset \neq \mathcal{B} \subset \mathcal{H}^{(s)}$ is simple (we prove that this is not the case for $r = 1$), and we suggest two conjectures about the extremal structures in general.

1 Introduction

We shall use small letters such as x to denote elements of a set or positive integers, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose members are sets themselves). Unless otherwise stated, it is to be assumed that sets and families are *finite*.

$\mathbb{N}$ is the set $\{1, 2, ...\}$ of positive integers. For $m, n \in \mathbb{N}$ with $m \leqslant n$, the set $\{i \in \mathbb{N} \colon m \leqslant i \leqslant n\}$ is denoted by $[m, n]$, and if $m = 1$ then we also write $[n]$. The *power set* $\{A : A \subseteq X\}$ of a set X is denoted by 2^X, and $\{A \subseteq X \colon |A| = r\}$ is denoted by $\binom{X}{r}$.

We next develop some notation for certain sets and families defined on a family $\mathcal{F} \subseteq 2^X$. Let $U(\mathcal{F})$ denote the *union* of all sets in $\mathcal{F}$. Let $\mathcal{F}^{(r)} := \{F \in \mathcal{F} \colon |F| = r\}$. For a set Y, let $\mathcal{F}(Y) := \{F \in \mathcal{F} \colon F \cap Y \neq \emptyset\}$ and $\mathcal{F}[Y] := \{F \in \mathcal{F} \colon Y \subseteq F\}$. For a single-element set $\{y\}$, we may abbreviate the notation $\mathcal{F}(\{y\})$ to $\mathcal{F}(y)$, and we set $\mathcal{F}\langle y \rangle := \{F \backslash \{y\} \colon F \in \mathcal{F}(y)\}$.

For $i, j \in [n]$, let $\Delta_{i,j} \colon 2^{2^{[n]}} \to 2^{2^{[n]}}$ be the *compression operation* (see [4]) defined by

$$\Delta_{i,j}(\mathcal{F}) := \{\delta_{i,j}(F) \colon F \in \mathcal{F}, \delta_{i,j}(F) \notin \mathcal{F}\} \cup \{F \in \mathcal{F} \colon \delta_{i,j}(F) \in \mathcal{F}\},$$

where $\delta_{i,j} \colon 2^{[n]} \to 2^{[n]}$ is defined by

$$\delta_{i,j}(F) := \begin{cases} (F \backslash \{j\}) \cup \{i\} & \text{if } i \notin F \text{ and } j \in F; \\ F & \text{otherwise.} \end{cases}$$

A family $\mathcal{F}$ is said to be
- a *hereditary family* (or an *ideal* or a *downset*) if all subsets of any set in $\mathcal{F}$ are in $\mathcal{F}$;
- *uniform* if the sets in $\mathcal{F}$ have the same size;
- *intersecting* if any set in $\mathcal{F}$ intersects any other set in $\mathcal{F}$;
- *centred* if the sets in $\mathcal{F}$ contain a common element;
- *compressed* if $\mathcal{F} \subseteq 2^{[n]}$ and $\Delta_{i,j}(\mathcal{F}) = \mathcal{F}$ for any $i, j \in [n]$ with $i < j$;
- *compressed with respect to* $x \in U(\mathcal{F})$ if $\Delta_{x,y}(\mathcal{F}) = \mathcal{F}$ for any $y \in U(\mathcal{F})$.

Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting* if any set in $\mathcal{A}$ intersects any set in $\mathcal{B}$. We say that $(\mathcal{A}, \mathcal{B})$ is a *cross-intersection pair* (*cip*) if $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting. We say that a cip $(\mathcal{A}, \mathcal{B})$ is *simple* if at least one of $\mathcal{A}$ and $\mathcal{B}$ contains only one set.

Hilton and Milner [7] proved that if $r \leqslant n/2$ and $\mathcal{A}, \mathcal{B}$ are non-empty cross-intersecting sub-families of $\binom{[n]}{r}$, then $|\mathcal{A}| + |\mathcal{B}| \leqslant \binom{n}{r} - \binom{n-r}{r} + 1 = |\mathcal{A}_0| + |\mathcal{B}_0|$, where $\mathcal{A}_0$ is $\{[r]\}$ and $\mathcal{B}_0$ is $\{B \in \binom{[n]}{r} \colon B \cap [r] \neq \emptyset\}$. A streamlined proof of this result was later obtained by Simpson [10] by means of the compression (also known as *shifting*) technique introduced in the seminal paper [4] (see [5] for a good survey on the uses of this technique in extremal set theory). Frankl and Tokushige [6] instead used the Kruskal-Katona Theorem [8, 9] to establish the following extension.

Theorem 1.1 (Frankl and Tokushige [6]) *If* $r \leqslant s$, $n \geqslant r + s$, *and* $(\mathcal{A}, \mathcal{B})$ *is a cip with* $\emptyset \neq \mathcal{A} \subseteq \binom{[n]}{r}$ *and* $\emptyset \neq \mathcal{B} \subseteq \binom{[n]}{s}$, *then* $|\mathcal{A}| + |\mathcal{B}| \leqslant \binom{n}{s} - \binom{n-r}{s} + 1 = |\mathcal{A}_0| + |\mathcal{B}_0|$, *where* $(\mathcal{A}_0, \mathcal{B}_0)$ *is the simple cip* $\left(\{[r]\}, \{B \in \binom{[n]}{s} \colon B \cap [r] \neq \emptyset\}\right)$.

In this paper we are interested in cip's $(\mathcal{A}, \mathcal{B})$ having a maximum value of $|\mathcal{A}| + |\mathcal{B}|$ under the condition that both $\mathcal{A}$ and $\mathcal{B}$ are non-empty uniform sub-families of a hereditary family $\mathcal{H}$. Note that Theorem 1.1 deals with the special case when $\mathcal{H}$ is the power set $2^{[n]}$, which is the simplest example of a hereditary family. It is easy to see that a family is hereditary if and only if it is a union of power sets. There are many interesting examples of hereditary families, such as the family of independent sets of a graph or matroid.

Before stating our results, we shall introduce a few more definitions.

We say that a set M is $\mathcal{F}$*-maximal* if M is not a subset of any set in $\mathcal{F}\backslash\{M\}$. We denote the size of a smallest $\mathcal{F}$-maximal set in $\mathcal{F}$ by $\mu(\mathcal{F})$.

For a family $\mathcal{F}$, we denote the set $\{(\mathcal{A},\mathcal{B})\colon (\mathcal{A},\mathcal{B})$ is a cip with a maximum value of $|\mathcal{A}|+|\mathcal{B}|$ under the condition that $\emptyset \neq \mathcal{A} \subset \mathcal{F}^{(r)}$ and $\emptyset \neq \mathcal{B} \subset \mathcal{F}^{(s)}\}$ by $C(\mathcal{F}, r, s)$.

Using the compression technique, we generalise Theorem 1.1 as follows.

Theorem 1.2 *If $r \leqslant s$, $n \geqslant r+s$, and $\mathcal{H}$ is a compressed hereditary sub-family of $2^{[n]}$ with $\mu(\mathcal{H}) \geqslant r+s$, then the simple cip $\big(\{[r]\}, \{B \in \mathcal{H}^{(s)}\colon B \cap [r] \neq \emptyset\}\big)$ is in $C(\mathcal{H}, r, s)$.*

Theorem 1.1 is the case $\mathcal{H} = 2^{[n]}$, in which $[n]$ is the only $\mathcal{H}$-maximal set in $\mathcal{H}$ and hence $\mu(\mathcal{H}) = n$. Note that we cannot relax the condition that $\mu(\mathcal{H}) \geqslant r+s$. Indeed, if $\mathcal{H} = 2^{[n]}$ and $s \leqslant \mu(\mathcal{H}) < r+s$, then any set in $\mathcal{H}^{(r)} = \binom{[n]}{r}$ intersects any set in $\mathcal{H}^{(s)} = \binom{[n]}{s}$ (since $n = \mu(\mathcal{H}) < r+s$), and hence $\big(\mathcal{H}^{(r)}, \mathcal{H}^{(s)}\big)$ is the only cip in $C(\mathcal{H}, r, s)$. Note that if $\mathcal{H} = 2^{[n]}$ and $\mu(\mathcal{H}) < s$, then $C(\mathcal{H}, r, s) = \emptyset$ (since $n = \mu(\mathcal{H}) < s$ and hence $\mathcal{H}^{(s)} = \emptyset$).

Remark 1.3 One of the central problems in extremal set theory is the famous Chvátal Conjecture [2], which claims that at least one of the largest intersecting sub-families of any hereditary family $\mathcal{H}$ is centred. Chvátal [3] proved his conjecture for the case when $\mathcal{H}$ is compressed. Snevily [11] improved Chvátal's result to the case when $\mathcal{H}$ is compressed with respect to an element of $U(\mathcal{H})$. In the next section we show that no similar improvement can be made to Theorem 1.2 for $r \geqslant 2$; more precisely, we show that for any $2 \leqslant r \leqslant s$ and $m \geqslant r+s$ there are hereditary families $\mathcal{H}$ with $\mu(\mathcal{H}) = m$ such that $\mathcal{H}$ is compressed with respect to an element of $U(\mathcal{H})$ and no cip in $C(\mathcal{H}, r, s)$ is simple. We then suggest two conjectures about the structure of at least one of the cip's in $C(\mathcal{H}, r, s)$ for any hereditary family $\mathcal{H}$ with $\mu(\mathcal{H}) \geqslant r+s$.

For $r = 1$ we do have the desired general result.

Theorem 1.4 *If $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geqslant 1+s$, then $C(\mathcal{H}, 1, s)$ has a simple cip $(\mathcal{A}_0, \mathcal{B}_0)$ with $\mathcal{A}_0 = \{\{x\}\}$ and $\mathcal{B}_0 = \{B \in \mathcal{H}^{(s)}\colon x \in B\}$ for some $x \in U(H)$.*

Proof. Let $(\mathcal{A},\mathcal{B}) \in C(\mathcal{H}, 1, s)$. Suppose $|\mathcal{A}| = 1$. Then, since $\mathcal{A} \subset \mathcal{H}^{(1)}$, $\mathcal{A} = \{\{x\}\}$ for some $x \in U(\mathcal{H})$. Since $\mathcal{B} \subset \mathcal{H}^{(s)}$ and $|\mathcal{A}|+|\mathcal{B}|$ is a maximum (under the cross-intersection condition), $\mathcal{B}$ must consist of all the sets in $\mathcal{H}^{(s)}$ which contain x.

Now suppose $|\mathcal{A}| > 1$. Let $Z := \{z \in U(\mathcal{H})\colon \{z\} \in \mathcal{A}\}$; so $|Z| = |\mathcal{A}|$ and hence $|Z| > 1$. Since every set in $\mathcal{B}$ must intersect every (single-element) set in $\mathcal{A}$, we clearly have $\mathcal{B} \subseteq \mathcal{H}^{(s)}[Z]$ $(= \{H \in \mathcal{H}^{(s)}\colon Z \subseteq H\})$. Let $B \in \mathcal{B}$. Since every (single-element) set in $\mathcal{A}$ must intersect B, we have $Z \subseteq B$ and hence $|Z| \leqslant s$. Let $x \in Z$ and let M be an $\mathcal{H}$-maximal set in $\mathcal{H}$ such that $B \subset M$. Then $|M| \geqslant 1+s$ (as $|M| \geqslant \mu(\mathcal{H})$), $Z \subset M$ (as $Z \subseteq B$), and $\binom{M}{s} \subseteq \mathcal{H}^{(s)}$ (as $\mathcal{H}$ is hereditary). Now let $(\mathcal{A}_0, \mathcal{B}_0)$ be the simple cip $\big(\{\{x\}\}, \mathcal{H}^{(s)}(x)\big)$. Since $(\mathcal{A},\mathcal{B}) \in C(\mathcal{H}, 1, s)$, $|\mathcal{A}_0|+|\mathcal{B}_0| \leqslant |\mathcal{A}|+|\mathcal{B}|$. Also,

$$\begin{aligned}|\mathcal{A}_0| + |\mathcal{B}_0| &= 1 + \left|\mathcal{H}^{(s)}(x)\right| \\ &\geqslant 1 + \left|\mathcal{H}^{(s)}[Z]\right| + \left|\left\{A \in \binom{M}{s} \colon x \in A, |A \cap Z| = |Z| - 1\right\}\right| \\ &= 1 + \left|\mathcal{H}^{(s)}[Z]\right| + \binom{|Z|-1}{|Z|-2}\binom{|M|-|Z|}{s-(|Z|-1)} \\ &\geqslant |Z| + \left|\mathcal{H}^{(s)}[Z]\right| = |\mathcal{A}| + \left|\mathcal{H}^{(s)}[Z]\right| \geqslant |\mathcal{A}| + |\mathcal{B}|.\end{aligned}$$

So we actually have $|\mathcal{A}_0| + |\mathcal{B}_0| = |\mathcal{A}| + |\mathcal{B}|$, and hence $(\mathcal{A}_0, \mathcal{B}_0) \in C(\mathcal{H}, 1, s)$. □

The above result will be used in the proof of Theorem 1.2. It is easy to see from its proof that if $\mu(\mathcal{H}) > 1 + s$, then any $(\mathcal{A}, \mathcal{B})$ in $C(\mathcal{H}, 1, s)$ is a simple cip as in the result.

2 A construction and two conjectures

The following is the proof of the claim in Remark 1.3.

Proposition 2.1 *Let $2 \leqslant l + 1 \leqslant r \leqslant s$, $m \geqslant r + s$ and $p > \left(\binom{m-l}{s} - \binom{m-r}{s} + 1\right) / \binom{m-l}{r-l}$. For each $i \in [p]$, let $M_i := [l] \cup [(i-1)(m-l)+l+1, i(m-l)+l]$. Let $\mathcal{E} = \bigcup_{i=1}^{p} 2^{M_i}$. Then $\mathcal{E}$ is hereditary, $\mathcal{E}$ is compressed with respect to 1, $\mu(\mathcal{E}) = m$, and no cip in $C(\mathcal{E}, r, s)$ is simple.*

Proof. It is straightforward that $\mathcal{E}$ is hereditary, $\mathcal{E}$ is compressed with respect to 1, and $\mu(\mathcal{E}) = |M_1| = \ldots = |M_p| = m$. Let $(\mathcal{A}, \mathcal{B})$ be a simple cip with $\emptyset \neq \mathcal{A} \subseteq \mathcal{E}^{(r)}$ and $\emptyset \neq \mathcal{B} \subseteq \mathcal{E}^{(s)}$. Let $L := [l]$, $\mathcal{A}_1 := \{L \cup C \colon C \in \binom{M_i \backslash L}{r-l}$ for some $i \in [p]\}$, $\mathcal{B}_1 = \mathcal{E}^{(s)}(L)$ $(= \{E \in \mathcal{E}^{(s)} \colon E \cap L \neq \emptyset\})$. Since $(\mathcal{A}_1, \mathcal{B}_1)$ is a non-simple cip with $\emptyset \neq \mathcal{A}_1 \subseteq \mathcal{E}^{(r)}$ and $\emptyset \neq \mathcal{B}_1 \subseteq \mathcal{E}^{(s)}$, the result follows if we show that $|\mathcal{A}| + |\mathcal{B}| < |\mathcal{A}_1| + |\mathcal{B}_1|$.

Let $R := [r]$, $\mathcal{A}_0 := \{R\}$, $\mathcal{B}_0 := \mathcal{E}^{(s)}(R)$. We will show that

$$|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}_0| + |\mathcal{B}_0|. \tag{1}$$

Let us first assume this. Note that $\mathcal{B}_0$ is the disjoint union of $\mathcal{B}_1$ and the family $\mathcal{R}$ of sets in $\mathcal{E}^{(s)}$ that intersect R but not L. Since R is a subset of M_1 but not a subset of any other set M_i, we clearly have $\mathcal{R} = \{A \in \binom{M_1 \backslash L}{s} \colon A \cap (R \backslash L) \neq \emptyset\}$. We have

$$\begin{aligned}(|\mathcal{A}_1| + |\mathcal{B}_1|) - (|\mathcal{A}| + |\mathcal{B}|) &\geqslant (|\mathcal{A}_1| + |\mathcal{B}_1|) - (|\mathcal{A}_0| + |\mathcal{B}_0|) \quad \text{(by (1))} \\ &= (|\mathcal{A}_1| + |\mathcal{B}_1|) - (|\mathcal{A}_0| + |\mathcal{B}_1| + |\mathcal{R}|) = |\mathcal{A}_1| - |\mathcal{A}_0| - |\mathcal{R}| \\ &= p\binom{m-l}{r-l} - \binom{m-l}{s} + \binom{m-r}{s} - 1 \\ &> 0 \quad \text{(by choice of } p\text{)}\end{aligned}$$

and hence $|\mathcal{A}| + |\mathcal{B}| < |\mathcal{A}_1| + |\mathcal{B}_1|$ as required.

We now prove (1). Suppose $\mathcal{A}$ contains only one set A. Then $\mathcal{B} \subseteq \mathcal{E}^{(s)}(A)$. Since $l < r$ and $M_i \cap M_j = L$ for any distinct i and j in $[p]$, there is a unique k in $[p]$ such that $A \subset M_k$, and it is therefore easy to see that $\left|\mathcal{E}^{(s)}(A)\right| \leqslant |\mathcal{B}_0|$; so (1) holds in this case. Now suppose $|\mathcal{A}| > 1$. Then, since $(\mathcal{A}, \mathcal{B})$ is a simple cip, $\mathcal{B}$ contains only one set B and $\mathcal{A} \subseteq \mathcal{E}^{(r)}(B)$. Let $S := [s]$, $\mathcal{C}_0 := \mathcal{E}^{(r)}(S)$, $\mathcal{D}_0 := \{S\}$. Similarly to the above, it is easy to see that $\left|\mathcal{E}^{(r)}(B)\right| \leqslant |\mathcal{C}_0|$; so $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{C}_0| + |\mathcal{D}_0|$. If $r = s$ then $|\mathcal{C}_0| + |\mathcal{D}_0| = |\mathcal{A}_0| + |\mathcal{B}_0|$ and hence (1) holds again. Suppose $r < s$. For each $i \in [p]$, let $\mathcal{F}_i := \binom{M_i}{s}$ and $\mathcal{G}_i := \binom{M_i}{r}$. Since $R \subset M_1$ and $R \cap M_i = S \cap M_i = L$ for each $i \in [2, p]$, we clearly have $|\mathcal{B}_0| = |\mathcal{F}_1(R)| + \sum_{i=2}^{p} |\mathcal{F}_i(L)|$ and $|\mathcal{C}_0| = |\mathcal{G}_1(S)| + \sum_{i=2}^{p} |\mathcal{G}_i(L)|$. We have $|\mathcal{G}_1(S)| < |\mathcal{F}_1(R)|$ since

$$|\mathcal{F}_1(R)| - |\mathcal{G}_1(S)| = \left(\binom{m}{s} - \binom{m-r}{s}\right) - \left(\binom{m}{r} - \binom{m-s}{r}\right)$$
$$= \left(\binom{m}{s} - \binom{m}{r}\right) - \left(\binom{m-r}{s} - \binom{m-s}{r}\right)$$
$$= \binom{m}{r}\left(\frac{r!(m-r)...(m-s+1)}{s!} - 1\right) - \binom{m-s}{r}\left(\frac{r!(m-r)...(m-s+1)}{s!} - 1\right) > 0.$$

By a similar calculation, we obtain that $|\mathcal{G}_i(L)| < |\mathcal{F}_i(L)|$ for each $i \in [2, p]$. So we have

$$|\mathcal{C}_0| + |\mathcal{D}_0| = |\mathcal{G}_1(S)| + \sum_{i=2}^{p} |\mathcal{G}_i(L)| + 1 < |\mathcal{F}_1(R)| + \sum_{i=2}^{p} |\mathcal{F}_i(L)| + 1 = |\mathcal{A}_0| + |\mathcal{B}_0|$$

and hence, since $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{C}_0| + |\mathcal{D}_0|$, (1) holds again. □

Something common to the cip $(\mathcal{A}_1, \mathcal{B}_1)$ in the above proof and the extremal structures determined in Theorems 1.2 and 1.4 is that the first family in the pair is centred. We conjecture that there always exist cip's $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ centred that are extremal under the conditions we have been considering, where by extremal we mean that $|\mathcal{A}| + |\mathcal{B}|$ is a maximum.

Conjecture 2.2 (Weak Form) *If $r \leqslant s$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geqslant r+s$, then for some $(\mathcal{A}_0, \mathcal{B}_0) \in C(\mathcal{H}, r, s)$, $\mathcal{A}_0$ is centred.*

Conjecture 2.3 (Strong Form) *If $r \leqslant s$ and $\mathcal{H}$ is a hereditary family with $\mu(\mathcal{H}) \geqslant r+s$, then there exists a set H in $\mathcal{H}$ with $1 \leqslant |H| \leqslant r$ such that for some $(\mathcal{A}_0, \mathcal{B}_0) \in C(\mathcal{H}, r, s)$, $\mathcal{A}_0 = \{A \in \mathcal{H}^{(r)} \colon H \subseteq A\}$ and $\mathcal{B}_0 = \{B \in \mathcal{H}^{(s)} \colon B \cap H \neq \emptyset\}$.*

Note that the families $\mathcal{A}_1$ and $\mathcal{B}_1$ in the proof of Proposition 2.1 have the structure of $\mathcal{A}_0$ and $\mathcal{B}_0$ in the above conjecture.

3 Some tools

This section provides the main tools we need for the proof of Theorem 1.2. We start with a crucial lemma concerning the levels of a hereditary family (see [1, Corollary 3.2]).

Lemma 3.1 (Borg [1]) *If $\mathcal{H}$ is a hereditary family and $r < s \leqslant \mu(\mathcal{H}) - r$, then*

$$\left|\mathcal{H}^{(r)}\right| \leqslant \frac{\binom{s}{s-r}}{\binom{\mu(\mathcal{H})-r}{s-r}} \left|\mathcal{H}^{(s)}\right|.$$

The following is our second important lemma, which purely concerns the parameter $\mu(\mathcal{F})$ of a family $\mathcal{F}$.

Lemma 3.2 *Let $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$ and $a \in [n]$. Let $\mathcal{D} := \mathcal{F}\backslash\mathcal{F}(a)$ and $\mathcal{E} := \mathcal{F}\backslash\mathcal{F}(n)$.*
(i) If $\mathcal{F}(a) \neq \emptyset$, then $\mu(\mathcal{F}\langle a\rangle) \geqslant \mu(\mathcal{F}) - 1$.
(ii) If $\mathcal{F}$ is hereditary, then $\mu(\mathcal{D}) \geqslant \mu(\mathcal{F}) - 1$.
(iii) If $\mathcal{F}$ is compressed and $[n] \notin \mathcal{F}$, then $\mu(\mathcal{E}) \geqslant \mu(\mathcal{F})$.

Proof. Suppose $\mathcal{F}(a) \neq \emptyset$. Let M be an $\mathcal{F}\langle a\rangle$-maximal set in $\mathcal{F}\langle a\rangle$. Then $M' := M \cup \{a\}$ is an $\mathcal{F}$-maximal set in $\mathcal{F}$. So $|M| = |M'| - 1 \geqslant \mu(\mathcal{F}) - 1$. Hence (i).

Suppose $\mathcal{F}$ is hereditary. Then, since $\mathcal{F} \neq \emptyset$, $\emptyset \in \mathcal{F}$. So $\mathcal{D} \neq \emptyset$. Let M be a $\mathcal{D}$-maximal set in $\mathcal{D}$. Suppose also that $|M| < \mu(\mathcal{F})$. So M is not $\mathcal{F}$-maximal, and hence there exists a set $M' \in \mathcal{F}(a)$ such that $M \subset M'$ and M' is $\mathcal{F}$-maximal. Since $\mathcal{F}$ is hereditary, $M'' := M'\backslash\{a\} \in \mathcal{F}$. Since M is $\mathcal{D}$-maximal and $M \subseteq M'' \in \mathcal{D}$, $M = M''$. So $M' = M \cup \{a\}$. Therefore $|M| = |M'| - 1 \geqslant \mu(\mathcal{F}) - 1$. Hence (ii).

Suppose $\mathcal{F}$ is compressed and $[n] \notin \mathcal{F}$. Let M be an $\mathcal{E}$-maximal set in $\mathcal{E}$. Suppose $|M| < \mu(\mathcal{F})$. Then there exists a set $M' \in \mathcal{F}(n)$ such that $M \subset M'$. Since $[n] \notin \mathcal{F}$, $X := [n]\backslash M' \neq \emptyset$. Let $x \in X$ and $M'' := \delta_{x,n}(M') = (M'\backslash\{n\}) \cup \{x\}$. Since $\mathcal{F}$ is compressed, $M'' \in \mathcal{F}$. But then $M \subsetneq M'' \in \mathcal{E}$, a contradiction (as M is $\mathcal{E}$-maximal). So $|M| \geqslant \mu(\mathcal{F})$. Hence (iii). □

We remark that the inequalities above cannot be replaced by equalities. An example for (iii) is that if $n \geqslant 3$ and $\mathcal{F}$ is the compressed (hereditary) family $2^{[n-1]} \cup 2^{[n-3]\cup\{n\}}$, then $\mu(\mathcal{E}) = n-1 > n-2 = \mu(\mathcal{F})$.

We shall say that a family $\mathcal{F} \subseteq 2^{[n]}$ is *quasi-compressed* if $\delta_{i,j}(F) \in \mathcal{F}$ for any $F \in \mathcal{F}$ and any $i, j \in U(\mathcal{F})$ with $i < j$. Therefore a quasi-compressed family $\mathcal{F} \subseteq 2^{[n]}$ is isomorphic to a compressed sub-family of $2^{[|U(\mathcal{F})|]}$, and the isomorphism is induced by the bijection $\beta\colon U(\mathcal{F}) \to [|U(\mathcal{F})|]$ defined by $\beta(u_i) := i$, $i = 1, ..., |U(\mathcal{F})|$, where $\{u_1, ..., u_{|U(\mathcal{F})|}\} = U(\mathcal{F})$ and $u_1 < ... < u_{|U(\mathcal{F})|}$.

The next lemma is straightforward, so we omit its proof.

Lemma 3.3 *Let $\mathcal{H} \subseteq 2^{[n]}$ and $a \in [n]$.*
(i) If $\mathcal{H}$ is hereditary, then $\mathcal{H}\backslash\mathcal{H}(a)$ and $\mathcal{H}\langle a\rangle$ are hereditary.
(ii) If $\mathcal{H}$ is quasi-compressed, then $\mathcal{H}\backslash\mathcal{H}(a)$ and $\mathcal{H}\langle a\rangle$ are quasi-compressed.

We shall frequently use the following property of quasi-compressed families.

Lemma 3.4 *Let $\mathcal{F} \subseteq 2^{[n]}$ be a quasi-compressed family with $U(\mathcal{F}) \neq \emptyset$. Let $Z \subseteq [n]$ and let $i, j \in U(\mathcal{F})$, $i \leqslant j$. Then $|\mathcal{F}(Z)| \leqslant |\mathcal{F}(\delta_{i,j}(Z))|$.*

Proof. Let $Y := \delta_{i,j}(Z)$. Suppose $Y \neq Z$, and let $W := Z \cap Y$. Then $i < j$, $Z = W \cup \{j\} \neq W$ and $Y = W \cup \{i\} \neq W$. Let $\mathcal{D} := \{F \in \mathcal{F} \colon i \in F, F \cap W = \emptyset\}$, $\mathcal{E} := \{F \in \mathcal{F} \colon j \in F, F \cap W = \emptyset\}$. Since $\mathcal{F}$ is quasi-compressed and $i, j \in U(\mathcal{F})$, we have $\Delta_{i,j}(\mathcal{E} \backslash \mathcal{E}(i)) \subseteq \mathcal{D} \backslash \mathcal{D}(j)$; so $|\mathcal{D} \backslash \mathcal{D}(j)| \geqslant |\Delta_{i,j}(\mathcal{E} \backslash \mathcal{E}(i))| = |\mathcal{E} \backslash \mathcal{E}(i)|$. Note that $\mathcal{D}(j) = \mathcal{E}(i)$. Thus, since $|\mathcal{F}(Y)| - |\mathcal{F}(Z)| = (|\mathcal{F}(W)| + |\mathcal{D}|) - (|\mathcal{F}(W)| + |\mathcal{E}|) = (|\mathcal{D}(j)| + |\mathcal{D} \backslash \mathcal{D}(j)|) - (|\mathcal{E}(i)| + |\mathcal{E} \backslash \mathcal{E}(i)|) = |\mathcal{D} \backslash \mathcal{D}(j)| - |\mathcal{E} \backslash \mathcal{E}(i)| \geqslant 0$, the result follows. □

For a set $X := \{x_1, ..., x_n\} \subset \mathbb{N}$ with $x_1 < ... < x_n$ and $r \in [n]$, call $\{x_1, ..., x_r\}$ the *initial r-segment of X*. For convenience, we call $\emptyset$ the *initial 0-segment of X*.

Corollary 3.5 *Let $\mathcal{F} \subseteq 2^{[n]}$ be quasi-compressed. Let $\emptyset \neq Z \subseteq [n]$ and let $Y \in \binom{[n]}{|Z|}$ such that Y contains the initial $|Z \cap U(\mathcal{F})|$-segment of $U(\mathcal{F})$. Then $|\mathcal{F}(Z)| \leqslant |\mathcal{F}(Y)|$.*

Proof. Let $Z' := Z \cap U(\mathcal{F})$. If $Z' = \emptyset$ then $|\mathcal{F}(Z)| = 0 \leqslant |\mathcal{F}(Y)|$. Suppose $Z' \neq \emptyset$. Let Y' be the initial $|Z'|$-segment of $U(\mathcal{F})$. Since $\mathcal{F}$ is quasi-compressed and $Z' \subseteq U(\mathcal{F})$, we can construct a composition of compressions $\delta_{i,j}$ with $i, j \in U(\mathcal{F})$, $i \leqslant j$, that yields Y' when applied on Z'. Thus $|\mathcal{F}(Z')| \leqslant |\mathcal{F}(Y')|$ by repeated application of Lemma 3.4. Since $Y' \subseteq Y$ and $|\mathcal{F}(Z)| = |\mathcal{F}(Z')|$, we have $|\mathcal{F}(Z)| \leqslant |\mathcal{F}(Y')| \leqslant |\mathcal{F}(Y)|$. □

The following is a well-known fundamental property of compressions that emerged in [4] and that is not difficult to prove.

Lemma 3.6 *If $\mathcal{A} \subset 2^{[n]}$ is intersecting and $i, j \in [n]$, then $\Delta_{i,j}(\mathcal{A})$ is intersecting.*

4 Proof of Theorem 1.2

Lemma 4.1 *Let r, s, n and $\mathcal{H}$ be as in Theorem 1.2, and let $(\mathcal{A}, \mathcal{B})$ be a cip with $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(r)}$ and $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(s)}$. Let $1 \leqslant i < j \leqslant n$. Then:*
(i) $\Delta_{i,j}(\mathcal{A})$ and $\Delta_{i,j}(\mathcal{B})$ are cross-intersecting;
(ii) if either $\Delta_{m,n}(\mathcal{A}) = \mathcal{A}$ for all $m \in [n-1]$ or $\Delta_{m,n}(\mathcal{B}) = \mathcal{B}$ for all $m \in [n-1]$, then $(A \cap B) \backslash \{n\} \neq \emptyset$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proof. Let $\mathcal{A}' := \{A \cup \{n+1\} \colon A \in \mathcal{A}\}$, $\mathcal{A}'' := \{A^* \cup \{n+1\} \colon A^* \in \Delta_{i,j}(\mathcal{A})\}$, $\mathcal{B}' := \{B \cup \{n+2\} \colon B \in \mathcal{B}\}$, $\mathcal{B}'' := \{B^* \cup \{n+2\} \colon B^* \in \Delta_{i,j}(\mathcal{B})\}$. Clearly, the family $\mathcal{C} := \mathcal{A}' \cup \mathcal{B}'$ is intersecting, and hence $\Delta_{i,j}(\mathcal{C})$ is intersecting by Lemma 3.6. Since $\Delta_{i,j}(\mathcal{C}) = \mathcal{A}'' \cup \mathcal{B}''$, (i) clearly follows.

Suppose without loss of generality that $\Delta_{m,n}(\mathcal{A}) = \mathcal{A}$ for all $m \in [n-1]$. Suppose $A \cap B = \{n\}$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then, since $|(A \cup B) \backslash \{n\}| = r + s - 2 < n - 1$, the set $X := [n-1] \backslash (A \cup B)$ is non-empty. Let $x \in X$. Since $\Delta_{x,n}(\mathcal{A}) = \mathcal{A}$, $\delta_{x,n}(A) \in \mathcal{A}$. But $\delta_{x,n}(A) \cap B = \emptyset$, a contradiction. Hence (ii). □

Proof of Theorem 1.2. Let $R := [r]$ and let $(\mathcal{A}_0, \mathcal{B}_0)$ be the simple cip $(\{R\}, \mathcal{H}^{(s)}(R))$. We clearly have $[\mu(\mathcal{H})] \in \mathcal{H}$ (since $\mathcal{H}$ is compressed) and hence

$$2^{[\mu(\mathcal{H})]} \subseteq \mathcal{H} \tag{2}$$

(since $\mathcal{H}$ is hereditary). So $R \in \mathcal{H}^{(r)}$. We therefore have $\emptyset \neq \mathcal{A}_0 \subset \mathcal{H}^{(r)}$ and $\emptyset \neq \mathcal{B}_0 \subset \mathcal{H}^{(s)}$. It remains to show that $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}_0| + |\mathcal{B}_0|$ for any cip $(\mathcal{A}, \mathcal{B})$ with $\emptyset \neq \mathcal{A} \subset \mathcal{H}^{(r)}$ and $\emptyset \neq \mathcal{B} \subset \mathcal{H}^{(s)}$, and we do this using induction on r.

Consider the base case $r = 1$. By Theorem 1.4, there exists a (single-element) set $Z \in \mathcal{H}^{(1)}$ such that $\big(\{Z\}, \mathcal{H}^{(s)}(Z)\big) \in C(\mathcal{H}, 1, s)$ and hence $|\mathcal{A}| + |\mathcal{B}| \leqslant 1 + |\mathcal{H}^{(s)}(Z)|$. By Corollary 3.5, $|\mathcal{H}^{(s)}(Z)| \leqslant |\mathcal{B}_0|$. So $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}_0| + |\mathcal{B}_0|$.

Now consider $r \geqslant 2$. Suppose $n = r + s$. So $\mu(\mathcal{H}) = n$ and hence $[n] \in \mathcal{H}$. Thus, since $\mathcal{H}$ is hereditary, $\mathcal{H}^{(p)} = \binom{[n]}{p}$ for each $p \in [n]$. Having $n = r + s$ means that for every $A \in \binom{[n]}{r}$ there is only one set $B \in \binom{[n]}{s}$ such that $A \cap B = \emptyset$, so $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}| + \big(\binom{n}{s} - |\mathcal{A}|\big) = |\mathcal{A}_0| + |\mathcal{B}_0|$.

We now consider $n \geqslant r + s + 1$ and proceed by induction on n. Let $n' := n - 1$.

In view of Lemma 4.1(i) and the assumption that $\mathcal{H}$ is compressed, if $\Delta_{m,n}(\mathcal{A}) \neq \mathcal{A}$ or $\Delta_{m,n}(\mathcal{B}) \neq \mathcal{B}$ for some $m \in [n-1]$, then we can replace $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}' := \Delta_{m,n}(\mathcal{A})$ and $\mathcal{B}' := \Delta_{m,n}(\mathcal{B})$, respectively, and repeat the procedure until we obtain families $\mathcal{A}^* \subset \mathcal{H}^{(r)}$ and $\mathcal{B}^* \subset \mathcal{H}^{(s)}$ such that $\Delta_{m,n}(\mathcal{A}^*) = \mathcal{A}^*$ and $\Delta_{m,n}(\mathcal{B}^*) = \mathcal{B}^*$ for all $m \in [n-1]$ (it is well-known and easy to see that such a procedure indeed takes a finite number of steps). We can therefore assume that

$$\Delta_{m,n}(\mathcal{A}) = \mathcal{A} \text{ and } \Delta_{m,n}(\mathcal{B}) = \mathcal{B} \text{ for all } m \in [n-1]. \tag{3}$$

Thus, by Lemma 4.1(ii),

$$(A \cap B)\backslash\{n\} \neq \emptyset \text{ for any } A \in \mathcal{A} \text{ and } B \in \mathcal{B}. \tag{4}$$

Let $\mathcal{I} := \mathcal{H}\backslash\mathcal{H}(n) = \{H \in \mathcal{H} \colon n \notin H\}$. Similarly, let $\mathcal{C} := \mathcal{A}\backslash\mathcal{A}(n)$, $\mathcal{D} := \mathcal{B}\backslash\mathcal{B}(n)$, $\mathcal{E} := \mathcal{B}_0\backslash\mathcal{B}_0(n)$. So $\mathcal{C} \subset \mathcal{I}^{(r)}$ and $\mathcal{D}, \mathcal{E} \subset \mathcal{I}^{(s)}$. Note that $\mathcal{C} \neq \emptyset$ and $\mathcal{D} \neq \emptyset$ by (3). Since $\mathcal{H}$ is hereditary, if $[n] \in \mathcal{H}$ then $\mu(\mathcal{I}) = n - 1$. Thus, if $[n] \in \mathcal{H}$ then $\mu(\mathcal{I}) \geqslant r + s$, and if $[n] \notin \mathcal{H}$ then, since $\mu(\mathcal{H}) \geqslant r + s$, it follows by Lemma 3.2(iii) that $\mu(\mathcal{I}) \geqslant r + s$. Clearly $\mathcal{I}$ is a compressed hereditary sub-family of $2^{[n-1]}$. Therefore, by the inductive hypothesis,

$$|\mathcal{C}| + |\mathcal{D}| \leqslant |\mathcal{A}_0| + |\mathcal{E}|. \tag{5}$$

Let $\mathcal{J} := \mathcal{H}\langle n\rangle$. Clearly $\mathcal{J}$ is a compressed hereditary sub-family of $2^{[n-1]}$, and $\mu(\mathcal{J}) \geqslant \mu(\mathcal{H}) - 1$ by Lemma 3.2(i). Let $r' := r - 1$ and $s' := s - 1$. So

$$r' \leqslant s' \quad \text{and} \quad \mu(\mathcal{J}) \geqslant \mu(\mathcal{H}) - 1 \geqslant r + s - 1 > r' + s'. \tag{6}$$

We have $\mathcal{A}\langle n\rangle \subset \mathcal{J}^{(r')}$ and $\mathcal{B}\langle n\rangle \subset \mathcal{J}^{(s')}$. By (4), $\mathcal{A}\langle n\rangle$ and $\mathcal{B}\langle n\rangle$ are cross-intersecting.

Suppose $\mathcal{A}\langle n\rangle \neq \emptyset$ and $\mathcal{B}\langle n\rangle \neq \emptyset$. Let $R' := [r'] = R\backslash\{r\}$. By the inductive hypothesis, $|\mathcal{A}\langle n\rangle| + |\mathcal{B}\langle n\rangle| \leqslant 1 + \big|\mathcal{J}^{(s')}(R')\big|$. Similarly to (2), $2^{[\mu(\mathcal{J})]} \subseteq \mathcal{J}$; so $\binom{[\mu(\mathcal{J})]}{s'} \subseteq \mathcal{J}^{(s')}$. Since $\mathcal{B}_0\langle n\rangle = \mathcal{J}^{(s')}(R)$,

$$\begin{aligned} |\mathcal{B}_0\langle n\rangle| &= \Big|\mathcal{J}^{(s')}(R')\Big| + \Big|\Big\{B \in \mathcal{J}^{(s')} \colon B \cap R' = \emptyset, r \in B\Big\}\Big| \\ &\geqslant \Big|\mathcal{J}^{(s')}(R')\Big| + \left|\left\{B \in \binom{[\mu(\mathcal{J})]\backslash R'}{s'} \colon r \in B\right\}\right| = \Big|\mathcal{J}^{(s')}(R')\Big| + \binom{\mu(\mathcal{J}) - r' - 1}{s' - 1} \end{aligned}$$

and hence, by (6), $|\mathcal{B}_0\langle n\rangle| \geqslant \left|\mathcal{J}^{(s')}(R')\right|+1$. So $|\mathcal{A}\langle n\rangle|+|\mathcal{B}\langle n\rangle| \leqslant |\mathcal{B}_0\langle n\rangle|$. Since $|\mathcal{A}|+|\mathcal{B}| = |\mathcal{C}|+|\mathcal{D}|+|\mathcal{A}\langle n\rangle|+|\mathcal{B}\langle n\rangle|$, (5) and the last inequality give us $|\mathcal{A}|+|\mathcal{B}| \leqslant |\mathcal{A}_0|+|\mathcal{E}|+|\mathcal{B}_0\langle n\rangle| = |\mathcal{A}_0| + |\mathcal{B}_0|$.

Next, suppose $\mathcal{A}\langle n\rangle = \emptyset$. Let $A \in \mathcal{C}$ (recall that $\mathcal{C} \neq \emptyset$). By (4), $|\mathcal{B}\langle n\rangle| \leqslant \left|\mathcal{J}^{(s')}(A)\right|$. It is easy to see that $U(\mathcal{J}^{(s')}) = [l]$ for some $l \in [n']$ (since $\mathcal{J}$ is compressed); so $\left|\mathcal{J}^{(s')}(A)\right| \leqslant \left|\mathcal{J}^{(s')}(R)\right|$ by Corollary 3.5. Since $|\mathcal{A}| + |\mathcal{B}| = |\mathcal{C}| + |\mathcal{D}| + |\mathcal{A}\langle n\rangle| + |\mathcal{B}\langle n\rangle|$, where $\mathcal{A}\langle n\rangle = \emptyset$ and $|\mathcal{B}\langle n\rangle| \leqslant \left|\mathcal{J}^{(s')}(R)\right| = |\mathcal{B}_0\langle n\rangle|$, it follows by (5) that $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}_0| + |\mathcal{B}_0|$.

Finally, suppose $\mathcal{B}\langle n\rangle = \emptyset$. If $r' = s'$ (i.e. $r = s$) then $|\mathcal{A}| + |\mathcal{B}| \leqslant |\mathcal{A}_0| + |\mathcal{B}_0|$ follows by an argument similar to the one for the previous case. Suppose $r' < s'$. Let $\mathcal{K}_0 := \mathcal{J}\backslash\mathcal{J}(1) := \{J \in \mathcal{J} \colon 1 \notin J\}$ and $\mathcal{K}_1 := \mathcal{J}\langle 1\rangle$. So $\mathcal{K}_0, \mathcal{K}_1 \subseteq 2^{[2,n-1]}$. By Lemma 3.3, $\mathcal{K}_0$ and $\mathcal{K}_1$ are hereditary and quasi-compressed. By (i) and (ii) of Lemma 3.2, $\mu(\mathcal{K}_0) \geqslant \mu(\mathcal{J}) - 1$ and $\mu(\mathcal{K}_1) \geqslant \mu(\mathcal{J}) - 1$. Thus, by (6), $\mu(\mathcal{K}_0) \geqslant r' + s'$. Let $R^* := [2, r]$ and $S^* := [2, s]$. It is clear from (2) that $R^*, S^* \in \mathcal{K}_0$. Note that therefore R^* and S^* are initial segments of $U(\mathcal{K}_0)$. Since $\left(\mathcal{K}_0^{(r')}(S^*), \{S^*\}\right)$ is a cip with the first family contained in $\mathcal{K}_0^{(r')}$ and the second family contained in $\mathcal{K}_0^{(s')}$, the inductive hypothesis gives us $\left|\mathcal{K}_0^{(r')}(S^*)\right| + |\{S^*\}| \leqslant |\{R^*\}| + \left|\mathcal{K}_0^{(s')}(R^*)\right|$ and hence

$$\left|\mathcal{K}_0^{(r')}(S^*)\right| \leqslant \left|\mathcal{K}_0^{(s')}(R^*)\right|. \tag{7}$$

Let $\mathcal{L}_0 := \{A \in \mathcal{J}^{(r')}(S) \colon 1 \notin A\}$ and $\mathcal{L}_1 := \{A\backslash\{1\} \colon 1 \in A \in \mathcal{J}^{(r')}(S)\}$. Let $\mathcal{M}_0 := \{B \in \mathcal{B}_0\langle n\rangle \colon 1 \notin B\}$ and $\mathcal{M}_1 := \{B\backslash\{1\} \colon 1 \in B \in \mathcal{B}_0\langle n\rangle\}$. Note that $\mathcal{L}_0 = \mathcal{K}_0^{(r')}(S^*)$ and $\mathcal{M}_0 = \mathcal{K}_0^{(s')}(R^*)$. So $|\mathcal{L}_0| \leqslant |\mathcal{M}_0|$ by (7). Let $r'' := r' - 1$ and $s'' := s' - 1$. Similarly to (6), $\mu(\mathcal{K}_1) > r'' + s''$. By Lemma 3.1, $|\mathcal{K}_1^{(r'')}| < |\mathcal{K}_1^{(s'')}|$. Thus, since $\mathcal{L}_1 = \mathcal{K}_1^{(r'')}$ and $\mathcal{M}_1 = \mathcal{K}_1^{(s'')}$, $|\mathcal{L}_1| < |\mathcal{M}_1|$. We therefore have

$$\left|\mathcal{J}^{(r')}(S)\right| = |\mathcal{L}_0| + |\mathcal{L}_1| < |\mathcal{M}_0| + |\mathcal{M}_1| = |\mathcal{B}_0\langle n\rangle|. \tag{8}$$

Now let $D \in \mathcal{D}$. By (4), $|\mathcal{A}\langle n\rangle| \leqslant \left|\mathcal{J}^{(r')}(D)\right|$. It is easy to see that $U(\mathcal{J}^{(r')}) = [l]$ for some $l \in [n']$ (since $\mathcal{J}$ is compressed); so $\left|\mathcal{J}^{(r')}(D)\right| \leqslant \left|\mathcal{J}^{(r')}(S)\right|$ by Corollary 3.5. Thus, by (8), $|\mathcal{A}\langle n\rangle| < |\mathcal{B}_0\langle n\rangle|$. Together with (5) and $\mathcal{B}\langle n\rangle = \emptyset$, this gives us $|\mathcal{A}| + |\mathcal{B}| < |\mathcal{A}_0| + |\mathcal{B}_0|$. □

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