A Hilton-Milner Theorem for Vector Spaces

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Submitted: Nov 1, 2009; Accepted: May 4, 2010; Published: May 14, 2010
Mathematics Subject Classification: 05D05, 05A30

Abstract

We show for $k \geqslant 2$ that if $q \geqslant 3$ and $n \geqslant 2k+1$, or $q = 2$ and $n \geqslant 2k+2$, then any intersecting family $\mathcal{F}$ of k-subspaces of an n-dimensional vector space over $GF(q)$ with $\bigcap_{F \in \mathcal{F}} F = 0$ has size at most $\left[{n-1 \atop k-1}\right] - q^{k(k-1)}\left[{n-k-1 \atop k-1}\right] + q^k$. This bound is sharp as is shown by Hilton-Milner type families. As an application of this result, we determine the chromatic number of the corresponding q-Kneser graphs.

1 Introduction

1.1 Sets

Let X be an n-element set and, for $0 \leqslant k \leqslant n$, let $\binom{X}{k}$ denote the family of all subsets of X of cardinality k. A family $\mathcal{F} \subset \binom{X}{k}$ is called *intersecting* if for all $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 \neq \emptyset$. Erdős, Ko, and Rado [5] determined the maximum size of an intersecting family, and introduced the so-called shifting technique.

Theorem 1.1 (Erdős-Ko-Rado) *Suppose $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n \geqslant 2k$. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$. Excepting the case $n = 2k$, equality holds only if $\mathcal{F} = \left\{F \in \binom{X}{k} : x \in F\right\}$ for some $x \in X$.*

For any family $\mathcal{F} \subset \binom{X}{k}$, the *covering number* $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. Theorem 1.1 shows that if $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family of maximum size and $n > 2k$, then $\tau(\mathcal{F}) = 1$.

Hilton and Milner [15] determined the maximum size of an intersecting family with $\tau(\mathcal{F}) \geqslant 2$. Later, Frankl and Füredi [9] gave an elegant proof of Theorem 1.2 using the shifting technique.

Theorem 1.2 (Hilton-Milner) *Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $k \geqslant 2$, $n \geqslant 2k+1$, and $\tau(\mathcal{F}) \geqslant 2$. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Equality holds only if*

(i) $\mathcal{F} = \{F\} \cup \{G \in \binom{X}{k} : x \in G,\ F \cap G \neq \emptyset\}$ *for some k-subset F and $x \in X \setminus F$.*

(ii) $\mathcal{F} = \{F \in \binom{X}{3} : |F \cap S| \geqslant 2\}$ *for some 3-subset S if $k = 3$.*

1.2 Vector spaces

Theorem 1.1 and Theorem 1.2 have natural extensions to vector spaces. We let V always denote an n-dimensional vector space over the finite field $GF(q)$. For $k \in \mathbb{Z}^+$, we write $\begin{bmatrix} V \\ k \end{bmatrix}_q$ to denote the family of all k-dimensional subspaces of V. For $a, k \in \mathbb{Z}^+$, define the *Gaussian binomial coefficient* by

$$\begin{bmatrix} a \\ k \end{bmatrix}_q := \prod_{0 \leqslant i < k} \frac{q^{a-i} - 1}{q^{k-i} - 1}.$$

A simple counting argument shows that the size of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From now on, we will omit the subscript q.

If two subspaces of V intersect in the zero subspace, then we say they are *disjoint* or that they *trivially intersect*; otherwise we say the subspaces *non-trivially intersect*. A family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is called intersecting if any two k-spaces in $\mathcal{F}$ non-trivially intersect. The maximum size of an intersecting family of k-spaces was first determined by Hsieh [16]. For alternate proofs of Theorem 1.3, see [4] and [11]. We remark that there is as yet no analog of the shifting technique for vector spaces.

Theorem 1.3 (Hsieh) *Suppose $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is intersecting and $n \geqslant 2k$. Then $|\mathcal{F}| \leqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. Equality holds if and only if $\mathcal{F} = \left\{F \in \begin{bmatrix} V \\ k \end{bmatrix} : v \subset F\right\}$ for some one-dimensional subspace $v \subset V$, unless $n = 2k$.*

Let the *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be defined as the minimum dimension of a subspace of V that intersects all elements of $\mathcal{F}$ nontrivially. Theorem 1.3 shows that, as in the set case, if $\mathcal{F}$ is a maximum intersecting family of k-spaces, then $\tau(\mathcal{F}) = 1$. Families satisfying $\tau(\mathcal{F}) = 1$ are known as *point-pencils*.

In this paper, we will extend Theorem 1.2 to vector spaces, and determine the maximum size of an intersecting family $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ with $\tau(\mathcal{F}) \geqslant 2$. For two subspaces $S, T \leqslant V$, we let $S+T \leqslant V$ denote their linear span. We observe that for a fixed 1-subspace $E \leqslant V$ and a k-subspace U with $E \not\leqslant U$, the family

$$\mathcal{F}_{E,U} = \{U\} \cup \{W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \leqslant W, \ \dim(W \cap U) \geqslant 1\}$$

is not maximal as we can add all subspaces in $\begin{bmatrix} E+U \\ k \end{bmatrix}$ that are not in $\mathcal{F}_{E,U}$. We will say that $\mathcal{F}$ is an *HM-type family* if

$$\mathcal{F} = \left\{W \in \begin{bmatrix} V \\ k \end{bmatrix} : E \leqslant W, \ \dim(W \cap U) \geqslant 1\right\} \cup \begin{bmatrix} E+U \\ k \end{bmatrix}$$

for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ and $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ with $E \not\leqslant U$. If $\mathcal{F}$ is an HM-type family, then its size is

$$|\mathcal{F}| = f(n,k,q) := \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k. \tag{1.1}$$

The main result of the paper is the following theorem.

Theorem 1.4 *Suppose $k \geqslant 3$, and either $q \geqslant 3$ and $n \geqslant 2k+1$, or $q = 2$ and $n \geqslant 2k+2$. For any intersecting family $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with $\tau(\mathcal{F}) \geqslant 2$, we have $|\mathcal{F}| \leqslant f(n,k,q)$ (with $f(n,k,q)$ as in (1.1)). Equality holds only if*

(i) $\mathcal{F}$ is an HM-type family,

(ii) $\mathcal{F} = \mathcal{F}_3 = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \dim(S \cap F) \geqslant 2\}$ for some $S \in \begin{bmatrix} V \\ 3 \end{bmatrix}$ if $k = 3$.

Furthermore, if $k \geqslant 4$, then there exists an $\epsilon > 0$ (independent of n, k, q) such that if $|\mathcal{F}| \geqslant (1-\epsilon) f(n,k,q)$, then $\mathcal{F}$ is a subfamily of an HM-type family.

If $k = 2$, then a maximal intersecting family $\mathcal{F}$ of k-spaces with $\tau(\mathcal{F}) > 1$ is the family of all 2-subspaces of a 3-subspace, and the conclusion of the theorem holds.

After proving Theorem 1.4 in Section 2, we apply this result to determine the chromatic number of q-Kneser graphs. The vertex set of the q-Kneser graph $qK_{n:k}$ is $\begin{bmatrix} V \\ k \end{bmatrix}$. Two vertices of $qK_{n:k}$ are adjacent if and only if the corresponding k-subspaces are disjoint. In [3], the chromatic number of the q-Kneser graph $qK_{n:2}$ is determined, and the minimum colorings are characterized. In [18], the chromatic number of the q-Kneser graph is determined in general for $q > q_k$. In Section 4, we prove the following theorem.

Theorem 1.5 *If $k \geqslant 3$, and either $q \geqslant 3$ and $n \geqslant 2k+1$, or $q = 2$ and $n \geqslant 2k+2$, then the chromatic number of the q-Kneser graph is $\chi(qK_{n:k}) = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Moreover, each color class of a minimum coloring is a point-pencil and the points determining a color are the points of an $(n-k+1)$-dimensional subspace.*

In Section 5, we prove the non-uniform version of the Erdős-Ko-Rado theorem.

Theorem 1.6 *Let $\mathcal{F}$ be an intersecting family of subspaces of V.*

(i) *If n is even, then* $|\mathcal{F}| \leqslant \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix} + \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix}$.

(ii) *If n is odd, then* $|\mathcal{F}| \leqslant \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix}$.

For even n, equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \{F \in \begin{bmatrix} V \\ n/2 \end{bmatrix} : E \leqslant F\}$ *for some* $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$*, or if* $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \begin{bmatrix} U \\ n/2 \end{bmatrix}$ *for some* $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$*. For odd n, equality holds only if* $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix}$.

Note that Theorem 1.6 follows from the profile polytope of intersecting families which was determined implicitly by Bey [1] and explicitly by Gerbner and Patkós [12], but the proof we present in Section 5 is simple and direct.

2 Proof of Theorem 1.4

This section contains the proof of Theorem 1.4 which we divide into two cases.

2.1 The case $\tau(\mathcal{F}) = 2$

For any $A \leqslant V$ and $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, let $\mathcal{F}_A = \{F \in \mathcal{F} : A \leqslant F\}$. First, let us state some easy technical lemmas.

Lemma 2.1 *Let $a \geqslant 0$ and $n \geqslant k \geqslant a+1$ and $q \geqslant 2$. Then*

$$\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-a-1 \\ k-a-1 \end{bmatrix} < \frac{1}{(q-1)q^{n-2k}} \begin{bmatrix} n-a \\ k-a \end{bmatrix}.$$

Proof. The inequality to be proved simplifies to

$$(q^{k-a}-1)(q^k-1)q^{n-2k} < q^{n-a}-1. \qquad \square$$

Lemma 2.2 *Let $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$. If $E \not\leqslant L \leqslant V$, where L is an l-subspace, then the number of k-subspaces of V containing E and intersecting L is at least* $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} l \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ *(with equality for $l = 2$), and at most* $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.

Proof. The k-spaces containing E and intersecting L in a 1-dimensional space are counted exactly once in the first term. Those subspaces that intersect L in a 2-dimensional space are counted $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = q+1$ times in the first term and $-q$ times in the second term, thus once overall. If a subspace intersects L in a subspace of dimension $i \geqslant 3$, then it is counted $\begin{bmatrix} i \\ 1 \end{bmatrix}$ times in the first term and $-q\begin{bmatrix} i \\ 2 \end{bmatrix}$ times in the second term, and hence a negative number of times overall. $\square$

Our next lemma gives bounds on the size of an HM-type family that are easier to work with than the precise formula mentioned in the introduction.

Lemma 2.3 *Let $n \geqslant 2k+1$, $k \geqslant 3$ and $q \geqslant 2$. If $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ is an HM-type family, then* $(1 - \frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} k \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \leqslant f(n,k,q) = |\mathcal{F}| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.

Proof. Since $q\begin{bmatrix}k\\2\end{bmatrix} = \begin{bmatrix}k\\1\end{bmatrix}(\begin{bmatrix}k\\1\end{bmatrix} - 1)/(q+1)$ and $n \geqslant 2k+1$, the first inequality follows from Lemma 2.1. Let $\mathcal{F}$ be the HM-type family defined by the 1-space E and the k-space U. Then $\mathcal{F}$ contains all k-subspaces of V containing E and intersecting U, so that the second inequality follows from Lemma 2.2. For the last inequality, Lemma 2.2 almost suffices, but we also have to count the k-subspaces of $\begin{bmatrix}E+U\\k\end{bmatrix}$ that do not contain E. Each $(k-1)$-subspace W of U is contained in $q+1$ such subspaces, one of which is $E+W$. On the other hand, $E+W$ was counted at least $q+1$ times since $k \geqslant 3$. This proves the last inequality. □

Lemma 2.4 *If a subspace S does not intersect each element of $\mathcal{F} \subset \begin{bmatrix}V\\k\end{bmatrix}$, then there is a subspace $T > S$ with $\dim T = \dim S + 1$ and $|\mathcal{F}_T| \geqslant |\mathcal{F}_S|/\begin{bmatrix}k\\1\end{bmatrix}$.*

Proof. There is an $F \in \mathcal{F}$ such that $S \cap F = 0$. Average over all $T = S + E$ where E is a 1-subspace of F. □

Lemma 2.5 *If an s-dimensional subspace S does not intersect each element of $\mathcal{F} \subset \begin{bmatrix}V\\k\end{bmatrix}$, then $|\mathcal{F}_S| \leqslant \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix}$.*

Proof. There is an $(s+1)$-space T with $\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix} \geqslant |\mathcal{F}_T| \geqslant |\mathcal{F}_S|/\begin{bmatrix}k\\1\end{bmatrix}$. □

Corollary 2.6 *Let $\mathcal{F} \subseteq \begin{bmatrix}V\\k\end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geqslant s$. Then for any i-space $L \leqslant V$ with $i \leqslant s$ we have $|\mathcal{F}_L| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^{s-i}\begin{bmatrix}n-s\\k-s\end{bmatrix}$.* □

Proof. If $i = s$, then clearly $|\mathcal{F}_L| \leqslant \begin{bmatrix}n-s\\k-s\end{bmatrix}$. If $i < s$, then there exists an $F \in \mathcal{F}$ such that $F \cap L = 0$; now apply Lemma 2.4 $s-i$ times. □

Before proving the q-analogue of the Hilton-Milner theorem, we describe the essential part of maximal intersecting families $\mathcal{F} \subset \begin{bmatrix}V\\k\end{bmatrix}$ with $\tau(\mathcal{F}) = 2$.

Proposition 2.7 *Let $n \geqslant 2k$ and let $\mathcal{F} \subset \begin{bmatrix}V\\k\end{bmatrix}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Define $\mathcal{T}$ to be the family of 2-spaces of V that intersect all subspaces in $\mathcal{F}$. One of the following three possibilities holds:*

(i) $|\mathcal{T}| = 1$ *and* $\begin{bmatrix}n-2\\k-2\end{bmatrix} < |\mathcal{F}| < \begin{bmatrix}n-2\\k-2\end{bmatrix} + (q+1)\left(\begin{bmatrix}k\\1\end{bmatrix} - 1\right)\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}$;

(ii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, *and there is an* $(l+1)$*-space* W *(with* $2 \leqslant l \leqslant k$*) and a 1-space* $E \leqslant W$ *so that* $\mathcal{T} = \{M : E \leqslant M \leqslant W,\ \dim M = 2\}$*. In this case,*
$\begin{bmatrix}l\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} - q\begin{bmatrix}l\\2\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix} \leqslant |\mathcal{F}| \leqslant \begin{bmatrix}l\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}(\begin{bmatrix}k\\1\end{bmatrix} - \begin{bmatrix}l\\1\end{bmatrix})\begin{bmatrix}n-3\\k-3\end{bmatrix} + q^l\begin{bmatrix}n-l\\k-l\end{bmatrix}$.
For $l = 2$, the upper bound can be strengthened to
$|\mathcal{F}| \leqslant (q+1)\begin{bmatrix}n-2\\k-2\end{bmatrix} - q\begin{bmatrix}n-3\\k-3\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}(\begin{bmatrix}k\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix})\begin{bmatrix}n-3\\k-3\end{bmatrix} + q^2\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}$;

(iii) $\mathcal{T} = \begin{bmatrix}A\\2\end{bmatrix}$ *for some 3-subspace A and* $\mathcal{F} = \{U \in \begin{bmatrix}V\\k\end{bmatrix} : \dim(U \cap A) \geqslant 2\}$*. In this case,*
$|\mathcal{F}| = (q^2+q+1)(\begin{bmatrix}n-2\\k-2\end{bmatrix} - \begin{bmatrix}n-3\\k-3\end{bmatrix}) + \begin{bmatrix}n-3\\k-3\end{bmatrix}$.

Proof. Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. By maximality, $\mathcal{F}$ contains all k-spaces containing a $T \in \mathcal{T}$. Since $n \geqslant 2k$ and $k \geqslant 2$, two disjoint elements of $\mathcal{T}$ would be contained in disjoint elements of $\mathcal{F}$, which is impossible. Hence, $\mathcal{T}$ is intersecting.

Observe that if $A, B \in \mathcal{T}$ and $A \cap B < C < A + B$, then $C \in \mathcal{T}$. As an intersecting family of 2-spaces is either a family of 2-spaces containing some fixed 1-space E or a family of 2-subspaces of a 3-space, we get the following:

$(*)$: $\mathcal{T}$ is either a family of all 2-subspaces containing some fixed 1-space E that lie in some fixed $(l+1)$-space with $k \geqslant l \geqslant 1$, or $\mathcal{T}$ is the family of all 2-subspaces of a 3-space.

(i) : If $|\mathcal{T}| = 1$, then let S denote the only 2-space in $\mathcal{T}$ and let $E \leqslant S$ be any 1-space. Since $\tau(\mathcal{F}) > 1$, there exists an $F \in \mathcal{F}$ with $E \nleqslant F$, for which we must have $\dim(F \cap S) = 1$. As S is the only element of $\mathcal{T}$, for any 1-subspace E' of F different from $F \cap S$, we have $\mathcal{F}_{E+E'} \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ by Lemma 2.5. Hence the number of subspaces containing E but not containing S is at most $(\begin{bmatrix} k \\ 1 \end{bmatrix} - 1) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$. This gives the upper bound.

(ii) : Assume that $\tau(\mathcal{T}) = 1$ and $|\mathcal{T}| > 1$. By $(*)$, $\mathcal{T}$ is the set of 2-spaces in an $(l+1)$-space W (with $l \geqslant 2$) containing some fixed 1-space E. Every $F \in \mathcal{F} \setminus \mathcal{F}_E$ intersects W in a hyperplane. Let L be a hyperplane in W not on E. Then $\mathcal{F}$ contains all k-spaces on E that intersect L. Hence the lower bound and the first term in the upper bound come from Lemma 2.2. The second term comes from using Lemma 2.5 to count the k-spaces of $\mathcal{F}$ that contain E and intersect a given $F \in \mathcal{F}$ (not containing E) in a point of $F \setminus W$. If $l \geqslant 3$, then there are q^l hyperplanes in W not containing E and there are $\begin{bmatrix} n-l \\ k-l \end{bmatrix}$ k-spaces through such a hyperplane; this gives the last term. For $l = 2$, we use the tight lower bound in Lemma 2.2 to count the number of k-spaces on E that intersect L. There are q^2 hyperplanes in W, and they cannot be in $\mathcal{T}$, so Lemma 2.5 gives the bound.

(iii) : This is immediate. □

Corollary 2.8 *Let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Suppose $q \geqslant 3$ and $n \geqslant 2k+1$, or $q = 2$ and $n \geqslant 2k+2$. If $\mathcal{F}$ is at least as large as an HM-type family and $k > 3$, then $\mathcal{F}$ is an HM-type family. If $k = 3$, then $\mathcal{F}$ is an HM-type family or an $\mathcal{F}_3$-type family.*

There exists an $\epsilon > 0$ (independent of n, k, q) such that if $k \geqslant 4$ and $|\mathcal{F}|$ is at least $(1-\epsilon)$ times the size of an HM-type family, then $\mathcal{F}$ is an HM-type family.

Proof. Apply Proposition 2.7. Note that the HM-type families are precisely those from case (ii) with $l = k$.

Let $n = 2k + r$ where $r \geqslant 1$. We have $|\mathcal{F}| / \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < 1 + \frac{q+1}{(q-1)q^r} \begin{bmatrix} k \\ 1 \end{bmatrix}$ in case (i) of Proposition 2.7 by Lemma 2.1. We have $|\mathcal{F}| / \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < (\frac{1}{q} + \frac{1}{(q-1)q^r}) \begin{bmatrix} k \\ 1 \end{bmatrix} + \frac{q^2}{(q-1)q^r}$ in case (ii) when $l < k$. In both cases, for $q \geqslant 3$ and $k \geqslant 3$, or $q = 2$, $k \geqslant 4$, and $r \geqslant 2$, this is less than $(1-\epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3. Using the stronger estimate in Lemma 2.3, we find the same conclusion for $q = 2$, $k = 3$, and $r \geqslant 2$.

In case (iii), $|\mathcal{F}_3| = \begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} - \frac{q^3-q}{q-1}\begin{bmatrix}n-3\\k-3\end{bmatrix}$. For $k \geqslant 4$, this is much smaller than the size of the HM-type families. For $k = 3$, the two families have the same size. $\square$

Proposition 2.9 *Suppose that $k \geqslant 3$ and $n \geqslant 2k$. Let $\mathcal{F} \subseteq \begin{bmatrix}V\\k\end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geqslant 2$. Let $3 \leqslant l \leqslant k$. If there is an l-space that intersects each $F \in \mathcal{F}$ and*

$$|\mathcal{F}| > \begin{bmatrix}l\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{l-1}\begin{bmatrix}n-l\\k-l\end{bmatrix}, \tag{2.2}$$

then there is an $(l-1)$-space that intersects each $F \in \mathcal{F}$.

Proof. By averaging, there is a 1-space P with $|\mathcal{F}_P| \geqslant |\mathcal{F}|/\begin{bmatrix}l\\1\end{bmatrix}$. If $\tau(\mathcal{F}) = l$, then by Corollary 2.6, $|\mathcal{F}| \leqslant \begin{bmatrix}l\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{l-1}\begin{bmatrix}n-l\\k-l\end{bmatrix}$, contradicting the hypothesis. $\square$

Corollary 2.10 *Suppose $k \geqslant 3$ and either $q \geqslant 3$ and $n \geqslant 2k+1$, or $q = 2$ and $n \geqslant 2k+2$. Let $\mathcal{F} \subseteq \begin{bmatrix}V\\k\end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geqslant 2$. If $|\mathcal{F}| > \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix}$, then $\tau(\mathcal{F}) = 2$; that is, $\mathcal{F}$ is contained in one of the systems in Proposition 2.7, which satisfy the bound on $|\mathcal{F}|$.*

Proof. By Lemma 2.1 and the conditions on n and q, the right hand side of (2.2) decreases as l increases, where $3 \leqslant l \leqslant k$. Hence, by Proposition 2.9, we can find a 2-space that intersects each $F \in \mathcal{F}$. $\square$

Remark 2.11 For $n \geqslant 3k$, all systems described in Proposition 2.7 occur.

2.2 The case $\tau(\mathcal{F}) > 2$

Suppose that $\mathcal{F} \subset \begin{bmatrix}V\\k\end{bmatrix}$ is an intersecting family and $\tau(\mathcal{F}) = l > 2$. We shall derive a contradiction from $|\mathcal{F}| \geqslant f(n,k,q)$, and even from $|\mathcal{F}| \geqslant (1-\epsilon)f(n,k,q)$ for some $\epsilon > 0$ (independent of n,k,q).

2.2.1 The case $l = k$

First consider the case $l = k$. Then $|\mathcal{F}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^k$ by Corollary 2.6. On the other hand,

$$|\mathcal{F}| \geqslant \left(1 - \frac{1}{q^3-q}\right)\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} > \left(1 - \frac{1}{q^3-q}\right)\begin{bmatrix}k\\1\end{bmatrix}^{k-1}\left((q-1)q^{n-2k}\right)^{k-2}$$

by Lemma 2.3 and Lemma 2.1. If either $q \geqslant 3$, $n \geqslant 2k+1$ or $q = 2$, $n \geqslant 2k+2$, then either $k = 3$, $(n,k,q) = (9,4,3)$, or $(n,k,q) = (10,4,2)$. If $(n,k,q) = (9,4,3)$ then $f(n,k,q) = 3837721$, and $40^4 = 2560000$, which gives a contradiction. If $(n,k,q) = (10,4,2)$, then $f(n,k,q) = 153171$, and $15^4 = 50625$, which again gives a contradiction. Hence $k = 3$. Now $|\mathcal{F}| \geqslant (1 - \frac{1}{q^3-q})\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix}$ gives a contradiction for $n \geqslant 8$, so $n = 7$. Therefore, if we assume that $n \geqslant 2k+1$ and either $q \geqslant 3$, $(n,k) \neq (7,3)$ or $q = 2$, $n \geqslant 2k+2$ then we are not in the case $l = k$.

It remains to settle the case $n = 7$, $k = l = 3$, and $q \geqslant 3$. By Lemma 2.4, we can choose a 1-space E such that $|\mathcal{F}_E| \geqslant |\mathcal{F}|/\begin{bmatrix}3\\1\end{bmatrix}$ and a 2-space S on E such that $|\mathcal{F}_S| \geqslant |\mathcal{F}_E|/\begin{bmatrix}3\\1\end{bmatrix}$.

Then $|\mathcal{F}_S| > q+1$ since $|\mathcal{F}| > {2 \brack 1}{3 \brack 1}^2$. Pick $F' \in \mathcal{F}$ disjoint from S and define $H := S+F'$. All $F \in \mathcal{F}_S$ are contained in the 5-space H. Since $|\mathcal{F}| > {5 \brack 3}$, there is an $F_0 \in \mathcal{F}$ not contained in H. If $F_0 \cap S = 0$, then each $F \in \mathcal{F}_S$ is contained in $S+(H \cap F_0)$; this implies $|\mathcal{F}_S| \leqslant q+1$, which is impossible. Thus, all elements of $\mathcal{F}$ disjoint from S are in H.

Now F_0 must meet F' and S, so F_0 meets H in a 2-space S_0. Since $|\mathcal{F}_S| > q+1$, we can find two elements F_1, F_2 of $\mathcal{F}_S$ with the property that S_0 is not contained in the 4-space F_1+F_2. Since any $F \in \mathcal{F}$ disjoint from S is contained in H and meets F_0, it must meet S_0 and also F_1 and F_2. Hence the number of such F's is at most q^5. Altogether $|\mathcal{F}| \leqslant q^5 + {2 \brack 1}{3 \brack 1}^2$; the first term comes from counting $F \in \mathcal{F}$ disjoint from S and the second term comes from counting $F \in \mathcal{F}$ on a given one-dimensional subspace $E < S$. This contradicts $|\mathcal{F}| \geqslant (1-\frac{1}{q^3-q}){3 \brack 1}{5 \brack 1}$.

2.2.2 The case $l < k$

Assume, for the moment, that there are two l-subspaces in V that non-trivially intersect all $F \in \mathcal{F}$, and that these two l-spaces meet in an m-space, where $0 \leqslant m \leqslant l-1$. By Corollary 2.6, for each 1-subspace P we have $|\mathcal{F}_P| \leqslant {k \brack 1}^{l-1}{n-l \brack k-l}$, and for each 2-subspace L we have $|\mathcal{F}_L| \leqslant {k \brack 1}^{l-2}{n-l \brack k-l}$. Consequently,

$$|\mathcal{F}| \leqslant {m \brack 1}{k \brack 1}^{l-1}{n-l \brack k-l} + \left({l \brack 1} - {m \brack 1}\right)^2 {k \brack 1}^{l-2}{n-l \brack k-l}. \tag{2.3}$$

The upper bound (2.3) is a quadratic in $x = {m \brack 1}$ and is largest at one of the extreme values $x = 0$ and $x = {l-1 \brack 1}$. The maximum is taken at $x = 0$ only when ${l \brack 1} - \frac{1}{2}{k \brack 1} > \frac{1}{2}{l-1 \brack 1}$; that is, when $k = l$. Since we assume that $l < k$, the upper bound in (2.3) is largest for $m = l-1$. We find

$$|\mathcal{F}| \leqslant {l-1 \brack 1}{k \brack 1}^{l-1}{n-l \brack k-l} + \left({l \brack 1} - {l-1 \brack 1}\right)^2 {k \brack 1}^{l-2}{n-l \brack k-l}.$$

On the other hand,

$$|\mathcal{F}| \geqslant (1-\tfrac{1}{q^3-q}){k \brack 1}{n-2 \brack k-2} > (1-\tfrac{1}{q^3-q}){k \brack 1}^{l-1}{n-l \brack k-l}((q-1)q^{n-2k})^{l-2}.$$

Comparing these, and using $k > l$, $n \geqslant 2k+1$, and $n \geqslant 2k+2$ if $q = 2$, we find either $(n,k,l,q) = (9,4,3,3)$ or $q = 2$, $n = 2k+2$, $l = 3$, and $k \leqslant 5$. If $(n,k,l,q) = (9,4,3,3)$ then $f(n,k,q) = 3837721$, while the upper bound is 3508960, which is a contradiction. If $(n,k,l,q) = (12,5,3,2)$ then $f(n,k,q) = 183628563$, while the upper bound is 146766865, which is a contradiction. If $(n,k,l,q) = (10,4,3,2)$ then $f(n,k,q) = 153171$, while the upper bound is 116205, which is a contradiction. Hence, under our assumption that there are two distinct l-spaces that meet all $F \in \mathcal{F}$, the case $2 < l < k$ cannot occur.

We now assume that there is a unique l-space T that meets all $F \in \mathcal{F}$. We can pick a 1-space $E < T$ such that $|\mathcal{F}_E| \geqslant |\mathcal{F}|/{l \brack 1}$. Now there is some $F' \in \mathcal{F}$ not on E, so E is in ${k \brack 1}$ lines such that each $F \in \mathcal{F}_E$ contains at least one of these lines. Suppose L is one of these lines and L does not lie in T; we can enlarge L to an l-space that still does not meet all elements of $\mathcal{F}$, so $|\mathcal{F}_L| \leqslant \left[{k \atop 1}\right]^{l-1}\left[{n-l-1 \atop k-l-1}\right]$ by Lemma 2.4 and Lemma 2.5. If L does lie on T, we have $|\mathcal{F}_L| \leqslant \left[{k \atop 1}\right]^{l-2}\left[{n-l \atop k-l}\right]$ by Corollary 2.6. Hence,

$$|\mathcal{F}| \leqslant \left[{l \atop 1}\right]|\mathcal{F}_E| \leqslant \left[{l \atop 1}\right]\left(\left[{l-1 \atop 1}\right](\left[{k \atop 1}\right]^{l-2}\left[{n-l \atop k-l}\right]) + (\left[{k \atop 1}\right] - \left[{l-1 \atop 1}\right])(\left[{k \atop 1}\right]^{l-1}\left[{n-l-1 \atop k-l-1}\right])\right).$$

On the other hand, we have $|\mathcal{F}| > \left(1 - \frac{1}{q^3-q}\right)((q-1)q^{n-2k})^{l-2}\left[{k \atop 1}\right]^{l-1}\left[{n-l \atop k-l}\right]$. Under our standard assumptions $n \geqslant 2k+1$ and $n \geqslant 2k+2$ if $q = 2$, this implies $q = 2$, $n = 2k+2$, $l = 3$, which gives a contradiction. We showed: If $q \geqslant 3$ and $n \geqslant 2k+1$ or if $q = 2$ and $n \geqslant 2k+2$, then an intersecting family $\mathcal{F} \subset \left[{V \atop k}\right]$ with $|\mathcal{F}| \geqslant f(n,k,q)$ must satisfy $\tau(\mathcal{F}) \leqslant 2$. Together with Corollary 2.8, this proves Theorem 1.4.

3 Critical families

A subspace will be called a *hitting subspace* (and we shall say that the subspace intersects $\mathcal{F}$), if it intersects each element of $\mathcal{F}$.

The previous results just used the parameter τ, so only the hitting subspaces of smallest dimension were taken into account. A more precise description is possible if we make the intersecting system of subspaces critical.

Definition 3.1 An intersecting family $\mathcal{F}$ of subspaces of V is *critical* if for any two distinct $F, F' \in \mathcal{F}$ we have $F \not\subset F'$, and moreover for any hitting subspace G there is a $F \in \mathcal{F}$ with $F \subset G$.

Lemma 3.2 *For every non-extendable intersecting family $\mathcal{F}$ of k-spaces there exists some critical family $\mathcal{G}$ such that*

$$\mathcal{F} = \{F \in \left[{V \atop k}\right] : \exists\, G \in \mathcal{G},\, G \subseteq F\}.$$

Proof. Extend $\mathcal{F}$ to a maximal intersecting family $\mathcal{H}$ of subspaces of V, and take for $\mathcal{G}$ the minimal elements of $\mathcal{H}$. □

The following construction and result are an adaptation of the corresponding results from Erdős and Lovász [6]:

Construction 3.3 *Let $A_1, \ldots, A_k$ be subspaces of V such that* $\dim A_i = i$ *and* $\dim(A_1 + \cdots + A_k) = \binom{k+1}{2}$. *Define*

$$\mathcal{F}_i = \{F \in \left[{V \atop k}\right] : A_i \subseteq F,\ \dim A_j \cap F = 1 \text{ for } j > i\}.$$

Then $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ is a critical, non-extendable, intersecting family of k-spaces, and $|\mathcal{F}_i| = \left[{i+1 \atop 1}\right]\left[{i+2 \atop 1}\right]\cdots\left[{k \atop 1}\right]$ for $1 \leqslant i \leqslant k$.

For subsets Erdős and Lovász proved that a critical, non-extendable, intersecting family of k-sets cannot have more than k^k members. They conjectured that the above construction is best possible but this was disproved by Frankl, Ota and Tokushige [10]. Here we prove the following analogous result.

Theorem 3.4 *Let $\mathcal{F}$ be a critical, intersecting family of subspaces of V of dimension at most k. Then $|\mathcal{F}| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}^k$.*

Proof. Suppose that $|\mathcal{F}| > \begin{bmatrix} k \\ 1 \end{bmatrix}^k$. By induction on i, $0 \leqslant i \leqslant k$, we find an i-dimensional subspace A_i of V such that $|\mathcal{F}_{A_i}| > \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-i}$. Indeed, since by induction $|\mathcal{F}_{A_i}| > 1$ and $\mathcal{F}$ is critical, the subspace A_i is not hitting, and there is an $F \in \mathcal{F}$ disjoint from A_i. Now all elements of $\mathcal{F}_{A_i}$ meet F, and we find $A_{i+1} > A_i$ with $|\mathcal{F}_{A_{i+1}}| > |\mathcal{F}_{A_i}|/\begin{bmatrix} k \\ 1 \end{bmatrix}$. For $i = k$ this is a contradiction. $\square$

Remark 3.5 For $l \leqslant k$ this argument shows that there are not more than $\begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1}$ l-spaces in $\mathcal{F}$.

If $l = 3$ and $\tau > 2$ then for the size of $\mathcal{F}$ the previous remark essentially gives $\begin{bmatrix} 3 \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^2\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$, which is the bound in Corollary 2.10.

Modifying the Erdős-Lovász construction (see Frankl [7]), one can get intersecting families with many l-spaces in the corresponding critical family.

Construction 3.6 *Let $A_1, \ldots, A_l$ be subspaces with $\dim A_1 = 1$, $\dim A_i = k + i - l$ for $i \geqslant 2$. Define $\mathcal{F}_i = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : A_i \leqslant F, \dim(F \cap A_j) \geqslant 1$ for $j > i\}$. Then $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l$ is intersecting and the corresponding critical family has at least $\begin{bmatrix} k-l+2 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} k \\ 1 \end{bmatrix}$ l-spaces.*

For n large enough the Erdős-Ko-Rado theorem for vector spaces follows from the obvious fact that no critical, intersecting family can contain more than one 1-dimensional member. The Hilton-Milner theorem and the stability of the systems follow from $(*)$ which was used to describe the intersecting systems with $\tau = 2$. As remarked above, the fact that the critical family has to contain only spaces of dimension 3 or more limits its size to $O(\begin{bmatrix} n \\ k-3 \end{bmatrix})$, if k is fixed and n is large enough. Stronger and more general stability theorems can be found in Frankl [8] for the subset case.

4 Coloring q-Kneser graphs

In this section, we prove Theorem 1.5. We will need the following result of Bose and Burton [2] and its extension by Metsch [17].

Theorem 4.1 (Bose-Burton) *If $\mathcal{E}$ is a family of 1-subspaces of V such that any k-subspace of V contains at least one element of $\mathcal{E}$, then $|\mathcal{E}| \geqslant \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Furthermore, equality holds if and only if $\mathcal{E} = \begin{bmatrix} H \\ 1 \end{bmatrix}$ for some $(n-k+1)$-subspace H of V.*

Proposition 4.2 (Metsch) *If $\mathcal{E}$ is a family of $\left[{n-k+1 \atop 1}\right] - \varepsilon$ 1-subspaces of V, then the number of k-subspaces of V that are disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k)}$.*

Proof of Theorem 1.5. Suppose that we have a coloring with at most $\left[{n-k+1 \atop 1}\right]$ colors. Let G (the good colors) be the set of colors that are point-pencils and let B (the bad colors) be the remaining set of colors. Then $|G| + |B| \leqslant \left[{n-k+1 \atop 1}\right]$. Suppose $|B| = \varepsilon > 0$. By Proposition 4.2, the number of k-spaces with a color in B is at least $\varepsilon q^{(k-1)(n-k)}$, so that the average size of a bad color class is at least $q^{(k-1)(n-k)}$. This must be smaller than the size of a HM-type family. Thus, by Lemma 2.3,

$$q^{(k-1)(n-k)} \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}.$$

For $k \geqslant 3$ and $q \geqslant 3$, $n \geqslant 2k+1$ or $q = 2$, $n \geqslant 2k+2$, this is a contradiction. (The weaker form of Proposition 4.2, as stated in [17], suffices unless $q = 2$, $n = 2k+2$.) If $|B| = 0$, all color classes are point-pencils, and we are done by Theorem 4.1. □

5 Proof of Theorem 1.6

Let $a + b = n, a < b$ and let $\mathcal{F}_a = \mathcal{F} \cap \left[{V \atop a}\right]$ and $\mathcal{F}_b = \mathcal{F} \cap \left[{V \atop b}\right]$. We prove

$$|\mathcal{F}_a| + |\mathcal{F}_b| \leqslant \left[{n \atop b}\right] \tag{5.4}$$

with equality only if $\mathcal{F}_a = \emptyset$ and $\mathcal{F}_b = \left[{V \atop b}\right]$.

Adding up (5.4) for $n/2 < b \leqslant n$ gives the bound on $|\mathcal{F}|$ in Theorem 1.6 if n is odd; adding the result of Greene and Kleitman [14] that states $|\mathcal{F}_{n/2}| \leqslant \left[{n-1 \atop n/2-1}\right]$ proves it for even n. For the uniqueness part of Theorem 1.6, we only have to note that if n is even then, by results of Godsil and Newman [13], we must have $\mathcal{F}_{n/2} = \{F \in \left[{V \atop n/2}\right] : E \leqslant F\}$ for some $E \in \left[{V \atop 1}\right]$ or $\mathcal{F}_{n/2} = \left[{U \atop n/2}\right]$ for some $U \in \left[{V \atop n-1}\right]$.

Now we prove (5.4). Consider the bipartite graph with vertex set $(\left[{V \atop a}\right], \left[{V \atop b}\right])$ and join $A \in \left[{V \atop a}\right]$ and $B \in \left[{V \atop b}\right]$ if $A \cap B = 0$. Observe that $\mathcal{F}_a \cup \mathcal{F}_b$ is an independent set in this graph. Now, this graph is regular with degree q^{ab}. Therefore any independent set in this graph has size at most $\left[{n \atop b}\right]$ by König's Theorem. Moreover, independent sets of size $\left[{n \atop b}\right]$ can only be $\left[{V \atop a}\right]$ or $\left[{V \atop b}\right]$, but the former is not an intersecting family. This proves (5.4). □

Acknowledgements

Ameera Chowdhury thanks the NSF for supporting her and the Rényi Institute for hosting her while she was an NSF-CESRI fellow during the summer of 2008. Balázs Patkós's research was supported by NSF Grant #: CCF-0728928. Tamás Szőnyi gratefully acknowledges the financial support of NWO, including the support of the DIAMANT and Spinoza projects. He also thanks the Department of Mathematics at TU/e for the warm hospitality. He was partly supported by OTKA Grants T 49662 and NK 67867.

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